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New conservation laws in general-relativistic magnetohydrodynamics

Jacob D. Bekenstein and Eliezer Oron
Department of Physics, Ben Gurion University of the Negev, Beer-Sheva, 84120 Israel
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The flow of magnetized plasma is governed by a large number of coupled equations (Maxwell's, Euler's, conservation of energy and of baryon number) so that the solution of a problem in general-relativistic magnetohydrodynamics is very complicated, even if symmetries are present. We present here a number of new conservation laws which make the solution process easier. We obtain the general criteria for a flux conservation law to exist. We apply them to obtain the relativistic versions of the conservation of magnetic flux and of Kelvin's circulation theorem for an unmagnetized fluid, as well as a new flux conservation law for a charged fluid. For stationary and axial symmetry we find conservation laws for each component of the Maxwell tensor; these are valid even if the plasma is nonperfect. For perfect plasma we find magnetic generalizations of the relativistic Bernoulli theorems for an unmagnetized fluid. We also find a new conservation law without previous analog. As an application of our results we show that extraction of rotational energy from a black hole by interaction with a magnetized plasma is not possible in the stationary state. This contradicts previous conclusions based on the approximation of geodesic flow. Finally, still for stationary and axial symmetry, we find the magnetic generalization of Kelvin's circulation theorem. With its help we reduce the problem of solving for the field of flow and for the magnetic field to the solution of two equations: baryon conservation and a Hamilton-Jacobi-type equation. A by-product of our derivations is an explicit formula for the strength of the magnetic field in terms of fluid variables.

I. INTRODUCTION

There are a number of astrophysical situations in which general-relativistic magnetohydrodynamic (GRM) effects may be important. Neutron stars such as are found in pulsars and in some of the compact x-ray sources are composed of strongly magnetized degenerate plasma subject to strong gravitational fields. The magnetosphere of such a neutron star is again a highly conducting plasma entrained in the star's magnetic field and, in regions close to the star, subject to its strong gravitational field. In the environment of an accreting black hole, such as may exist in some of the x-ray sources and in quasars, plasma carrying a frozen-in magnetic field falls in the strong gravitational field of the hole. The relevance of GRM to astrophysics is thus clear.

The equations of GRM have been developed over the years by a number of people.[1-5] Unfortunately there has been as yet little application of these to astrophysical calculations. For example, pulsar magnetospheres are still treated by special relativity.[6] Accretion onto black holes has been treated by general relativity, but in the approximation of geodesic motion which neglects magnetic forces.[7,8] It appears that this trend resulted from the lack of conservation laws in GRM which might have made interesting calculations tractable. In this paper we present a number of new conservation laws in GRM, both for the general case and for the case of stationary axisymmetric flow, which, we believe, will be of help in GRM calculations for astrophysical processes.

In Sec. II we collect the relevant equations for GRM flow. We include a justification of the often-used approximation of a vanishing electric field in the comoving frame even when the anisotropy of the conductivity due to the magnetic field is taken into account. In Sec. III we obtain the conditions under which a flux conservation law exists for fluid flow. As applications we obtain the law of magnetic-flux conservation in GRM, the general-relativistic version of Kelvin's theorem on the conservation of circulation, and a new flux conservation law for charged nonconducting fluids. From the last two we show that the general-relativistic flow of a perfect fluid (charged or neutral unmagnetized) may be described by a Hamilton-Jacobi-type equation.

In Sec. IV we present five conservation laws for components of the electromagnetic field in the case of a stationary axisymmetric GRM flow. These laws are valid even if the fluid is imperfect. In Sec. V we derive a conservation law without previous analog for a stationary axisymmetric GRM flow of perfect plasma. It involves the chemical potential and the covariant time and axial velocity components. In Sec. VI we obtain the GRM generalizations of the Bernoulli theorems for stationary axisymmetric flow. These generalize the previously known results for unmagnetized fluid.[7] In Sec. VII we derive a conservation law for a combination of components of the magnetic field. It gives a connection between the "constants of the motion" involved in the conservation laws of Secs.

V and VI.

In Sec. VIII we apply our results on stationary axisymmetric flow to the question of whether energy can be extracted from a rotating black hole by interaction with an enveloping plasma. We find that extraction is impossible. This conclusion is at variance with that of Ruffini and Wilson,[8] who reached their conclusion on the basis of the geodesic-motion approximation.

In Sec. IX we use our previous results to prove a GRM generalization of Kelvin's theorem valid for a stationary axisymmetric flow. From this it follows that under suitable boundary conditions GRM flow may be described by a Hamilton-Jacobi-type equation.

A word about units. We use Gaussian units for electromagnetism and set $c=1$. The signature for the metric is +2.

II. THE BASIC EQUATIONS OF GENERAL-RELATIVISTIC MAGNETOHYDRODYNAMICS

In GRM one is interested in the relativistic flow of fluid interacting with the electromagnetic field. The motion of the fluid is governed by the equations of motion

$$T^{\alpha\beta}{}_{;\beta}=0\,, \tag{1}$$

where $T^{\alpha\beta}$ is the total energy-momentum tensor of the fluid and electromagnetic field. The electromagnetic field evolves according to the Maxwell equations

$$F_{[\alpha\beta;\gamma]}=0 \tag{2}$$

and

$$F^{\alpha\beta}{}_{;\beta}=4\pi J^{\alpha}\,, \tag{3}$$

where J^α is the electric-current 4-vector. We have assumed that the permittivity and permeability of the plasma are unity, which is a good approximation in astrophysical contexts. In addition to all the above, baryon number must be conserved. Let n be the proper baryon density. One can always define a velocity field u^α, with $u^\alpha u_\alpha = -1$, such that nu^α is the baryon current. Thus

$$(nu^\alpha)_{;\alpha}=0\,. \tag{4}$$

The u^α may be called the fluid 4-velocity, although it must be remembered that it may not be representative of the electron velocity.

