Affine connections in special relativity

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Affine geometry, implemented by a "Lorentz connection" for accelerated frames of reference in pseudo-Euclidean space-times, a "Fourier connection" for abstract Hilbert spaces associated with classical Fourier analysis, and a "quantum connection" for quantum-mechanical Hilbert spaces, gives new perspectives on special relativity, where affine connections are usually interpreted phenomenologically, in a manner which obscures their geometric significance. The connections are determined by "absolute constants," whose covariant derivatives, expressed in terms of the connection coefficients, vanish identically. In the deterministic theory of prequantum physics, the space-time connection is expressed in terms of "local Lorentz transformations," which represent the motion of accelerated frames relative to inertial frames. In quantum theory, relative motion is not well defined, but local Lorentz transformations, represented by locally isomorphic symmetry groups, remain well defined, and become the basis of quantum field theory.

I. INTRODUCTION

It is well known that the affine-connection formalism of general relativity can be used in special relativity, but the existence of inertial frames makes its use optional. However, there are ways in which the connection concept can be even more useful in special relativity, where its physical significance is more readily interpreted for problems of practical importance.

Relativistic dynamics can be formulated on a nonholonomic basis (see Ref. 1, Misner *et al.*, p. 210, and Weinberg) on which the space-time metric remains the Lorentz metric $\eta_{\alpha\beta}$ (the pseudo-Euclidean metric of Minkowski space), and the space-time connection is expressed in terms of an orthonormal tetrad or vierbein e_α^μ where α,β $(=0,1,2,3)$ are Lorentz tensor indices,[2] while μ $(=0,1,2,3)$ is a coordinate tensor index subject to general coordinate transformations which do not affect α or β, whereas α and β are subject to local Lorentz transformations which do not effect μ.

The e_α^μ have vanishing covariant derivatives, $e^\mu_{\alpha;\nu}=0$, with respect to any space-time coordinates x^ν for which the ordinary partial derivatives are $e^\mu_{\alpha,\nu}\equiv\partial e^\mu_\alpha/\partial x^\nu$, and

$$e^\mu_{\alpha;\nu}=e^\mu_{\alpha,\nu}+\Gamma^\mu_{\nu\sigma}e^\sigma_\alpha-\omega^\beta_{\alpha\nu}e^\mu_\beta\ , \qquad (1.1)$$

relating the Christoffel symbols $\Gamma^\mu_{\nu\sigma}$ (connection coefficients on a holonomic basis) to the connection ω on the nonholonomic basis (summation convention is assumed for repeated indices).

In this respect the e^μ_α have a useful role as "absolute constants" whose 16 components represent the 16 degrees of freedom emphasized by Einstein,[3] and whose transformation properties illustrate the principles of *general covariance* and *local Lorentz covariance*, which are the basic symmetries of Einstein's relativity, and are equivalent according to the equivalence principle (see Ref. 1, Misner *et al.*, p. 386), which allows the laws of nature to be formulated on a holonomic or a nonholonomic basis with equal validity.

Under a general coordinate transformation $dx^\mu\to\lambda^\mu_\nu dx^\nu$, where the λ^μ_ν satisfy the integrability condition $(\lambda^\mu_{\nu,\sigma}=\lambda^\mu_{\sigma,\nu})$, together with a local Lorentz transformation Λ, whose components Λ^β_α are not constant (and do not satisfy the integrability condition), the e^μ_α undergo the combined transformation

$$e^\mu_\alpha\to\lambda^\mu_\nu\Lambda^\beta_\alpha e^\nu_\beta\ , \qquad (1.2)$$

which can impose up to ten constraints because λ is defined in terms of four functions, and Λ is defined in terms of six functions (six Lie parameters, generalized to six functions of the x^μ). The remaining six independent e^μ_α represent the six components emphasized by Rindler,[4] indicating how the vierbein serves in lieu of the usual ten-component metric.

