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# Thermodynamics of the statistical model of atoms in a very strong magnetic field: An approach to the properties of condensed matter in the outer crust of pulsars

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Condensed matter at very high pressure is viewed as a collection of squeezed atoms, which are described in terms of a statistical model. A very strong magnetic field is supposed to be present, and temperature effects are taken into account, but exchange is neglected, because at nonzero temperature its effect tends to be canceled by long-range correlations. The free energy of such an atom can be calculated explicitly as a function of the appropriate variables, after which the thermodynamic properties are derived by standard methods. For canonical pulsars, with an iron crust having already cooled down to about $10^5$ K, and surface magnetic fields of the order of $10^{12}$ G, temperature corrections turn out to be negligible. However, they represent sizable effects in the case of very young and hot pulsars.

## I. INTRODUCTION

In a previous work[1] the pressure-density relationship for condensed matter in a very strong magnetic field and at zero temperature was derived under the simplifying assumption (valid at high pressure) that condensed matter consists of atoms squeezed together. These atoms were described in terms of a statistical model,[2-4] which incorporated the assumption that a very strong magnetic field was present. The procedure has obvious limitations: cohesion cannot be explained in such a simplified scheme, and spherical symmetry must be assumed in order to handle the equations more easily. This limits the applications to deformations preserving the macroscopic shapes (no shears), the model being thus able to yield information only on the diagonal elements of the stress tensor. Nevertheless, the simplicity of the method recommends it as a first step in investigating the properties of matter in the outer crust of magnetic neutron stars[5] and, possibly, of supercompressed matter in the laboratory.[6]

Our approach[1,7] was inspired by some early work on the equations of state for normal condensed matter,[8,9] derived from the conventional statistical model of atoms,[10,11] and aiming at understanding the behavior of matter under high pressure, e.g., inside white dwarfs or in nuclear explosions. Later, this work has been completed by taking into account temperature effects, treated either as perturbations,[12] or exactly.[9,13] In the present paper we extend in a similar manner the method of Ref. 1 to nonzero temperatures.

What distinguishes the statistical model of atoms used here[2-4] from the conventional one[11] is the fact that, because of the anisotropy introduced by the magnetic field, the transverse and longitudinal motions of the electrons (with respect to the direction of $\vec{B}$) must be treated differently. In the degenerate case, assuming that the magnetic field is sufficiently strong, the transverse Fermi momentum, $p_F$, is position-independent, being determined only by the magnetic field (adiabatic hypothesis); viz.,

$$p_F{}^2 = 2\hbar eB/c\,. \tag{1.1}$$

In this approximation the transverse magnetic motion decouples from the rest; after subtracting its contribution, the energy of a single electron is written as

$$\epsilon = q^2/2m + e\,\varphi(\vec{r}), \tag{1.2}$$

where $q$ is the longitudinal momentum and $\varphi(\vec{r})$ the electrostatic potential. The longitudinal Fermi momentum, $q_F(\vec{r})$, is locally related to the number density of electrons, $n(\vec{r})$, by the equation $n(\vec{r}) = (eB/2\pi^2\hbar^2c)q_F(\vec{r})$. Moreover, it is assumed that everywhere (except for the close neighborhood of the nucleus, where the statistical description is inadequate) $q_F(\vec{r}) \ll p_F$, which is a convenient expression of the fact that the magnetic field is very strong.

In extending the method to nonzero temperatures, we have in mind the conditions existing in the outer crust of pulsars, down to a few meters below the surface: temperatures[14] $T \simeq 10^5$ K (except for very young pulsars, which are much hotter), and magnetic fields[15] $B \simeq 10^{12}$ G. Even at much higher temperatures and lower field intensities than the typical figures quoted above, one has $kT \ll p_F^2/2m$. Therefore, as far as the transverse motion is concerned, the electrons continue to be degenerate; the magnetic motion is so fast that temperature, like Coulomb forces, practically does not affect it. This is not necessarily true for the much slower longitudinal motion, and it may happen that $kT \simeq q_F^2/2m$; in this case, from the point of view of the longitudinal motion, the electrons must be treated as nondegenerate. Neglecting the vanishingly small contribution of the excited magnetic levels,[1,4] the mean occupation number of a one-electron state with energy $\epsilon$ and transverse momentum $p$ will then be

$$\nu(\epsilon,p)=\begin{cases}\{\exp[(\epsilon-\mu)/kT]+1\}^{-1}, & p\leq p_F\\ 0 & , \ p>p_F\,.\end{cases} \tag{1.3}$$

These considerations are applied first to a gas of noninteracting electrons (Sec. II) in order to show how the method works in a very simple case and to prepare the ground for the more complex statistical model. The latter is examined in Sec. III, which represents the main part of the paper. After deriving the basic equation of the model and its range of validity, we use the March[11] variational principle to obtain the expression of the free energy, from which the thermodynamic properties are derived by standard methods. Numerical results are given in Sec. IV, followed by a short discussion (Sec. V). The low-temperature limit is examined briefly in Appendix A. The explicit expressions of the various contributions to the energy in terms of the solution of the basic equation are collected in Appendix B.

## II. NONINTERACTING ELECTRONS [16]

We consider first a simple, if unrealistic (except at very high pressure), example: a gas of electrons without Coulomb interactions, but interacting with a very strong external magnetic field $\vec{B}$. Apart from showing how the method works in the simplest possible situation, this is needed as a basis for the derivation of the free energy in the statistical model (Sec. III).