The electric field according to a comoving observer is defined as[1]

$$E_\alpha = F_{\alpha\beta}u^\beta\,, \tag{5}$$

and the corresponding magnetic field as[1]

$$B_\alpha=\tfrac{1}{2}\epsilon_{\alpha\beta\gamma\delta}u^\beta F^{\gamma\delta}\,, \tag{6}$$

where $\epsilon_{\alpha\beta\gamma\delta}$ is the Levi-Civita antisymmetric tensor. Clearly $E_\alpha u^\alpha = B_\alpha u^\alpha = 0$ so that E_α and B_α have only three independent components each, as required on physical grounds. The electric current may be decomposed into a component along u^α and components normal to it:

$$J^\alpha = \epsilon u^\alpha + j^\alpha\,, \tag{7}$$

where $j_\alpha u^\alpha = 0$. We see that $\epsilon = -J^\alpha u_\alpha$, and therefore ϵ is the charge density measured by the comoving observer. Then ϵu^α may be called the convection current while j^α is clearly the conduction current. We shall assume a linear relation between j^α and E^α (Ohm's law):

$$J^\alpha = \epsilon u^\alpha + \sigma^{\alpha\beta}E_\beta\,. \tag{8}$$

It is by no means true that $\sigma^{\alpha\beta}$ must be of the form $\sigma g^{\alpha\beta}$. In the presence of a magnetic field the conductivity of a plasma must be anisotropic.[9] To be specific let us consider the conductivity as calculated in the collision-time approximation.[10,11] The standard result may easily be rewritten in 4-tensor form:

$$\sigma^{\alpha\beta}=\sigma(g^{\alpha\beta}+\xi B^\alpha B^\beta+\zeta\epsilon^{\alpha\beta\gamma\delta}u_\gamma B_\delta)\,. \tag{9}$$

(The u_γ is required in order that $\sigma^{\alpha\beta}u_\alpha E_\beta=0$.) The coefficients in (9) are given by[10,11]

$$\sigma=n_e e^2\tau m^{-1}[1+(e\tau B/m)^2]^{-1}\,, \tag{10}$$

$$\xi=(e\tau/m)^2\,, \tag{11}$$

$$\zeta=e\tau/m\,, \tag{12}$$

where τ is the collision time, n_e is the electron density, e and m are the electron's charge and mass, and $B^2\equiv B_\alpha B^\alpha$.

Were $B=0$, the limit $\tau\to\infty$ (collisions rare) would correspond to the often-made approximation of infinite conductivity. In this case it would be legitimate to require $E_\beta=0$ in order that J^α be finite. This is the standard procedure in various treatments of GRM. But clearly when $B\neq 0$ in the limit $\tau\to\infty$,

$$\sigma^{\alpha\beta}\to n_e e^2 m^{-1}B^{-2}(\tau B^\alpha B^\beta + me^{-1}\epsilon^{\alpha\beta\gamma\delta}u_\gamma B_\delta)\,, \tag{13}$$

so that the finiteness of J^α only implies that $B^\alpha E_\alpha\to 0$, but says nothing about the components of E_α normal to B^α. However, one can justify the standard procedure under some conditions. Suppose $n_e e^2\tau/m$ is large (lots of free electrons) while at the same time $eB\tau/m$ is small compared to unity (τ small compared to electron Larmor period). Then it follows from (9) that

$$\sigma^{\alpha\beta}\approx(n_e e^2\tau/m)g^{\alpha\beta}\,, \tag{14}$$

so that we recover the case of an isotropic highly conducting fluid. In this case it is appropriate to assume $E_\alpha=0$. The conditions mentioned above for this to be a good approximation have a good chance to hold only if B is not too large, if n_e is large, and if the fluid is hot (and consequently τ is short while the product $n_e\tau$ is not small).

Since in some of the situations we enumerated in Sec. I hot dense plasma are involved, we shall have occasion to assume the condition

$$E_\alpha = F_{\alpha\beta}u^\beta = 0 \tag{15}$$

in what follows. When this is true the energy-momentum tensor for the electromagnetic field can be written as[7]

$$T^{\alpha\beta}_{\text{em}} = (B^2u^\alpha u^\beta + B^2h^{\alpha\beta} - 2B^\alpha B^\beta)/8\pi\,, \tag{16}$$

where

$$h^{\alpha\beta} = g^{\alpha\beta} + u^\alpha u^\beta \tag{17}$$

is the projection tensor into the space orthogonal to u^α.