Under the coordinate transformation λ the $\omega^\alpha_{\beta\mu}$ transform as a covariant tensor on the subscript μ. However, under the local Lorentz transformation Λ (with inverse $\kappa=\Lambda^{-1}$),

$$\omega^\alpha_{\beta\mu}\to\kappa^\alpha_\zeta\Lambda^\gamma_\beta(\omega^\zeta_{\gamma\mu}-\Lambda^\zeta_\xi\kappa^\xi_{\gamma,\mu})\ , \qquad (1.3)$$

indicating how ω differs from a tensor when $\kappa^\alpha_{\beta,\mu}\neq0$.

In special relativity any problem can be analyzed relative to an inertial frame in which $e^\mu_\alpha=\delta^\mu_\alpha$ (the 4-index Kronecker δ), and $\Gamma=\omega=0$. The x^μ are then the Cartesian coordinates of a global Lorentz frame, and the resulting formalism is Lorentz covariant, i.e., form invariant under constant κ and space-time translations of the 10-parameter Poincare isometry group of Minkowski space.[5] Equivalently, the formalism is covariant under linear coordinate transformations for which the constants λ^μ_ν are Lorentz transformation coefficients.

For the transformation to an arbitrary frame of reference, Eq. (1.2) then gives

$$\delta^\mu_\alpha \to e^\mu_\alpha = \lambda^\mu_\nu \Lambda^\nu_\alpha \ . \tag{1.4}$$

If comoving coordinates are chosen for an accelerated frame, the e^μ_α cannot reduce to Lorentz-transformation coefficients, because λ must satisfy the integrability condition, whereas Λ cannot. The same restriction applies to the reciprocal tetrad f^α_μ, which satisfies the conditions $f^\alpha_\mu e^\mu_\beta = \delta^\alpha_\beta$ and $e^\mu_\alpha f^\alpha_\nu = \delta^\mu_\nu$, and in terms of which the metric of the accelerated frame (expressed on a holonomic basis) has the form

$$g_{\mu\nu} = \eta_{\alpha\beta} f^\alpha_\mu f^\beta_\nu \ . \tag{1.5}$$

Hence, $g_{\mu\nu}$ cannot reduce to $\eta_{\mu\nu}$. Conversely, if $g_{\mu\nu}$ is erroneously assumed to reduce to $\eta_{\mu\nu}$, then this false assumption leads to a contradiction[6] which is associated with the "clock paradox."

If there is no coordinate transformation, i.e., if $\lambda^\mu_\nu = \delta^\mu_\nu$, then Eq. (1.4) implies $e^\mu_\alpha = \Lambda^\mu_\alpha$, Eq. (1.5) implies $g_{\mu\nu} = \eta_{\mu\nu}$, and Eq. (1.1) implies $\Lambda^\mu_{\alpha;\nu} = 0$, where

$$\Lambda^\mu_{\alpha;\nu} = \Lambda^\mu_{\alpha,\nu} - \omega^\beta_{\alpha\nu}\Lambda^\mu_\beta \ , \tag{1.6}$$

noting that the x^μ are comoving coordinates for the inertial frame (characterized by $\Gamma = 0$), and the Λ^μ_α are a comoving tetrad for the accelerated frame (characterized by $\omega \neq 0$). *Equation (1.3) is equivalent to Eq. (1.6), and quantifies the equivalence principle, which relates acceleration to inertia* via *local Lorentz covariance* (see Ref. 1, Misner *et al.*, 386).

$\Lambda^\mu_{\alpha;\nu}$ is a *generally covariant* derivative, in the sense that ν is a coordinate tensor index. It is also possible to define the *locally Lorentz-covariant* derivative:

$$\Lambda^\mu_{\alpha;\beta} = \Lambda^\mu_{\alpha,\beta} - G^\gamma_{\alpha\beta}\Lambda^\mu_\gamma \ , \tag{1.7}$$

where

$$\Lambda^\mu_{\alpha,\beta} \equiv \Lambda^\mu_{\alpha,\nu}\Lambda^\nu_\beta \ , \tag{1.8}$$

and

$$G^\gamma_{\alpha\beta} \equiv \omega^\gamma_{\alpha\mu}\Lambda^\mu_\beta \ , \tag{1.9}$$

noting the identity $\Lambda^\mu_{\alpha;\beta} = \Lambda^\mu_{\alpha;\nu}\Lambda^\nu_\beta$, which is a contravariant *coordinate vector*, and a covariant second-rank *Lorentz tensor*.