Let the gas consist of $Z$ electrons, occupying a volume $v$ at temperature $T$. To derive its macroscopic properties we start from the grand canonical ensemble and calculate the free energy

$$F(Z,v,T)=-kT\ln Q+\mu Z. \tag{2.1}$$

The grand potential and the total number of particles are

$$\ln Q=\sum \ln\{1+\exp[(\mu-\epsilon)/kT]\}, \tag{2.2}$$

$$Z=\sum\{\exp[(\epsilon-\mu)/kT]+1\}^{-1}, \tag{2.3}$$

where $\epsilon$ is given by Eq. (1.2) with $\varphi(\vec{r})=0$. The summation over the microscopic states amounts to integrations over the transverse and longitudinal momenta, $p$ and $q$, with a density-of-states factor $[v/(2\pi\hbar)^3]\,2\pi p\,dp\,dq$. Using Eqs. (1.1) and (1.3) one obtains

$$\ln Q=\frac{veB}{2\pi^2\hbar^2c}(2mkT)^{1/2}I_{1/2}(\mu/kT), \tag{2.4}$$

$$Z=\frac{veB}{4\pi^2\hbar^2c}(2mkT)^{1/2}I_{-1/2}(\mu/kT), \tag{2.5}$$

where

$$I_\alpha(\eta)=\int_0^\infty\frac{y^\alpha dy}{\exp(y-\eta)+1} \tag{2.6}$$

are the usual Fermi-Dirac functions.[17] Hence

$$F(Z,v,T)=ZkT\left(\eta-2\,\frac{I_{1/2}(\eta)}{I_{-1/2}(\eta)}\right), \tag{2.7}$$

with

$$\eta=\mu/kT. \tag{2.8}$$

Now the various thermodynamic quantities are calculated straightforwardly. The chemical potential, the longitudinal pressure,[18] and the entropy are given by

$$\mu=\left(\frac{\partial F}{\partial Z}\right)_{v,T},$$

$$P_\parallel=-\left(\frac{\partial F}{\partial v}\right)_{Z,T}=\frac{kTeB}{2\pi^2\hbar^2c}(2mkT)^{1/2}I_{1/2}(\eta), \tag{2.9}$$

$$S=-\left(\frac{\partial F}{\partial T}\right)_{Z,v}=Zk\left(3\frac{I_{1/2}(\eta)}{I_{-1/2}(\eta)}-\eta\right).$$

To obtain the latter expression one needs the relation

$$\frac{dI_\alpha(\eta)}{d\eta}=\alpha I_{\alpha-1}(\eta),\quad \alpha\geq\tfrac{1}{2}. \tag{2.10}$$

Then, one can compute the energy

$$E=F+TS=ZkT\,\frac{I_{1/2}(\eta)}{I_{-1/2}(\eta)} \tag{2.11}$$

and check the virial theorem[19]

$$6E=3P_\parallel v. \tag{2.12}$$

This standard game may be played further, e.g., one can calculate quantities like

$$C_v=\left(\frac{\partial E}{\partial T}\right)_{Z,v}=\tfrac{1}{2}S, \tag{2.13}$$

etc. However, we will stop it here, and go on to examine the more interesting and more realistic case of the statistical model.

## III. STATISTICAL MODEL

### A. Basic equation

When the Coulomb interactions are switched on, the single-electron energy is given by the complete expression (1.2). The statistical model treats the electrons as locally free, so Eq. (2.3) may still be applied to a small cell around point $\vec{r}$, yielding for $n = Z/v$

$$n(\vec{r}) = \frac{eB}{4\pi^2\hbar^2 c}(2mkT)^{1/2} I_{-1/2}(\eta(\vec{r})), \tag{3.1}$$

with

$$n(\vec{r}) = [\mu - e\varphi(\vec{r})]/kT. \tag{3.2}$$

This local relationship between the number density of electrons, $n(\vec{r})$, and the electrostatic potential, $\varphi(\vec{r})$, is the basic equation of the model.

For a neutral atom of atomic number $Z$, consisting of a pointlike nucleus at the origin and a gas of electrons described in terms of their number density, $n(\vec{r})$, one has

$$\varphi(\vec{r}) = -\frac{Ze}{r} + e\int \frac{n(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d\vec{r}\,', \tag{3.3}$$

with the normalization condition

$$\int n(\vec{r})\, d\vec{r} = Z. \tag{3.4}$$

Equations (3.1) and (3.3) may be combined, to yield one basic equation for one basic quantity, which determines all the physics of the model. The usual procedure is to replace first Eq. (3.3) by the Poisson equation

$$\Delta\varphi(\vec{r}) = -4\pi e n(\vec{r}) \tag{3.5}$$

with the boundary condition

$$\lim_{r\to 0} r\varphi(\vec{r}) = -Ze, \tag{3.6}$$

and then eliminate $n(\vec{r})$ between Eqs. (3.1) and (3.5).[20] At this stage we assume spherical symmetry,[21] and introduce the dimensionless quantities $s$ and $\beta(s)$ defined by

$$r = Cs, \tag{3.7}$$

$$[\mu - e\varphi(r)]/kT = \beta(s)/s. \tag{3.8}$$

It is convenient to choose

$$C = a_0 2^{-1/2}\pi^{1/2}\alpha L^{-1/2}\tau^{1/4}, \tag{3.9}$$

where $a_0 = \hbar^2/me^2$, $\alpha = e^2/\hbar c$, and

$$L = \frac{\hbar e B}{m^2 c^3}, \quad \tau = \frac{2kT}{mc^2\alpha^2}. \tag{3.10}$$

(In other words, the magnetic field is measured in units $B_c = m^2c^3/\hbar e$, and the temperature in $Ry/k$.) One obtains the differential equation

$$\beta'' = sI_{-1/2}(\beta/s), \tag{3.11}$$

to be solved subject to the conditions (3.6) and (3.4), which in the new variables read

$$\beta(0) = 2^{3/2}\pi^{-1/2}\alpha^{-1}ZL^{1/2}\tau^{5/4}, \tag{3.12}$$

$$\beta(s_0) - s_0\beta'(s_0) = 0. \tag{3.13}$$

Here $s_0$ corresponds, via Eq. (3.7), to the atomic radius $r_0$.