It must be remembered that in general[12]

$$T^{\alpha\beta}_{\text{em};\beta} = -F^{\alpha\beta}J_\beta\,. \tag{18}$$

Thus,

$$u_\alpha T^{\alpha\beta}_{\text{em};\beta} = E^\beta J_\beta = \sigma_{\alpha\beta}E^\alpha E^\beta \tag{19}$$

is seen to represent the Joule heating of the plasma by the electromagnetic field. Under the conditions that justify (15) we have $\sigma_{\alpha\beta}E^\beta$ finite, and therefore the Joule heating vanishes—there is no exchange of energy between electromagnetic field and the internal degrees of freedom of the plasma. In a sense, magnetic energy is conserved:

$$u_\alpha T^{\alpha\beta}_{\text{em};\beta} = 0\,. \tag{20}$$

When one is not interested in the dissipative effects in the plasma, it is appropriate to regard it as a perfect fluid. Then one can write the total energy-momentum tensor as

$$T^{\alpha\beta} = T^{\alpha\beta}_{\text{em}} + \rho u^\alpha u^\beta + p h^{\alpha\beta}\,, \tag{21}$$

where ρ and p are the proper energy density and pressure of the fluid, respectively. Substituting into (1) and contracting with u_α we have, in view of (20),

$$u^\alpha \rho_{,\alpha} + (p+\rho)u^\alpha{}_{;\alpha} = 0\,, \tag{22}$$

which represents the conservation of fluid energy. From the law of baryon conservation we have

$$u^\alpha n_{,\alpha} + n u^\alpha{}_{;\alpha} = 0\,. \tag{23}$$

Eliminating $u^\alpha{}_{;\alpha}$ between (22) and (23) gives

$$d\rho/d\tau = n^{-1}(\rho+p)dn/d\tau\,, \tag{24}$$

where $d/d\tau = u^\alpha\partial_\alpha$ represents the (convective) derivative along the flowline. Equation (24) is valid along an arbitrary flowline. We can thus view the relation[12]

$$\mu \equiv d\rho/dn = n^{-1}(\rho+p) \tag{25}$$

as valid generally; it is essentially a thermodynamic relation. The quantity μ is referred to as the chemical potential.

Projecting Eq. (1) with $h_{\alpha\beta}$ gives the (magnetic) Euler equations:

$$(\rho + p + B^2/4\pi)a^\alpha = -h^{\alpha\beta}[(p+B^2/8\pi)_{,\beta} - (B_\beta B^\gamma)_{;\gamma}/4\pi]\,, \tag{26}$$

where $a^\alpha = u^\alpha{}_{;\beta}u^\beta$ is the fluid's 4-acceleration (note that $a^\alpha u_\alpha = 0$). The equations are clearly very complicated and it would be handy to have first integrals for them. In Secs. V and VI we shall present such first integrals for the case of stationary axisymmetric flow.

Finally, we want to write down an expression for $B^\alpha{}_{;\alpha}$ that shall be of use to us in Sec. V. Taking the divergence of (6), we have

$$B^\alpha{}_{;\alpha} = \tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta}(u_{\beta;\alpha}F_{\gamma\delta} + u_\beta F_{\gamma\delta;\alpha})\,. \tag{27}$$

In view of (2) the second term in (27) vanishes. Now in the high-conductivity case $E_\alpha \to 0$. Thus, since B^α and u^α are the only vectors left, one must have up to a factor,

$$\tfrac{1}{2}\epsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta} = u^\alpha B^\beta - u^\beta B^\alpha\,. \tag{28}$$

It is easy to verify that (6) follows from (28); this verifies the numerical factor in (28). Substituting (28) into (27) we have

$$B^\alpha{}_{;\alpha} = a_\alpha B^\alpha\,, \tag{29}$$

which is the required result. It was earlier given by Yodzis[3] and Banerji.[4]

III. FLUX CONSERVATION LAWS

The conservation of magnetic flux through a contour comoving with a highly conducting fluid is a well-known result of magnetohydrodynamics.[10] Since it is a local law, it must also be true in GRM. Here we want to consider the general conditions under which one can obtain flux conservation laws (even others than magnetic-flux conservation) for fluid flow. Any such law will have to be of the form

$$\frac{d}{d\tau}(V^\alpha dA_\alpha) = 0\,, \tag{30}$$

where V^α is the vector whose flux is conserved while dA_α is a vectorial element of area carried by the fluid in its motion. It suffices to consider a quadrangular element whose sides are represented by two infinitesimal vectors ξ^α and η^α.

Then

$$dA_\alpha = \epsilon_{\alpha\beta\gamma\delta}u^\beta \xi^\gamma \eta^\delta \,, \tag{31}$$

as may be verified by going to the comoving frame where $u^\beta = (1, 0, 0, 0)$. The conservation law then takes the form

$$\frac{d}{d\tau}(w_{\alpha\beta}\xi^\alpha\eta^\beta) = 0 \,, \tag{32}$$

where

$$w_{\alpha\beta} = \epsilon_{\gamma\delta\alpha\beta}V^\gamma u^\delta . \tag{33}$$

The question, then, is what conditions must an antisymmetric tensor $w_{\alpha\beta}$ satisfy in order for a conservation law of the form (32) to exist.