Since $g_{\mu\nu} = \eta_{\mu\nu}$, for the combination of x^μ and e^μ_α used here, $G^\gamma_{\alpha\beta}$ can be regarded as the *Lorentz connection*, which is essential for analyzing problems relative to accelerated frames in special relativity.

Defining $G_{\alpha\beta\gamma} \equiv \eta_{\alpha\zeta}G^\zeta_{\beta\gamma}$, and noting that $\eta_{\alpha\beta;\gamma} = 0$, it follows that

$$G_{\alpha\beta\gamma} + G_{\beta\alpha\gamma} = 0 \ . \tag{1.10}$$

This is different from the symmetry $\Gamma^\mu_{\nu\sigma} = \Gamma^\mu_{\sigma\nu}$ of the Christoffel symbols.

Because of the asymmetry of G, the relation between an electromagnetic field tensor $F_{\alpha\beta}$ and a 4-potential A_α, expressed relative to an accelerated frame, must be kept in manifestly covariant form:

$$F_{\alpha\beta} = A_{\alpha;\beta} - A_{\beta;\alpha} \neq A_{\alpha,\beta} - A_{\beta,\alpha} \ , \tag{1.11}$$

because G does not drop out as Γ does.

For a particle having one unit of electric charge and one unit of proper mass with a proper time element $d\tau = (\eta_{\mu\nu}\, dx^\mu dx^\nu)^{1/2}$, the 4-velocity, $V^\alpha = (dx^\mu/d\tau)\kappa^\alpha_\mu$, relative to the accelerated frame, satisfies the Einstein-Lorentz equation

$$dV^\alpha/d\tau + G^\alpha_{\beta\gamma}V^\beta V^\gamma = V^\beta F^\alpha_\beta \ , \tag{1.12}$$

where $F^\alpha_\beta \equiv F_{\beta\gamma}\eta^{\gamma\alpha}$. The F term (the tensor force of electromagnetism) is the 4-acceleration, while the G term (the nontensor force derived from the connection) is the effective gravitational or inertial acceleration, including centrifugal and Coriolis terms, which, although indistinguishable from gravity (according to the equivalence principle), cannot be attributed to the metric since $g_{\mu\nu} = \eta_{\mu\nu}$ here.

Thus the Lorentz-covariant formalism of inertial frames is replaced by a locally Lorentz-covariant formalism for arbitrarily accelerated frames, in which the dynamical equations are form invariant under local Lorentz transformations, which preserve the Lorentz metric η but transform the Lorentz connection G, without entailing any coordinate transformation. Hence the x^μ can be the Cartesian coordinates of any convenient inertial frame, and the Λ^μ_α, which are functions of the x^μ, define the accelerated frame (and determine G) completely.

Defining the second covariant derivative $A^\alpha_{;\beta\gamma} \equiv (A^\alpha_{;\beta})_{;\gamma}$, it follows that $A^\alpha_{;\beta\gamma} = A^\alpha_{;\gamma\beta}$ because the space-time (i.e., the idealized model of special relativity) is intrinsically flat, and hence the curvature tensor vanishes.

For a charged-matter distribution of electric 4-current J^α, and energy-momentum tensor $T^{\alpha\beta}$ the accelerated-frame dynamics can be expressed by generalized Maxwell-Lorentz equations,

$$\eta^{\beta\gamma}A^\alpha_{;\beta\gamma} - \eta^{\alpha\gamma}A^\beta_{;\beta\gamma} = 4\pi J^\alpha \ , \tag{1.13a}$$

$$T^{\alpha\beta}_{,\beta} + G^\alpha_{\gamma\beta}T^{\gamma\beta} + G^\beta_{\gamma\beta}T^{\alpha\gamma} = J^\beta F^\alpha_\beta \ , \tag{1.13b}$$

using geometrized units (see Ref. 1, Misner *et al.*, p. 36) with $A^\alpha \equiv \eta^{\alpha\beta}A_\beta$. If a "proper frame" exists, it can be defined by the conditions $J^\alpha = J^0\delta^\alpha_0$, $T^{\alpha\beta} = T^{00}\delta^\alpha_0\delta^\beta_0$, $T^{\alpha\beta}_{,\beta} = 0$, for which Eqs. (1.13) give a balance of electric and inertial forces, equivalent (in the sense of the equivalence principle) to the balance of gravity and electromagnetism which maintains equilibrium on the surface of the earth.