The range of validity of the model is limited by several restrictions imposed on its parameters: atomic number ($Z$), field strength ($L$), temperature ($\tau$), and pressure (or, equivalently, atomic volume, determined by $s_0$). First, the statistical description holds only within the interval[22]

$$5\times 10^{-5}Z^{4/3} \ll L \ll 10^{-4}Z^3. \tag{3.14}$$

Second, the assumption that the transverse motion is unaffected by temperature requires that the average thermal energy per electron be small compared to the spacing of the Landau levels: $kT \ll \hbar eB/mc$, or

$$\tau \ll 2\alpha^{-2}L. \tag{3.15}$$

Third, the pressure can be neither too small (in which case the details of the electronic structure in the outer layers of the atom would become important, and the description in terms of squeezed Thomas-Fermi-like atoms would be inadequate), nor too high (the average kinetic energy per electron must remain small compared to the spacing of the Landau levels). Because at high compression the electrons become practically free, one may use for the energy Eq. (2.11); then, the latter condition reads

$$kTI_{1/2}(\eta)/I_{-1/2}(\eta) \ll \hbar eB/mc.$$

Typically, this must hold at the atomic boundary, where $\varphi(r_0) = 0$, and the argument of the Fermi-Dirac functions is $\eta = \beta(s_0)/s_0 \gg 1$. Using the leading asymptotic behavior given by Eq. (A1),[23] one obtains a lower bound for $s_0$ in implicit form:

$$\beta(s_0)/s_0 \ll 6\alpha^{-2}L\tau^{-1}. \tag{3.16}$$

This may be put in explicit form by using Eqs. (3.7), (3.9), and (A2), (A4) to obtain the relationship between $s_0$ and $x_0$, then combining it with Eq. (3.12) to get

$$\beta(0) = 4x_0^{-5}s_0^{5}. \tag{3.17}$$

Here $x_0$ is a universal quantity (independent of $Z$,

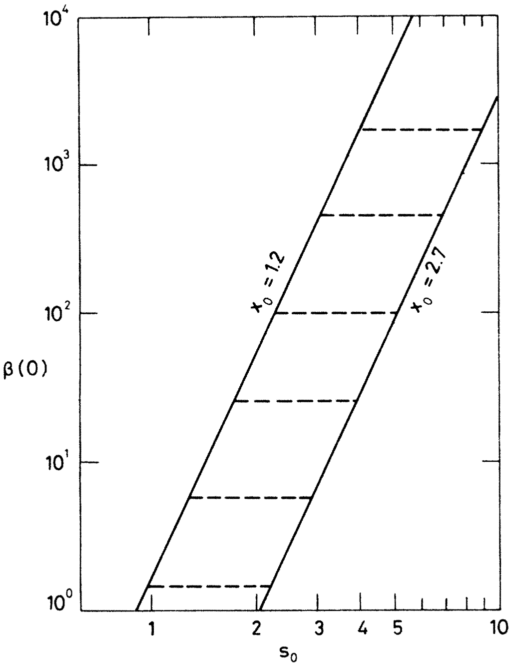

FIG. 1. The range of validity of Eqs. (3.11)–(3.13) is the oblique strip between the straight lines $x_0=1.2$ and $x_0=2.7$. The six series of calculations reported in Sec. IV were performed along the dashed horizontal lines; their ordinates correspond to iron ($Z=26$) in a magnetic field $B=1.4\times10^{12}$ G and at temperatures (downwards from top) $T=10^5$ K, $3\times10^5$ K, $10^6$ K, $3\times10^6$ K, $10^7$ K, $3\times10^7$ K.

$L$, and $\tau$) characterizing the compression of the atom. Its lower bound may be determined once for all from condition (3.16), by solving numerically Eq. (3.11) for arbitrary values of the other parameters. The upper bound on $x_0$ results from the requirement that the atoms be squeezed to the point where the statistical model becomes applicable.[1] One obtains approximately

$$1.2<x_0<2.7\ . \tag{3.18}$$

In a log-log plot of $\beta(0)$ vs $s_0$, the allowed range resulting from Eqs. (3.17) and (3.18) is an oblique strip of constant width (Fig. 1).

### B. Free energy

To extract physical information from Eqs. (3.11)–(3.13) we calculate first the free energy $F(Z,v,T)$ in terms of their solution, $\beta(s)$. One possibility of doing that is to start from the total energy (see Appendix B) and integrate the Gibbs-Helmholtz equation.[24] We find more instructive the procedure which sets up the expression of the free energy on the basis of simple intuitive arguments compatible with the assumptions of the statistical model. This is not a rigorous derivation, and it leads to an ansatz for $F$, which must be checked afterwards to show that it is indeed compatible with the basic equation (3.1). Such a "heuristic derivation" can be made either at the microscopic level, to obtain first the partition function,[13] or, even simpler, at the phenomenological level, yielding directly the expression of the free energy.[25] It is this latter alternative that we shall adopt here.

The electrons being regarded as locally free, one can imagine the following prescription for setting up the free energy in the statistical model: take the analytic expression of the density of free energy as given by the case of noninteracting electrons, make it a function of the position, then add the terms representing the Coulomb interactions. Using Eq. (2.7) we are thus led to the ansatz

$$F(Z,v,T)=kT\int n(\vec{r})\left[\eta(\vec{r})-2\frac{I_{1/2}(\eta(\vec{r}))}{I_{-1/2}(\eta(\vec{r}))}\right]d\vec{r}$$
$$-Ze^2\int\frac{n(\vec{r})}{r}d\vec{r}+\frac{e^2}{2}\int\frac{n(\vec{r})n(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}d\vec{r}\,d\vec{r}\,', \tag{3.19}$$

where $\eta(\vec{r})$ is a function related to the number density $n(\vec{r})$ by Eq. (2.5) applied locally, i.e., by Eq. (3.1).