Carrying out the differentiation in (32) explicitly, and replacing $w_{\alpha\beta;\gamma}$ in terms of $w_{[\alpha\beta;\gamma]}$, we get

$$\frac{d}{d\tau}(w_{\alpha\beta}\xi^\alpha\eta^\beta) = (3w_{[\alpha\beta;\gamma]} - w_{\gamma\alpha;\beta} - w_{\beta\gamma;\alpha})u^\gamma\xi^\alpha\eta^\beta + w_{\alpha\beta}\xi^\alpha{}_{;\gamma}u^\gamma\eta^\beta + w_{\alpha\beta}\xi^\alpha\eta^\beta{}_{;\gamma}u^\gamma . \tag{34}$$

Now, since the area element is carried by the fluid, so must the vectors ξ^α and η^α. Mathematically speaking, this means that the Lie derivative of ξ^α (or η^α) along u^α must vanish.[13] Thus,

$$\xi^\alpha{}_{;\gamma}u^\gamma = u^\alpha{}_{;\gamma}\xi^\gamma , \tag{35}$$

and a similar relation for η^α. Substituting these into (34), and shifting indices around we can write the result as

$$\frac{d}{d\tau}(w_{\alpha\beta}\xi^\alpha\eta^\beta) = [(w_{\alpha\gamma}u^\gamma)_{;\beta} - (w_{\beta\gamma}u^\gamma)_{;\alpha} + 3w_{[\alpha\beta;\gamma]}u^\gamma]\xi^\alpha\eta^\beta . \tag{36}$$

We now see that the conservation law will hold if

$$w_{\alpha\beta}u^\beta = 0 \tag{37}$$

and

$$w_{[\alpha\beta;\gamma]} = 0 . \tag{38}$$

We note that (37) is consistent with (33), which means that the conservation law can also be written in the form (30).

If we choose $w_{\alpha\beta}$ as the electromagnetic field, then (38) is satisfied by virtue of Maxwell's equations (2) while (37) follows from (15) in the high-conductivity case. It is easy to see by comparing (33) with (6) that V^α is proportional to B^α. Thus we obtain a conservation law which, in the form (30), is evidently the law of magnetic-flux conservation in GRM. The existence of such a law was well known earlier.[2] Applications of this law are manifold. For example, from it one deduces correctly the expected strength of the magnetic field of a pulsar starting from the typical magnetic field in a massive star. For an alternative way of stating this law see Appendix A.

As a second application of our conclusion, consider a plasma in which the magnetic pressure $B^2/8\pi$ is negligible compared to the fluid pressure. This may be relevant to the study of accretion of plasma from the interstellar medium by a lone black hole. It then follows from (26) that

$$(\rho + p)a^\alpha \approx -h^{\alpha\beta}p_{,\beta} , \tag{39}$$

which are the ordinary Euler equations. Now differentiate (25) and divide by μ to obtain

$$p_{,\beta} = (\rho + p)\mu_{,\beta}/\mu . \tag{40}$$

Therefore (39) can be written as

$$\mu a^\alpha + (u^\alpha u^\beta + g^{\alpha\beta})\mu_{,\beta} = 0 . \tag{41}$$

This equation can be thrown into a suggestive form, due originally to Khalatnikov,[14] be defining the antisymmetric tensor

$$w_{\alpha\beta} = (\mu u_\beta)_{;\alpha} - (\mu u_\alpha)_{;\beta} . \tag{42}$$

Then (41) is equivalent to $w_{\alpha\beta}u^\beta = 0$.

In addition, because $w_{\alpha\beta}$ is a curl, it automatically satisfies (38). Thus we again have a flux conservation law of the form (32). The vector V^γ is in this case

$$V^\delta = \tfrac{1}{2}\epsilon^{\gamma\delta\alpha\beta}u_\gamma w_{\alpha\beta} = \mu\epsilon^{\gamma\delta\alpha\beta}u_\gamma u_{\beta;\alpha} = \mu\omega^\delta , \tag{43}$$

which is just the vorticity vector of the fluid times μ. Thus (30) represents the generalization of Kelvin's theorem of the conservation of circulation[15]:

$$\frac{d}{d\tau}(\mu\omega^\alpha dA_\alpha) = 0 . \tag{44}$$

These results may be used as follows. In a situation where the flow at large distances from the region of interest is uniform and nonrelativistic, $w_{\alpha\beta}$ will vanish asymptotically (μ is nearly constant since p and the internal energy density are negligible compared to the rest energy density). By the conservation law, $w_{\alpha\beta}$ must vanish along all flowlines, even into the relativistic region (so long as $B^2/8\pi \ll p$). By (42) it then follows that

$$\mu u_\alpha = \Phi_{,\alpha} , \tag{45}$$

where Φ is some scalar function. Then by the normalization of u^α,

$$\nabla_\alpha\Phi\nabla^\alpha\Phi = -\mu^2 , \tag{46}$$

which is a Hamilton-Jacobi equation with "variable mass" μ. To solve for the flow, then, one has to solve only two equations, (46) and (4) together with an equation of state, instead of the five equations (1) and (4). Further simplification results if there are symmetries. For example, if the flow is stationary and axisymmetric, then in order that μu_α

share the symmetries we must have

$$\Phi = S(x^1, x^2) + Lx^3 - Ex^0, \tag{47}$$

where L and E are constants, and where x^0 (x^3) is the time (axial) coordinate. Thus the problem reduces to solving two partial-differential equations in two variables (x^1 and x^2). It follows from (45) and (47) that

$$\mu u_0 = -E, \tag{48a}$$

$$\mu u_3 = L. \tag{48b}$$

These are general relativistic generalizations of Bernoulli's theorem for potential flow.[15]

As a third application of our conditions, consider the relativistic flow of a uniformly charged nonconducting ideal fluid. We might meet such a flow in a massive neutron star's interior where, according to calculations,[6] the free protons form a charged superfluid. Now a superfluid flows through other material without impediment; it is also devoid of viscosity. Thus we may regard the proton component as a uniformly charged fluid flowing by itself and interacting only with the electromagnetic and gravitational fields. In such a description there is no place for a conduction current. Thus we may write (8) as

$$J^\alpha = enu^\alpha, \tag{49}$$

where e is the proton charge. The energy-momentum tensor will be of the form (21) where the electromagnetic part must now include the contribution of E_α which does not vanish here.