This approach can be useful for analyzing formal relationships, but does not necessarily simplify the mathematical problem. Other approaches are investigated here for that reason. In particular, the Hilbert-space connection,[7] useful in general-relativistic quantum theory, can also be useful in special relativity for a geometric interpretation of classical physics and quantum mechanics.

II. FOURIER CONNECTION

When the problem is analyzed relative to an inertial frame, with $e_\alpha^\mu = \delta_\alpha^\mu$, it can often be simplified by Fourier expansion,

$$A^\mu = A_n^\mu f^n \ , \tag{2.1}$$

equivalent to Hilbert-space superposition, where the summation convention over n (the Hilbert-space index) implies Lebesgue-Stieltjes integration[8] which reduces to summation over discrete indices and Riemann integration over continuous n,[9] noting that n is a real variable, A_n^μ can be a complex variable (but not a function of the x^μ), and f^n can be a complex function of the x^μ.

Since $A_{n,\nu}^\mu = 0$, the covariant derivative has the form

$$A_{n;\nu}^\mu = -K_{n\nu}^m A_m^\mu \ , \tag{2.2}$$

where K is the *Fourier connection*, i.e., the affine connection of the Hilbert space spanned by the f^n, which are assumed to have a reciprocal set f_n, with reciprocity relations

$$f_n(x) f^n(y) = \delta(x-y) \ , \tag{2.3a}$$

$$\int f_m(x) f^n(x) d^4x = I_m^n \ , \tag{2.3b}$$

where the integral is taken over all values of the x^μ (symbolized by x and y), $\delta(x-y)$ is the 4-dimensional Dirac δ function, and I_m^n is the Hilbert-space identity operator, which reduces to the Kronecker δ for discrete indices, and the Dirac δ function for continuous indices.

The f^n and f_n, like the e_α^μ and f_μ^α, have vanishing covariant derivatives, $f^n_{;\mu} = 0$, where

$$f^n_{;\mu} = f^n_{,\mu} + K^n_{m\mu} f^m \ . \tag{2.4}$$

Equations (2.3) and (2.4) then determine K, which, like A_n^μ, can be complex, but which, unlike ω, is not a function of the x^μ. Thus the Hilbert space has a well-defined affine structure with a relativity and an equivalence principle analogous to Eq. (1.6) even though the transformation is not relativistic in the Einsteinian sense.

Maxwell's equations (1.13a) can be written in the form

$$K_{n\sigma}^m K_{m\nu}^l (\eta^{\nu\sigma} A_l^\mu - \eta^{\mu\sigma} A_l^\nu) = 4\pi J_n^\mu \ , \tag{2.5}$$

where $J^\mu = J_n^\mu f^n$ and, to avoid extraneous terms, it is assumed that the f^n and $f^n_{,\mu}$ vanish on the boundary of the integral (2.3b).

Radiation modes (which carry the photon momentum)[10] satisfy the homogeneous Maxwell equation [(2.5) with $J_n^\mu = 0$], which gives an abstract geometric interpretation of radiation states and their characteristic parameters (frequencies, angular momenta, etc.).