To check the correctness of Eq. (3.19) we start from the variational principle[25] which requires the free energy, regarded as a functional of $n(\vec{r})$, to be stationary with respect to variations of $n(\vec{r})$, subject to the additional condition (3.4). Introducing a Lagrange multiplier $\mu$, we therefore require that

$$\delta(F-\mu Z)=0 \tag{3.20}$$

for arbitrary variations $\delta n(\vec{r})$ or, equivalently, $\delta\eta(\vec{r})$. Using Eq. (2.10) and the formula

$$\frac{d\ln I_{-1/2}(\eta)}{d\eta}=\frac{1}{n}\frac{dn}{d\eta} \tag{3.21}$$

[the latter is obtained by differentiating Eq. (3.1)], Eq. (3.20) may be cast into the form

$$\int[kT\eta(\vec{r})+e\varphi(\vec{r})-\mu]\frac{dn}{d\eta}\,\delta\eta\,d\vec{r}=0. \tag{3.22}$$

Hence we obtain Eq. (3.2) which, substituted into (3.1), yields indeed the basic equation of the statistical model.

It is convenient to write Eq. (3.19) in the form

$$F(Z,v,T)=2K_{\parallel}-W-\mu Z, \tag{3.23}$$

where the expressions of $K_{\parallel}$, $W$, and $\mu$ are given in Appendix B. The analog of Eq. (3.23) in the conventional model has been established by Brachman.[24]

### C. Thermodynamics

Within the limits of the thermodynamic description and of the specific model adopted, Eq. (3.23) contains implicitly all the physical information on our system. To extract it in explicit form one must follow the standard procedure of calculating the successive partial derivatives of the thermodynamic potential with respect to the appropriate variables (in this case, the derivatives of the free energy with respect to $Z$, $v$, and $T$).

First, and trivially, one has $\mu = (\partial F/\partial Z)_{v,T}$, which confirms the interpretation of $\mu$ as chemical potential.[26] Second, the longitudinal pressure[18] is given by $P_{\parallel} = -(\partial F/\partial v)_{Z,T}$. To obtain the macroscopic pressure, this quantity must be calculated at the atomic boundary, where the electrons are free and the $W$ term in Eq. (3.23) may be dropped; the result is twice the density of longitudinal kinetic energy:

$$P_{\parallel} = 2k_{\parallel}(r_0). \tag{3.24}$$

[See Eq. (B3); a comparison with Eq. (2.9) shows that, in accordance with the philosophy of the statistical model, the expression of the pressure in terms of the electron density is the same as for noninteracting electrons, with the difference that it has become a function of position. This holds also for the transverse pressure, $P_{\perp}$.[18]] The result of Eq. (3.24) may also be obtained from the virial theorem, Eq. (B7), or by a simple kinetic argument.[1]

Third, the entropy is calculated as $S = -(\partial F/\partial T)_{Z,v}$. Bearing in mind that $n(r)$, and therefore $\varphi(r)$, depend on the temperature, and using Eqs. (2.10), (3.1), (3.3), and the relevant formulas in Appendix B, one obtains

$$S = \frac{1}{T}(3K_{\parallel} + V + 2W - \mu Z). \tag{3.25}$$

It is easy to check that in the completely degenerate case Eq. (3.25) reduces to $S = 0$.[4]

The higher-order derivatives of the thermodynamic potential (specific heat, compressibility, etc.) do not have simple analytic expressions; if necessary, they may be computed numerically. The explicit expressions of the various quantities appearing in Eqs. (3.23)–(3.25), in terms of the solution of Eqs. (3.11)–(3.13), are given in Appendix B.

## IV. NUMERICAL RESULTS

We wrote a computer program for the numerical integration of Eqs. (3.11)–(3.13). The integration may be started either from the origin, with given values of $\beta(0)$ and $\beta'(0)$, ending at $s_0$ determined by Eq. (3.13), or in opposite direction, with given values of $s_0$ and $\beta(s_0)$, and ending at the origin. The relevant quantities computed from the solution, or assigned by the initial conditions, are $\beta(0)$, $\beta'(0)$, $s_0$, $\beta(s_0)$, and $\int \beta'^2(s)ds$, in terms of which the basic physics of the model can be expressed (see Sec. III and Appendix B).

The Fermi-Dirac functions $I_{\pm 1/2}(\eta)$, needed both during the integration and in expressing the physical output, were computed as follows. For $n < -2$, we used the truncated expansion[17]

$$I_{\alpha}(\eta) = \Gamma(\alpha+1) \sum_{k=1}^{5} (-1)^{k-1} \frac{e^{k\eta}}{k^{\alpha+1}}; \tag{4.1}$$

for $-2 \leqslant \eta < 10$, we used the Blakemore tables,[17] together with a fourth-order interpolation formula for calculating the functions at intermediate points; for $10 \leqslant \eta < 10^5$, we used the truncated asymptotic formula[27,17]

$$I_{\alpha}(\eta) = \frac{\eta^{\alpha+1}}{\alpha+1}\left(1 + \sum_{k=1}^{5} \frac{d_{2k}^{(\alpha)}}{\eta^{2k}}\right), \tag{4.2}$$

where

$$d_{2k}^{(\alpha)} = 2(1 - 2^{1-2k})\zeta(2k)(\alpha+1)\alpha \ldots (\alpha - 2k + 2); \tag{4.3}$$

finally, for $10^5 \leqslant \eta$, only the leading term in Eq. (4.2) was used. For any $\eta$, this procedure guarantees a relative error less than $10^{-5}$ on $I_{\pm 1/2}(\eta)$, sufficient for ensuring a relative accuracy of the physical results (some of which are obtained as differences of large numbers) better than $10^{-2}$.

The results reported hereafter refer to six series of calculations, each of them corresponding to a given $\beta(0)$ and a string of values for $s_0$, distributed throughout its entire range (Fig. 1). The values chosen for $\beta(0)$ correspond to iron ($Z = 26$) in a magnetic field $B = 1.4 \times 10^{12}$ G and at the following temperatures: $T = 10^5$ K, $3 \times 10^5$ K, $10^6$ K, $3 \times 10^6$ K, $10^7$ K, $3 \times 10^7$ K. Some results for zero temperature, taken from Ref. 1, have been added for comparison.