Taking the divergence of $T^{\alpha\beta}$, and substituting (18) and (49) we have as the Euler equations

$$(\rho + p)a^\alpha = -h^{\alpha\beta}p_{,\beta} + enF^{\alpha\beta}u_\beta. \tag{50}$$

Replacing $p_{,\beta}$ from (40) and $(\rho+p)$ from (25) we turn this into the form

$$\mu a^\alpha + (u^\alpha u^\beta + g^{\alpha\beta})\mu_{,\beta} - eF^{\alpha\beta}u_\beta = 0. \tag{51}$$

Defining

$$w_{\alpha\beta} = (\mu u_\beta)_{;\alpha} - (\mu u_\alpha)_{;\beta} + eF_{\alpha\beta}, \tag{52}$$

we see that (51) is equivalent to $w_{\alpha\beta}u^\beta = 0$.

In addition, since $F_{\alpha\beta}$ is a curl, the full $w_{\alpha\beta}$ is a curl and it automatically satisfies (38). Thus we have a flux conservation law like (32). Passing to the form (30) we have on the basis of previous results

$$\frac{d}{d\tau}[(\mu\omega^\alpha - eB^\alpha)dA_\alpha] = 0. \tag{53}$$

Thus for a uniformly charged fluid only a combination of circulation and magnetic flux is conserved.

In precise analogy to our treatment for the uncharged fluid we conclude that if in the region where the flow originates conditions are such that $w_{\alpha\beta} = 0$, then throughout the region one can take

$$\mu u_\alpha + eA_\alpha = \Phi_{,\alpha}, \tag{54}$$

where A_α is the electromagnetic vector potential. Then it follows from $u^\alpha u_\alpha = -1$ that

$$(\nabla_\alpha - eA_\alpha)\Phi(\nabla^\alpha - eA^\alpha)\Phi = -\mu^2, \tag{55}$$

which is the Hamilton-Jacobi equation for a charged particle with variable mass μ.

IV. CONSERVATION LAWS FOR THE ELECTROMAGNETIC FIELD IN THE STATIONARY AXISYMMETRIC CASE

We now specialize our considerations to the case of stationary axisymmetric flow in a stationary axisymmetric spacetime. This case would be of relevance in studying accretion by a rotating black hole, or the dynamics of a pulsar magnetosphere in the aligned magnetic axis model (Goldreich-Julian approach).[16] We take x^0 (x^3) to be the time (axial) coordinate and choose the other coordinates such that

$$g_{\alpha\beta,0} = g_{\alpha\beta,3} = 0. \tag{56}$$

We make no other assumption about the form of the metric. The condition that the electromagnetic field and the field of flow are stationary and axisymmetric means that the Lie derivatives of all relevant quantities along the Killing vectors $\xi^\alpha_t = \delta^\alpha_0$ and $\xi^\alpha_\varphi = \delta^\alpha_3$ must vanish. Equivalently, in our coordinates,

$$F_{\alpha\beta,0} = F_{\alpha\beta,3} = 0 \tag{57}$$

and

$$u^\alpha{}_{,0} = u^\alpha{}_{,3} = n_{,0} = n_{,3} = \rho_{,0} = \rho_{,3} = 0. \tag{58}$$

Our object is to display a number of conservation laws for the components of $F_{\alpha\beta}$ which follow from the symmetries.

In view of the symmetries, the Maxwell equations (2) give

$$F_{03,1} = F_{03,2} = 0, \tag{59}$$

$$F_{13,2} + F_{32,1} = 0, \tag{60}$$

$$F_{10,2} + F_{02,1} = 0. \tag{61}$$

It is clear from (57) and (59) that F_{03} is a constant. With reasonable asymptotic conditions it can be taken to vanish, and we shall assume this henceforth.

Assuming a large conductivity, we have condition (15) from which follow

$$F_{02} = -u^1F_{01}/u^2, \tag{62}$$

$$F_{32} = -u^1F_{31}/u^2, \tag{63}$$

$$F_{12} = (u^3F_{31} + u^0F_{01})/u^2. \tag{64}$$

Substituting (62) into (61) and dividing through by F_{01} we get

$$-u^2(u^1/u^2)_{,1}=(u^1F_{01,1}+u^2F_{01,2})/F_{01} = d(\ln F_{01})/d\tau. \quad (65)$$