Radiation reaction is due to the interaction of a particle with its own radiation field, which carries off momentum and causes a reaction in accordance with Newton's third law. The theory of this phenomenon remains problematical (Marx, and Mo and Papas)[11] because the results depend on the assumptions which are made in deriving the reactive force. The total 4-acceleration is given by Eq. (1.12). The reactive 4-acceleration is derived from a modified potential,

$$\mathcal{A}^\mu = A_m^\mu P_n^m f^n \ , \tag{2.6}$$

where P is a Hilbert-space projection operator for selecting radiation modes emitted by the particle, rejecting radiation modes not emitted by the particle, excluding the particle's nonradiative field (which carries no momentum), and excluding Coulomb fields due to external sources. Radiation reaction depends on the definition of P, which in turn depends on the characteristic modes n of the entire system as a whole. Hence the reactive force depends not only on the particle's motion, but also on its surroundings (on how the environment echoes the radiation). A particle radiating in a laser cavity experiences a different reaction from one accelerating through interstellar space. In this respect, rather than conforming entirely to Newton's third law, the effect has some analogy to Mach's principle,[12] which attributes natural laws to the entire state of the universe as a whole, thereby giving nature a coherence which is quantified by Einstein's theories. Radiation reaction, like friction and dispersion, can be treated approximately with phenomenological formulas, but its exact treatment requires detailed knowledge of the entire system of which the particle is a part.

The identity operator I_m^n is the mixed form of the Hilbert-space metric,

$$h_{mn} = \int f_m^* f_n d^4x \ , \tag{2.7}$$

where the asterisk indicates complex conjugation, and h is Hermitian with inverse $h^{mn} = (h^{nm})^*$.

The f^n and f_n are related by the condition

$$f^m = h^{mn} f_n^* \; . \quad (2.8)$$

Defining $K_{mn\mu} \equiv h_{ml} K^l_{n\mu}$, the conditions $h_{mn;\mu} = 0 = h_{mn,\mu}$ give

$$K^*_{mn\mu} = -K_{nm\mu} \; , \quad (2.9)$$

a consistency condition on K which is skew-Hermitian on the Hilbert-space indices.

For any Hilbert-space operator, such as the projection operator P, it is convenient to define an adjoint $\not{P}$, such that

$$\not{P}^m_n \equiv h^{ml} (P^q_l)^* h_{qn} \; . \quad (2.10)$$

The identity operator is self-adjoint, i.e., $\not{I}^n_m = I^n_m$. Equation (2.9) implies $\not{K}^m_{n\mu} = -K^m_{n\mu}$, i.e., K is skew-adjoint.

A linear automorphism (a mapping of the Hilbert space onto itself) is defined by the transformation $f_m \to U^n_m f_n$, where the U^n_m are complex coefficients with $U^n_{m,\mu} = 0$. Hilbert-space symmetries are defined by isometries, i.e., automorphisms which preserve h through the relation

$$(U^l_m)^* h_{lq} U^q_n = h_{mn} \; , \quad (2.11)$$

which gives $\not{U} = U^{-1}$. If h is expressed in diagonalized form (as is always possible), then by definition U is unitary if h is a positive-definite metric, and pseudounitary if h is indefinite (having both positive and negative eigenvalues).

In Eq. (2.6) P has the transformation property

$$P^m_n \to \not{U}^m_l P^l_q U^q_n \; , \quad (2.12)$$

but is not necessarily form invariant under transformations satisfying Eq. (2.11). In this respect, the formalism is not covariant under Hilbert-space symmetry groups. However, this is not a flaw in the theory, because Hilbert-space symmetry does not necessarily reflect any laws of nature. The asymmetry of P can actually be useful, because it implies that there may be a Fourier basis on which the theory reduces to a particularly simple form in which the states n can be regarded as canonical modes of the system.

Under a combined isometry of the Hilbert space and Minkowski space the A^μ_m undergo the transformation

$$A^\mu_m \to \Lambda^\mu_\nu A^\nu_n U^n_m \; , \quad (2.13)$$

where $A^\mu_{m,\nu} = 0$ implies $\Lambda^\mu_{\nu,\sigma} = 0$, so that the Fourier formalism here is covariant under the Poincaré group (the usual symmetry group of special relativity) but not under local Lorentz transformations. Hence the K and G formalisms based on the Fourier and Lorentz connections, respectively, are mutually complementary. Equation (2.2) assumes $\Gamma = \omega = 0$.