Figure 2 shows three solutions $\beta(s)$ for $T = 3 \times 10^7$ K; they refer, respectively, to a low, a moderate, and a high compression. The corresponding graphs of the volume density of electrons, $n(r)$, and the "radial density", $4\pi r^2 n(r)$, may be obtained from Eq. (3.1). As temperature is decreased,[28] the curvature of $\beta(s)$ gradually increases, the electron distribution being more and more determined by the Coulomb forces; of course, this fact is less apparent at high compression.

According to the discussion in Sec. I, the total free energy of the atom is obtained by adding to $F$ the kinetic energy of the transverse motion, $K_{\perp} = \hbar eBZ/2mc$.[1] In Fig. 3 the quantity $F + K_{\perp}$ is plot-

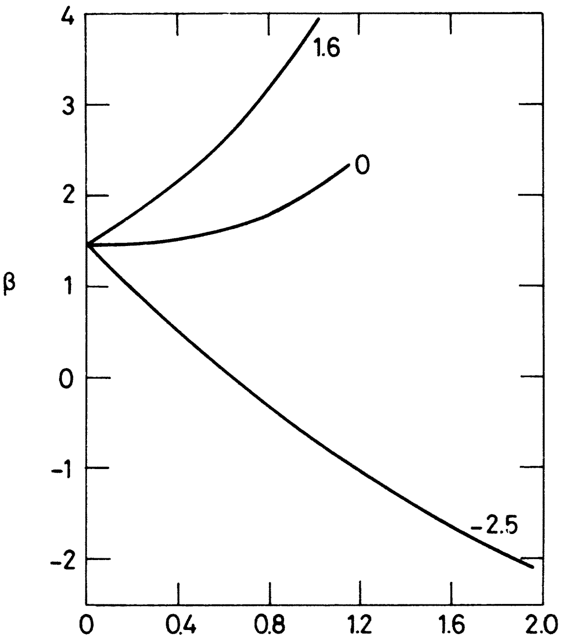

FIG. 2. Three solutions of Eqs. (3.11)–(3.13) for $Z=26$, $B=1.4\times10^{12}$ G, and $T=3\times10^{7}$ K. The numbers on the curves indicate the corresponding values of $\beta'(0)$.

ted against the mass density, $\rho=3AM/4\pi r_0^{\,3}$ ($A$ is the atomic weight, and $M$ is the proton mass). One sees that, for the given values of $Z$ and $B$, the temperature correction becomes a sizable effect only for temperatures of the order of a few times $10^6$ K; its relative importance decreases with increasing density.

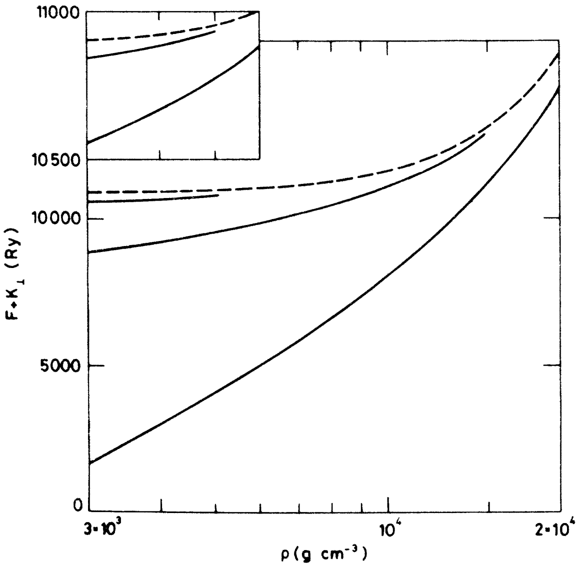

FIG. 3. Total free energy vs mass density at constant temperature, for $Z=26$ and $B=1.4\times10^{12}$ G. The corresponding temperatures are (downwards from top) $T=0$ K (dashed curve), $3\times10^{6}$ K, $10^{7}$ K, $3\times10^{7}$ K (solid curves); for the insert (downwards from top) $T=0$ K (dashed curve), $10^{6}$ K, $3\times10^{6}$ K (solid curves).

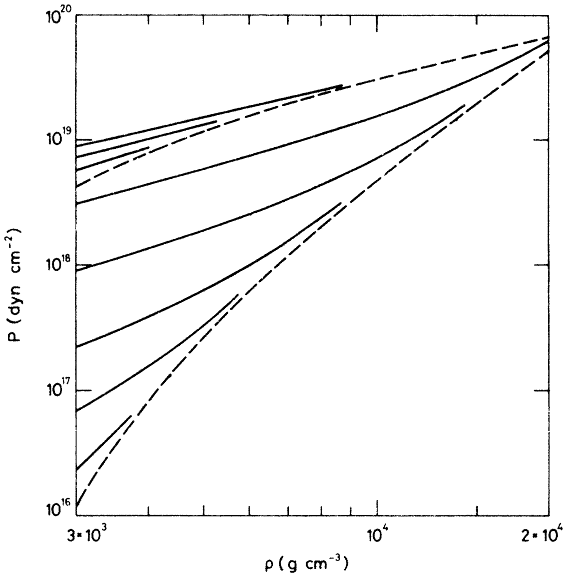

FIG. 4. Pressure vs mass density at constant temperature, for $Z=26$ and $B=1.4\times10^{12}$ G. Lower curves: longitudinal pressure at temperature (upwards from bottom) $T=0$ K (dashed curve), $3\times10^{5}$ K, $10^{6}$K, $3\times10^{6}$ K, $10^{7}$ K, $3\times10^{7}$ K (solid curves). Upper curves: transverse pressure at temperatures (upwards from bottom) $T=0$ K (dashed curve), $3\times10^{6}$ K, $10^{7}$ K, $3\times10^{7}$ K (solid curves).