Similarly from (63) and (60) one gets

$$-u^2(u^1/u^2)_{,1}=d(\ln F_{31})/d\tau. \quad (66)$$

Now by subtracting (65) from (66) we see that F_{01}/F_{31} is conserved along each flowline. Thus

$$F_{01}/F_{31}=A, \quad (67)$$

where A may vary only from flowline to flowline. From the ratio of (63) and (62) we see that

$$F_{02}/F_{32}=A. \quad (68)$$

Let us now return to (66) and write it as

$$d(\ln F_{31})d\tau=-u^\alpha{}_{,\alpha}+u^\alpha u^2{}_{,\alpha}/u^2, \quad (69)$$

where account has been taken of (58). From baryon-number conservation it follows that

$$u^\alpha{}_{,\alpha}=-\frac{d}{d\tau}\ln(\sqrt{-g}\,n). \quad (70)$$

With the aid of this we can cast (69) into the form

$$\frac{d}{d\tau}\ln\frac{F_{31}}{\sqrt{-g}\,nu^2}=0. \quad (71)$$

Thus

$$\frac{F_{31}}{\sqrt{-g}\,nu^2}=C, \quad (72)$$

where C may differ from flowline to flowline but is conserved along each flowline. Now it follows immediately from (63) that

$$\frac{F_{23}}{\sqrt{-g}\,nu^1}=C. \quad (73)$$

Finally, it follows from (64), (67), and (72) that

$$\frac{F_{12}}{\sqrt{-g}\,n(u^3+Au^0)}=C. \quad (74)$$

We have thus obtained five conservation laws (67), (68), and (72)–(74) for components of the electromagnetic field. Effectively, these are first integrals of the Maxwell equations (2). Given the field of flow (n and u^α) and initial conditions for $F_{\alpha\beta}$ (in the asymptotic region, for example), the first integrals determine $F_{\alpha\beta}$ everywhere. We point out that all results in this section are valid even if the fluid is viscous since we did not assume yet that the fluid is ideal.

V. CONSERVATION LAW FOR THE COMPONENT OF EULER'S EQUATION ALONG THE MAGNETIC FIELD

Let us contract Euler's equations (26) with B_α. Recalling that $u^\alpha B_\alpha=0$ we have

$$(\rho+p+B^2/4\pi)B_\alpha a^\alpha=-p_{,\alpha}B^\alpha+B^2B^\alpha{}_{;\alpha}/4\pi. \quad (75)$$

Replacing $B_\alpha a^\alpha$ by (29) and $p_{,\alpha}$ by (40) and canceling out terms, we can write the result as

$$(\mu B^\alpha)_{;\alpha}=0. \quad (76)$$

Although this looks like Gauss's law for the magnetic field, one must remember that it also contains information about the dynamics of the (perfect) fluid.

Writing out (76) explicitly with account taken of the symmetries [$B^\alpha{}_{,0}=B^\alpha{}_{,3}=0$ follows from (57), (58), and (6)], we have

$$(B^1\sqrt{-g})_{,1}+(B^2\sqrt{-g})_{,2}=-\sqrt{-g}\,[(\ln\mu)_{,1}B^1+(\ln\mu)_{,2}B^2]. \quad (77)$$

The B^1 and B^2 may be computed directly from the definition (6). Replacing each component of $F_{\alpha\beta}$ with the help of (67), (68), and (72)–(74), we get

$$\sqrt{-g}\,B^1=-C(u_0-Au_3)\sqrt{-g}\,nu^1, \quad (78)$$

$$\sqrt{-g}\,B^2=-C(u_0-Au_3)\sqrt{-g}\,nu^2. \quad (79)$$

When these are substituted into (77) and use is made of baryon conservation one gets (recall C is conserved)

$$-C\sqrt{-g}\,n[(u_0-Au_3)_{,1}u^1+(u_0-Au_3)_{,2}u^2] = C\sqrt{-g}\,n(u_0-Au_3)[(\ln\mu)_{,1}u^1+(\ln\mu)_{,2}u^2]. \quad (80)$$

Dividing this through by $C\sqrt{-g}\,n(u_0-Au_3)$ and remembering the symmetries, we have

$$\frac{d}{d\tau}\ln[\mu(u_0-Au_3)]=0. \quad (81)$$

Thus we have

$$\mu(u_0-Au_3)=\mu u_\alpha(\xi^\alpha_t-A\xi^\alpha_\varphi)=D, \quad (82)$$

where D is conserved along each flowline, though it may vary from flowline to flowline. This law has no analog in perfect-fluid dynamics. Now we shall consider some that do.

VI. GENERALIZATIONS OF BERNOULLI'S THEOREM

In Newtonian physics the sum of the specific enthalpy, specific kinetic energy and gravitational potential for an ideal fluid is conserved along each flowline for stationary flow.[15] For relativistic flow one has the following generalization of this Bernoulli theorem[7]:

$$\mu u_\alpha\xi^\alpha_t=-E, \quad (83)$$

where E is constant along each flowline. An analogous law applies for ξ^α_φ. Once a sizable magnetic field is present these conservation laws break down. We now show how to generalize them to include effects of the magnetic field.