III. SPINOR CONNECTION

In the first quantization when the problem is analyzed relative to an accelerated frame on the nonholonomic basis Λ^μ_α determined by the local Lorentz transformation from an inertial frame with Cartesian coordinates x^μ, the Maxwell-Dirac equations can be written in the form

$$F^{\alpha\beta}{}_{;\beta} = 4\pi \not{\varphi} \gamma^\alpha \varphi, \quad (3.1a)$$

$$\gamma^\alpha (i\nabla_\alpha - A_\alpha)\varphi = \varphi \; , \quad (3.1b)$$

where Eq. (3.1a) is equivalent to Eq. (1.13a), φ is a 4-component spinor field (expressed in closed form as a column matrix), $i^2 = -1$, the γ^α are the Dirac matrices (see Ref. 13, p. 70), and ∇_α is a covariant derivative operator such that $\nabla_\alpha \varphi = \varphi_{;\alpha}$. The locally Lorentz-covariant derivative

$$\varphi_{;\alpha} = \varphi_{,\mu} \Lambda^\mu_\alpha + \Upsilon_\alpha \varphi \; , \quad (3.2)$$

using the spinor connection[14] Υ_α expressed as a 4×4 matrix in geometrized units for a spin-$\frac{1}{2}$ particle of unit charge and mass. $\not{\varphi}$ here is $\bar{\varphi}$ defined by Schweber, with Planck's reduced constant $\hbar \equiv 1$.

Corresponding to the space-time basis Λ^μ_α there is the spinor basis $\mathcal{S}$ expressed in closed form as a complex 4×4 matrix field satisfying the homomorphic identity

$$\Lambda^\mu_\alpha \mathcal{S} \gamma^\alpha \mathcal{S}^{-1} = \gamma^\mu \; , \quad (3.3)$$

which relates the Lorentz group to its covering group.[15] As Eqs. (1.6)—(1.9) relate G to Λ, so Υ is related to $\mathcal{S}$ by noting $\mathcal{S}_{;\mu} = 0$, where

$$\mathcal{S}_{;\mu} = \mathcal{S}_{,\mu} - \Omega_\mu \mathcal{S} \; , \quad (3.4)$$

using spinor connection Ω_μ for the generally covariant derivative so that

$$\Upsilon_\alpha = \Omega_\mu \Lambda^\mu_\alpha \, . \quad (3.5)$$

G and Υ are related by noting that the γ^α are absolute constants,[16] so $\gamma^\alpha_{;\beta} = 0$, and on the basis used here $\gamma^\alpha_{,\mu} = 0$, giving

$$\gamma^\alpha_{;\beta} = G^\alpha_{\xi\beta} \gamma^\xi + \Upsilon_\beta \gamma^\alpha - \gamma^\alpha \Upsilon_\beta \; , \quad (3.6)$$

consistent with Eq. (3.3) in the sense that under local Lorentz transformations γ^α is a mixed second-rank spinor as well as a contravariant vector.

Spinor space, interpreted as the universal covering of Minkowski space,[15] has the Hermitian metric $\beta = \beta^\dagger = \beta^{-1} = \gamma^0$ in the representation used here in which

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \; , \quad (3.7)$$

and the dagger denotes Hermitian conjugation.

$\mathcal{S}$ is further defined by the isometry

$$\mathcal{S}^\dagger \beta \mathcal{S} = \beta , \tag{3.8}$$

which, together with Eq. (3.3), implies that β and the relation $\beta = \gamma^0$ are invariant under local Lorentz transformations; so the formalism is covariant.

φ is a complex vector in a pseudounitary metric space, with scalar product $\bar{\varphi}\varphi$ defined in terms of the adjoint vector $\bar{\varphi} \equiv \varphi^\dagger \beta$. For any matrix operator Θ it is useful to define the adjoint operator,

$$\bar{\Theta} \equiv \beta^{-1} \Theta^\dagger \beta . \tag{3.9}$$

It then follows that γ^α is self-adjoint ($\bar{\gamma}^\alpha = \gamma^\alpha$), Ω_μ is skew-adjoint ($\bar{\Omega}_\mu = -\Omega_\mu$), and $\mathcal{S}$ is pseudounitary ($\bar{\mathcal{S}} = \mathcal{S}^{-1}$).