In Fig. 4 the longitudinal and transverse pressures, $P_\parallel$ and $P_\perp$,[18] are plotted against $\rho$. A comparison with the completely degenerate case (dashed curves)[1] reveals the size of the temperature correction, which is much more important for $P_\parallel$ than for $P_\perp$. Indeed, in this model the effect of temperature on $P_\parallel$ is due to the increase of both the average longitudinal momentum and the electron density at the atomic boundary, whereas in the case of $P_\perp$ only the latter (and less important) factor is present, the transverse momentum remaining unaffected in the adiabatic approximation.

Finally, Fig. 5 presents the entropy as a function of the mass density.

## V. DISCUSSION

Let us first discuss the reasons for neglecting exchange corrections in the above treatment. At zero temperature their effect has been calculated and shown to be quite important at relatively low pressure.[1] In the conventional statistical model[11] exchange corrections have been considered also in the case of incomplete degeneracy.[29] However, we have deliberately ignored them here, because at nonzero temperatures it seems quite inconsis-

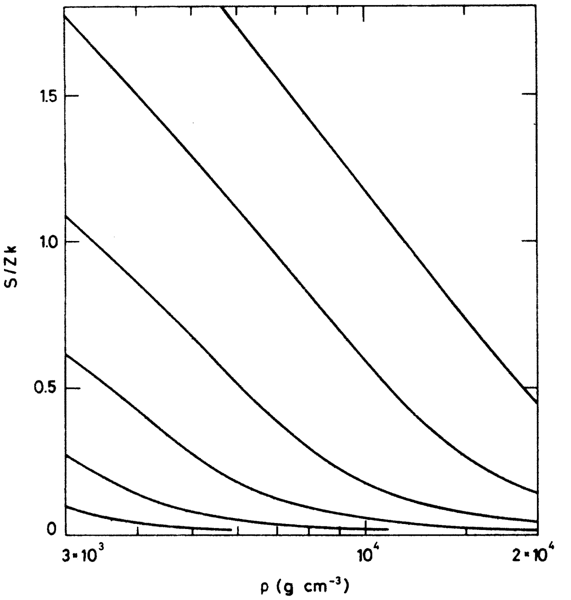

FIG. 5. Entropy vs mass density at constant temperature, for $Z=26$ and $B=1.4\times10^{12}$ G. The corresponding temperature are (upwards from bottom) $T=10^5$ K, $3\times10^5$ K, $10^6$ K, $3\times10^6$ K, $10^7$ K, $3\times10^7$ K.

tent to take into account exchange effects while neglecting electron correlations. If, however, one is willing to give away the simplicity of the statistical model in favor of a more realistic scheme which is able to incorporate both electron exchange and long-range correlations, these two effects are found to cancel almost exactly.[30] This suggests that in the incompletely degenerate case the simple statistical model may be closer to reality if exchange corrections are ignored—not to speak of the clumsiness which results from trying to incorporate them into the scheme.

The limitations of the "squeezed-atoms" model in describing the properties of matter at high pressure have been discussed in Ref. 1, and that discussion applies to the nonzero-temperature case as well.

As to the applications to matter in the outer crust of pulsars, the results of Sec. IV show that the effect of temperature on "standard" pulsars, which have already cooled down to about $10^5$ K, is irrelevant; in a magnetic field of $10^{12}$ G the properties of an iron crust are practically the same as at zero temperature. However, this is no longer true for very young pulsars, which might be as hot as $10^6$–$10^7$ K,[31] and are very interesting objects, hardly understood at all at this moment. It has already been pointed out[1] that the properties of the crust may be determinant for a series of observable effects which have their origin in the pulsar magnetosphere.

It is very difficult to speculate on the applications of this model to supercompressed matter on earth,[6] without knowing what field intensities and temperatures can one really expect to produce in the laboratory in the not-too-distant future.

## APPENDIX A: LOW TEMPERATURES

At low temperatures, viz., when $\mu - e\varphi(r_0) \gg kT$, one may use in Eq. (3.1) the asymptotic expansion[27,17]

$$I_\alpha(\eta)=\frac{\eta^{\alpha+1}}{\alpha+1}\left[1+\alpha(\alpha+1)\pi^2/6\eta^2+O(\eta^{-4})\right], \quad \eta\gg1. \tag{A1}$$

Making the change of variables[1,4]

$$r=Gx\,, \tag{A2}$$

$$r\left[\mu-e\varphi(r)\right]=Ze^2\Phi(x), \tag{A3}$$

with

$$G=a_0 2^{-3/5}\pi^{2/5}\alpha^{4/5}Z^{1/5}L^{-2/5}, \tag{A4}$$

and following the procedure described in Sec. III, one obtains the equation

$$\Phi''=(x\Phi)^{1/2}(1-\lambda^2x^2/\Phi^2), \tag{A5}$$

where

$$\lambda^2=2^{-31/5}3^{-1}\pi^{14/5}\alpha^{8/5}Z^{-8/5}L^{-4/5}T^2\,. \tag{A6}$$

The boundary conditions are the same as for zero temperature:

$$\Phi(0)=1,\quad \Phi(x_0)-x_0\Phi'(x_0)=0, \tag{A7}$$

where $x_0$ corresponds to the atomic radius. Eq. (A5) gives a correct description only if temperature effects may be regarded as small corrections to the Coulomb interactions; more precisely, only if

$$\lambda x_0/\left|\Phi(x_0)\right| \ll 1. \tag{A8}$$

This represents a much stronger limitation on the temperature than Eq. (3.15), which implies that temperature effects are regarded as small corrections to the magnetic forces.

In the conventional case, the analog of Eq. (A5)[12] and different perturbation schemes based on the expansion (A1) have been studied by several authors.[32] We will not make a similar investigation here because (i) the temperatures at which condition (A8) is satisfied are so low that the physics is essentially the same as at zero temperature (See Sec. IV), and (ii) solving numerically the exact equation (3.11) is a more direct and illuminating procedure than any perturbation expansion in the temperature.