Let ξ^α denote either ξ_t^α or ξ_φ^α. It satisfies Killing's equation[13]

$$\xi_{\alpha;\beta}+\xi_{\beta;\alpha}=0 \tag{84}$$

by virtue of the symmetry. One may define a current vector

$$P^\alpha = T^{\alpha\beta}\xi_\beta \tag{85}$$

which, by virtue of (1), the symmetry of $T^{\alpha\beta}$, and (84), is divergenceless:

$$P^\alpha{}_{;\alpha}=0. \tag{86}$$

For ξ_t^α, $-P^\alpha$ represents the (conserved) energy current; for ξ_φ^α, P^α represents the (conserved) angular momentum current. This is all very general. Specializing now to the case of a highly conducting magnetized perfect fluid we have from (21), with the notation

$$\chi=\mu+B^2/4\pi n, \tag{87}$$

that

$$P^\alpha=\chi u_\beta\xi^\beta n u^\alpha+(p+B^2/8\pi)\xi^\alpha-B_\beta\xi^\beta B^\alpha/4\pi. \tag{88}$$

In taking the divergence of P^α we must remember that the divergence of nu^α vanishes, that the gradient of $p+B^2/8\pi$ along ξ^α vanishes by the symmetry, and that $\xi^\alpha{}_{;\alpha}=0$ by the trace of (84). The result is

$$(\chi u_\beta\xi^\beta)_{,\alpha}nu^\alpha-(B_\beta\xi^\beta)_{,\alpha}B^\alpha/4\pi-(B_\beta\xi^\beta)B^\alpha{}_{;\alpha}/4\pi=0. \tag{89}$$

By virtue of (76) we may replace $B^\alpha{}_{;\alpha}$ by $-(\ln\mu)_{,\alpha}B^\alpha$. By the symmetries, only derivatives with respect to x^1 and x^2 can appear in (89). Replacing B^1 and B^2 wherever they appear by (78) and (79), and dividing through by n, we have

$$\frac{d}{d\tau}(\chi u_\beta\xi^\beta)+\frac{C}{4\pi}(u_0-Au_3)\times\left[\frac{d}{d\tau}(B_\beta\xi^\beta)-B_\beta\xi^\beta\frac{d}{d\tau}(\ln\mu)\right]=0. \tag{90}$$

The derivative of $\ln\mu$ may be evaluated by means of (81). The final result is

$$\frac{d}{d\tau}\left[\chi u_\beta\xi^\beta+\frac{C}{4\pi}(u_0-Au_3)B_\beta\xi^\beta\right]=0. \tag{91}$$

We thus have the following generalizations of the Bernoulli theorems:

$$\chi u_\alpha\xi_t^\alpha+Cu_\alpha(\xi_t^\alpha-A\xi_\varphi^\alpha)B_\beta\xi_t^\beta/4\pi=-E, \tag{92}$$

$$\chi u_\alpha\xi_\varphi^\alpha+Cu_\alpha(\xi_t^\alpha-A\xi_\varphi^\alpha)B_\beta\xi_\varphi^\beta/4\pi=L, \tag{93}$$

where E and L are conserved along each flowline, but may vary from flowline to flowline. The magnetic corrections implicit in (92) and (93) are significant when the magnetic pressure is comparable to p.

It is important to be sure that the conservation laws (82), (92), and (93) are really independent. The law (82) is effectively the projection of (1) along B_α. In view of (85), the laws (92) and (93) are the projection of (1) along the Killing vectors since

$$(T^{\alpha\beta}\xi_\beta)_{;\alpha}=T^{\alpha\beta}{}_{;\alpha}\xi_\beta. \tag{94}$$

But since B^α, ξ_t^α, and ξ_φ^α are clearly linearly independent, the laws must be independent. It is also important to note that each of the two Bernoulli-type laws (92) and (93) is valid only if both symmetries are present, unlike the case for a nonmagnetized fluid where each symmetry gives rise to its own law independently.

VII. CONSERVATION OF MAGNETIC ENERGY

We pointed out in Sec. II that in the high-conductivity limit, internal energy and magnetic energy are not exchanged, i.e., (20) is true. Our object here is to write this differential law as the conservation of a certain quantity along a flowline. From the new form we shall obtain a connection between the constants of integration D, A, E, and L.

Substituting (16) into (20) and remembering that $a^\alpha u_\alpha=B^\alpha u_\alpha=0$, we have

$$dB^2/d\tau=-2B^2u^\alpha{}_{;\alpha}-2B_{\alpha;\beta}B^\beta u^\alpha. \tag{95}$$

Now, from $B_\alpha u^\alpha=0$ it follows by covariant differentiation and contraction with B^β that

$$B_{\alpha;\beta}B^\beta u^\alpha=-B_\alpha B^\beta u^\alpha{}_{;\beta}, \tag{96}$$

whereas,

$$B_\alpha B^\beta u^\alpha{}_{;\beta}=B_\alpha B^\beta u^\alpha{}_{,\beta}+B_\alpha B^\beta\Gamma^\alpha_{\gamma\beta}u^\gamma. \tag{97}$$

The term involving the Christoffel symbols may be obtained from the result

$$dB^2/d\tau=(B_\alpha B^\alpha)_{,\beta}u^\beta=2B_\alpha B^\alpha{}_{,\beta}u^\beta+2B_\alpha B^\beta\Gamma^\alpha_{\gamma\beta}u^\gamma. \tag{98}$$

In making all the replacements in (95) it is also convenient to use

$$B_\alpha B^\alpha{}_{,\beta}u^\beta=dB^2/d\tau-B_{\alpha,\beta}u^\beta B^\alpha \tag{99}$$

and

$$B_\alpha B^\beta u^\alpha{}_{,\beta}=-B_{\alpha,\beta}u^\alpha B^\beta, \tag{100}$$

which follow directly from $B_\alpha B^\alpha=B^2$ and $B_\alpha u^\alpha=0$, respectively, as well as

$$u^\alpha{}_{;\alpha}=-n^{-1}dn/d\tau, \tag{101}$$

which follows from (4). The final result is

$$dB^2/d\tau-B^2n^{-1}dn/d\tau+(B_{\alpha,\beta}-B_{\beta,\alpha})u^\alpha B^\beta=0. \tag{102}$$