Under relativistic transformations spinor space does not have unitary symmetry, just as space-time (which is pseudo-orthogonal) does not have orthogonal symmetry. However, under nonrelativistic transformations (not entailing a change in the Lorentz frame) the 3-space of special relativity does have orthogonal symmetry.

If the analysis is confined to an inertial frame ($G = 0$) with $\Lambda^\mu_\alpha = \delta^\mu_\alpha$, then φ is a complex vector in a unitary Hilbert space, and in the representation used here the scalar product can be defined in integral form,

$$\int \varphi^\dagger \varphi \, d^3x = 1 , \tag{3.10}$$

where the integral is taken over all values of the spatial coordinates x^i, for $i = 1, 2, 3$; and the time coordinate x^0 is the independent variable of Eqs. (3.1). This is the result in a "coordinate representation" in which the x^i are quantum numbers, i.e., eigenvalues of observables,[17] and φ is assumed to have unit norm.

The formalism is no longer manifestly covariant, but, nevertheless, it is Lorentz covariant in the sense that the same results can be derived in any Lorentz frame.

This dual role of the Dirac field φ makes it possible to use the spinor connection to define the various "pictures" of quantum theory in terms of Hilbert-space geometry.

In the Schrodinger picture $\Omega = 0$, and Eq. (3.1b) can be expressed in the form

$$i\varphi_{,0} = H\varphi , \tag{3.11}$$

where H is the Schrödinger Hamiltonian

$$H = -i\,\alpha^j \partial_j + \alpha^\mu A_\mu + \beta , \tag{3.12}$$

using summation convention over $j = 1, 2, 3$ and $\mu = 0, 1, 2, 3$, with $\alpha^\mu \equiv \gamma^0 \gamma^\mu$, $\partial_\mu \equiv \partial / \partial x^\mu$, and $\beta = \gamma^0$.

The Dirac picture can be introduced by letting $\varphi = S\psi$, where S is a Hilbert-space operator, and ψ satisfies the modified Schrödinger equation,

$$i\psi_{,0} = S^{-1} \alpha^\mu A_\mu S \psi , \tag{3.13}$$

using a transformed interaction Hamiltonian. Equation (3.4), with $\Omega_\mu = \Omega_0 \delta^0_\mu$, then gives

$$\Omega_0 = -\alpha^j \partial_j - i\beta , \tag{3.14}$$

so that $i\Omega_0$ is the zeroth-order or "unperturbed" Hamiltonian, and S has the exponential form $S = \exp(\Omega_0 x^0)$, making the Dirac picture a viable approach to quantum field theory (see Ref. 13, p. 317).

The Heisenberg picture can be introduced by letting $\psi_{,0} = 0$, which gives

$$S_{,0} S^{-1} = \Omega_0 = -iH , \tag{3.15}$$

where H is the complete Hamiltonian (3.12). This approach to the Maxwell-Dirac equations is problematical,[18] because Eq. (3.15) cannot be solved explicitly for arbitrary A_μ, and there are covariance problems due to the ambiguity regarding whether spinor space is unitary or pseudounitary.

As long as the analysis is confined to a Lorentz frame, ψ can be treated as a complex vector in a unitary Hilbert space with scalar product (3.10). However, for the analysis relative to an accelerated frame which entails the transformation $\psi \to \mathcal{S}\psi$, the scalar product assumes the form

$$\int \psi^\dagger \mathcal{S}^\dagger \mathcal{S} \psi d^3x , \tag{3.16}$$

where $\mathcal{S}$ satisfies equation (3.8), so that the Hilbert space is no longer unitary. This problem can be avoided in special relativity, but it is an essential part of general relativity, where the Einstein-Dirac equations are problematical[19] in a manner which may be resolvable by the second quantization[20]; although the problem has not yet been solved, and may entail changes in the Hamiltonian formalism.