## APPENDIX B: ENERGY AND VIRIAL THEOREM

We collect here some formulas which give the various contributions to the energy of a $Z$-electron atom, in terms of the solution of Eqs. (3.11)–(3.13).

After subtracting the kinetic energy of the transverse motion, the energy of the atom is

$$E = K_{\parallel} + V + W, \tag{B1}$$

where $V$ and $W$ represent the contributions of the Coulomb interactions

$$\begin{aligned} V &= -Ze^2 \int \frac{n(\vec{r})}{r}\, d\vec{r}, \\ W &= \frac{e^2}{2} \int \frac{n(\vec{r})n(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\, d\vec{r}\, d\vec{r}\,', \end{aligned} \tag{B2}$$

and $K_{\parallel}$ is the kinetic energy of the longitudinal motion. The density of the latter, $k_{\parallel}(\vec{r})$, is obtained by averaging the quantity $q^2/2m$ over a small cell around $\vec{r}$. One finds

$$k_{\parallel}(\vec{r}) = \frac{kTeB}{4\pi^2\hbar^2 c}(2mkT)^{1/2} I_{1/2}(\eta(\vec{r})), \tag{B3}$$

with $\eta(\vec{r})$ given by Eq. (3.2). Hence

$$K_{\parallel} = \frac{kTeB}{4\pi^2\hbar^2 c}(2mkT)^{1/2} \int I_{1/2}(\eta(\vec{r}))\, d\vec{r}. \tag{B4}$$

Assuming spherical symmetry and making the change of variables given by Eqs. (3.7)–(3.9), one obtains, after some manipulations involving partial integrations and the use of Eqs. (3.11)–(3.13),

$$\begin{aligned} \frac{K_{\parallel}}{ZkT} &= \frac{1}{6\beta(0)}\Bigg(\frac{\beta^2(s_0)}{2s_0} - \beta(0)\beta'(0) + 2s_0{}^3 I_{1/2}(\beta(s_0)/s_0) \\ &\qquad - \tfrac{1}{2}\int_0^{s_0} \beta'^2(s)\,ds\Bigg), \\ \frac{V}{ZkT} &= \beta'(0) - \frac{\beta(s_0)}{s_0}, \\ \frac{W}{ZkT} &= \frac{\beta(s_0)}{s_0} - \frac{1}{2\beta(0)}\left(\frac{\beta^2(s_0)}{s_0} - \int_0^{s_0} \beta'^2(s)\,ds\right). \end{aligned} \tag{B5}$$

Once the quantities $\beta(0)$, $\beta'(0)$, $s_0$, $\beta(s_0)$, and $\int \beta'^2(s)ds$ are determined by solving Eqs. (3.11)–(3.13), $K_{\parallel}$, $V$, and $W$ may be calculated explicitly from Eqs. (B5). Also, using the same equations and

$$\mu = kT\beta(s_0)/s_0 \tag{B6}$$

[see Eq. (3.8) with $\varphi(r_0)=0$], one can check the virial theorem[1]

$$6K_{\parallel} + V + W = 3P_{\parallel}v, \tag{B7}$$

where $v = \frac{4}{3}\pi r_0{}^3$ is the atomic volume.

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[19] See Ref. 1. In the case of noninteracting electrons $E = K_{\parallel}$, the kinetic energy of the longitudinal motion.

[20] The low-temperature case, treated perturbatively in the lowest order, is examined in Appendix A.

[21] Spherical symmetry is exact at zero temperature (Ref. 3), but is slightly broken by temperature effects, as it is also by exchange corrections (Refs. 1 and 4); such factors act upon the longitudinal motion but not (if the magnetic field is sufficiently strong) on the

transverse motion. While considerably simplifying the equations, the sphericity assumption should have a negligible effect, compared to the overall error introduced by the various approximations of the model.

[22] See Ref. 4, Eq. (31) and Fig. 4.

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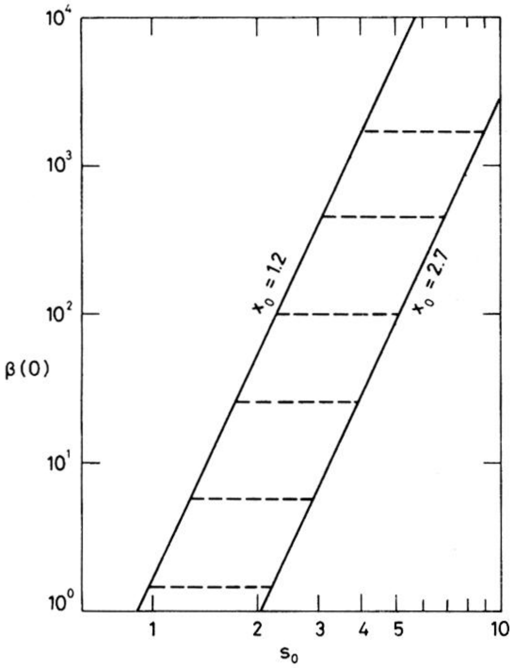

FIG. 1. The range of validity of Eqs. (3.11)–(3.13) is the oblique strip between the straight lines $x_0 = 1.2$ and $x_0 = 2.7$. The six series of calculations reported in Sec. IV were performed along the dashed horizontal lines; their ordinates correspond to iron ($Z = 26$) in a magnetic field $B = 1.4 \times 10^{12}$ G and at temperatures (downwards from top) $T = 10^5$ K, $3 \times 10^5$ K, $10^6$ K, $3 \times 10^6$ K, $10^7$ K, $3 \times 10^7$ K.

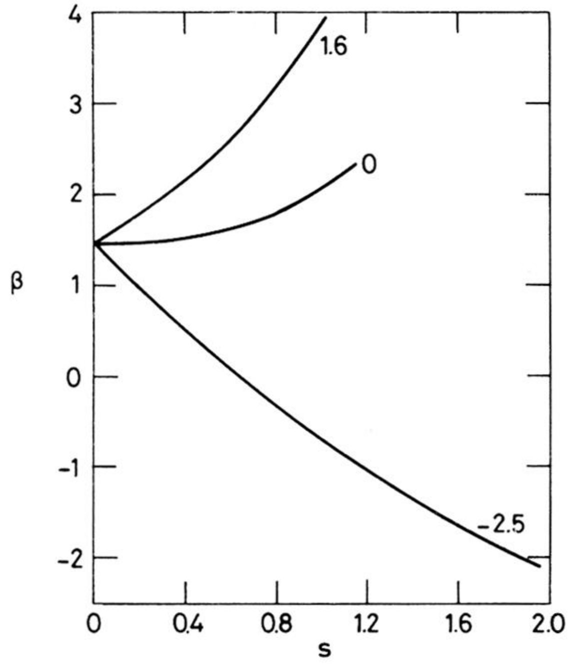

FIG. 2. Three solutions of Eqs. (3.11)–(3.13) for $Z = 26$, $B = 1.4\times10^{12}$ G, and $T = 3\times10^{7}$ K. The numbers on the curves indicate the corresponding values of $\beta'(0)$.

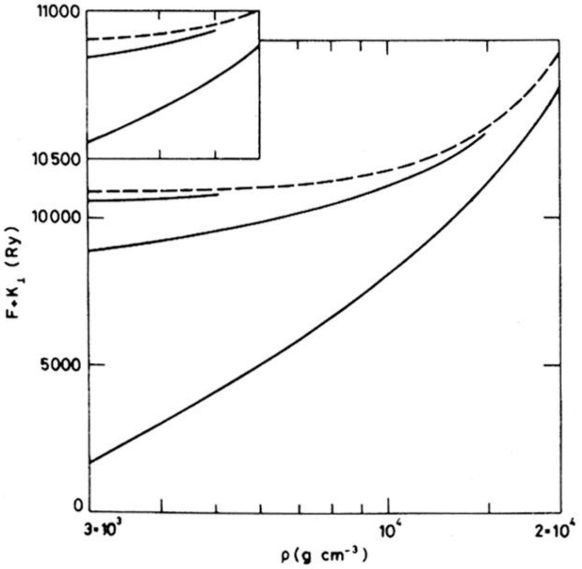

FIG. 3. Total free energy vs mass density at constant temperature, for $Z = 26$ and $B = 1.4 \times 10^{12}$ G. The corresponding temperatures are (downwards from top) $T = 0$ K (dashed curve), $3 \times 10^6$ K, $10^7$ K, $3 \times 10^7$ K (solid curves); for the insert (downwards from top) $T = 0$ K (dashed curve), $10^6$ K, $3 \times 10^6$ K (solid curves).

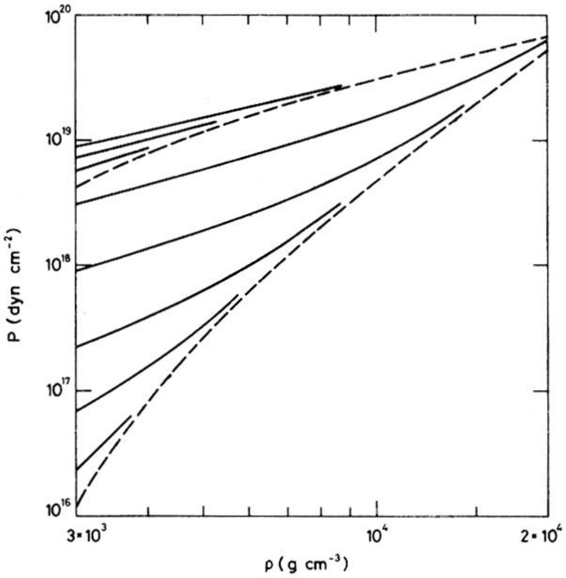

FIG. 4. Pressure vs mass density at constant temperature, for $Z = 26$ and $B = 1.4 \times 10^{12}$ G. Lower curves: longitudinal pressure at temperature (upwards from bottom) $T = 0$ K (dashed curve), $3 \times 10^5$ K, $10^6$K, $3 \times 10^6$ K, $10^7$ K, $3 \times 10^7$ K (solid curves). Upper curves: transverse pressure at temperatures (upwards from bottom) $T = 0$ K (dashed curve), $3 \times 10^6$ K, $10^7$ K, $3 \times 10^7$ K (solid curves).

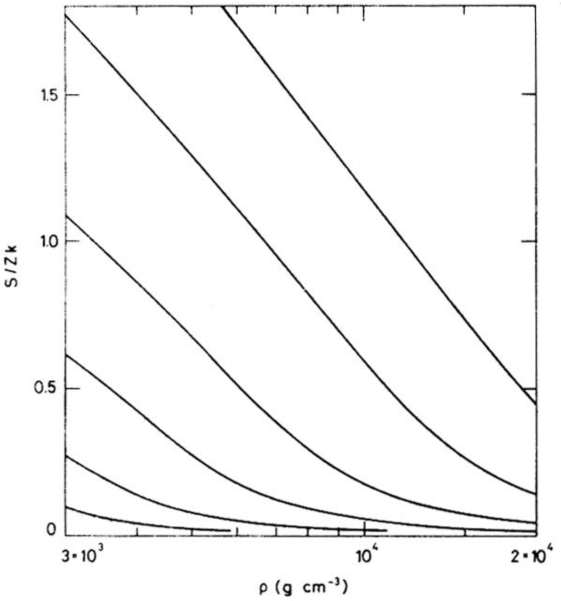

FIG. 5. Entropy vs mass density at constant temperature, for $Z=26$ and $B=1.4\times10^{12}$ G. The corresponding temperature are (upwards from bottom) $T=10^5$ K, $3\times10^5$ K, $10^6$ K, $3\times10^6$ K, $10^7$ K, $3\times10^7$ K.