In the Schrödinger picture, in which $\gamma^\mu_{,\nu} = 0$, it seems logical to generalize this to the condition $\gamma^\mu_{;\nu} = 0$ for other pictures. However, in the Heisenberg picture in which $\psi_{,0} = 0$, it seems logical to generalize this to the condition $\psi_{;\mu} = 0$, where

$$\psi_{;\mu} = \psi_{,\mu} + Q_\mu \psi , \tag{3.17}$$

and Q_μ is a "quantum connection," i.e., an affine connection for the Hilbert space of ψ. In special relativity it suffices to let $Q_j = -\partial_j$, and $Q_0 = iH$ for the Schrödinger picture, so that H is the essential part of the connection. When Heisenberg proved that a Hamiltonian can vanish, he proved it is not a tensor because a tensor cannot vanish in any frame of reference unless it vanishes in all frames. In effect H is the "quantum connection," and the Heisenberg picture is a quantum analog of an inertial frame. In classical physics the Hamilton-Jacobi method establishes similar results.[21]

In the second quantization the Q formalism is more applicable than in the first, and readily generalizes to general relativity[7] because it is manifestly covariant under conditions where Q_μ is a globally well-defined concept, but a Hamiltonian can only be defined locally (in accordance with the principle of local Lorentz covariance). Quantum-mechanical derivatives are reinterpretable as covariant derivatives, and Q can vanish locally (a local Heisenberg picture, i.e., a quantum analog of a local inertial frame) owing to its properties as an affine connection; but Q cannot vanish globally, except in the limiting case of special relativity.

IV. CONCLUSIONS

The affine connection is as important in special relativity as in general relativity, but its geometric significance is obscured by phenomenological interpretations, as in thermodynamics, where geometry is deemphasized in favor of chemical terminology.[22]

The various connections are conveniently determined from absolute constants, i.e., quantities whose covariant derivatives vanish identically, as illustrated by Eqs. (1.1), (1.6), (1.7), (2.4), (3.4), (3.6), and (3.17).

The quantum-mechanical equation (3.17) determines Q_μ for special relativity, where the Hamiltonian operator is essentially a phenomenological interpretation of the Hilbert-space connection. In the first quantization the spinor connection, devised for a covariant treatment of the Dirac equation (3.1b), becomes a natural vehicle for introducing the "quantum connection" Q_μ. In the second quantization Q_μ can be determined in such a way that Q_0 is the global Hamiltonian, and the remaining $Q_j = 0$.

In general relativity the "global Hamiltonian" is not covariant, but Q_μ is manifestly covariant, and allows a local Hamiltonian to be defined from a global "quantum connection."

The affine connection is defined independently of a metric (Eisenhart[23]), but in physical applications there are associated metric spaces related by a topological-group formalism.[15] However, nonmetric affine geometry, in which covariant derivatives are defined directly in terms of the connection, is an equally valid basis for theoretical physics.

The "quantum connection" Q_μ, which is trivially associated with the Hamiltonian in special relativity, remains undetermined and speculative in general relativity, where "absolute constants" determining Q are unknown.

The "Lorentz connection" (1.9) is useful for treating problematical aspects of accelerated frames in which the kinematics requires an affine connection, but the chronometry can be treated with the Cartesian coordinates x^μ of an inertial frame thereby obviating the clock paradox[24] without introducing comoving coordinates or a non-Lorentzian metric for the accelerated frame of reference.

The "Fourier connection" (2.2) allows a geometric interpretation of spectral analysis, and associated problems such as radiation reaction while the second quantization appears necessary to resolve the problems associated with the Dirac equation.

The results here are consistent with conclusions of Cartan,[25] who proved that the components of the spinor connection (the 4×4 matrix Ω_μ) cannot be restricted to the domain of ordinary complex numbers. Contrary to causing difficulty, this allows the connection formalism to be used for the definition of quantum-mechanical "pictures," and in other applications where there are transformations which do not necessarily involve any change in the relativistic frame of reference.

Much work remains to be done on this, e.g., in the geometric classification of elementary-particle spectra, where the problem of combining relativistic and nonrelativistic transformation properties has not yet been solved.

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