We may simplify (102) by recalling that all derivatives of B_α with respect to x^0 or x^3 must vanish. In view of the form of B^1 and B^2 given by (78) and (79) we get

$$B^{-2}dB^2/d\tau - n^{-1}dn/d\tau + B^{-2}[(fu^0 - B^0)dB_0/d\tau + (fu^3 - B^3)dB_3/d\tau] = 0, \tag{103}$$

where

$$f = -Cn(\mu_0 - Au_3). \tag{104}$$

By going back to the definition of B^α, (6), and computing the components of $F_{\alpha\beta}$ from (67), (68), and (72)–(74), we have

$$fu^0 - B^0 = Cn, \tag{105}$$

$$fu^3 - B^3 = -CAn. \tag{106}$$

Thus (103) reduces to

$$\frac{d}{d\tau}\left[\frac{B^2}{n} + C(B_0 - AB_3)\right] = 0, \tag{107}$$

which is the required conservation law.

We may also write (107) as

$$B^2/n + C(B_0 - AB_3) = F, \tag{108}$$

where F is conserved along each flowline. Let us now take (92) and subtract from it A times (93). In view of (82) and (108) we get

$$D + F(u_0 - Au_3)/4\pi = -E - AL. \tag{109}$$

Clearly [see (82)] $u_0 - Au_3$ is not in general conserved along a flowline. The other terms in (109) are conserved; thus we must have $F = 0$, and consequently

$$D = -E - AL. \tag{110}$$

Thus D is not an independent "constant of the motion." Once D is chosen as in (110), the conservation law (107) becomes a consequence of the laws (82) and (92), (93).

VIII. ON EXTRACTION OF ENERGY FROM A BLACK HOLE

The energy source of a pulsar is the rotation of a neutron star. Is there any astrophysical phenomenon which is powered by the rotational energy of a black hole? One might put forward some of the compact x-ray sources and the quasars as possible candidates. In both cases there is evidence consistent with the presence of a black hole,[17] but in both cases the energy may well be liberated by accretion of gas onto the hole, rather than being rotational in origin. Clearly the answer to our question depends on whether black-hole rotational energy is extractable in an astrophysically realistic way. That the extraction is possible in principle was first demonstrated by Penrose.[18] Recently Piran and Shaham[19] proposed an astrophysically realistic mechanism based on Penrose processes to extract black-hole rotational energy and produce the mysterious γ-ray bursts.[20] By its very nature this mechanism extracts energy in spurts. Is there a mechanism capable of extracting energy steadily?

The straightforward analog of the pulsar mechanism is ruled out because, unlike a pulsar, an astrophysical black hole is expected to be devoid of a magnetic field.[21] One might, however, hope that the magnetic field frozen into plasma surrounding a black hole might accomplish the extraction of energy, perhaps with the help of the dragging of inertial frames. In this context the question may be posed as follows: Is outflow of energy from a rotating black hole possible when the hole is surrounded by a magnetized plasma in a stationary state? In view of a general theorem of Hawking,[22] one may also assume the system to be axisymmetric.

Ruffini and Wilson[8] have claimed that the answer to the question posed earlier is positive on the basis of a calculation of the stationary axisymmetric flow of highly conducting magnetized plasma in the vicinity of a Kerr black hole. Their conclusion may be questioned since it was reached by neglecting the influence of the magnetic field on the motion (geodesic flowlines), while taking it into account in the energy transport. Now that we have several results about magnetohydrodynamic flow which take into account fully the effects of the magnetic field, we can check whether the result in question is valid in general, or is an artifact of the assumptions made.

We shall *not* have to assume that the black hole is of the Kerr type; thus our argument will hold even if the enveloping plasma is massive enough to perturb the metric significantly. Now consider a spacelike 2-surface Σ, sharing the symmetries of the problem, which encloses the black hole. It must be described by an equation of the form

$$g(x^1, x^2) = \text{const}, \tag{111}$$

where g is some function; there is no x^0 or x^3 dependence in view of the symmetries. The surface's normal vector, $N_\alpha = g_{,\alpha}$, clearly has vanishing N_0 and N_3 components. Recalling that the energy current is given by $-T_0^\alpha$ [see (85)], we see that the mass energy M of the black hole must change at a rate

$$\dot{M} = \int_\Sigma T_0^\alpha N_\alpha dA, \tag{112}$$

where dA is the area element of Σ, and N_α has been chosen as the appropriately normalized outward normal. It is understood that Σ is chosen close enough to the black-hole horizon. Were we to be interested in the rate at which the black hole absorbs baryon number b, we would write

$$\dot{b} = -\int_{\Sigma} nu^{\alpha} N_{\alpha} dA \tag{113}$$

because nu^{α} is the baryon current.

From (88), (78), and (79) we have for the components T^1_0 and T^2_0

$$T^i_0 = [\chi u_0 + C(u_0 - Au_3)B_0/4\pi]nu^i . \tag{114}$$

Thus by (92)

$$\dot{M} = -\int_{\Sigma} Enu^{\alpha} N_{\alpha} dA , \tag{115}$$

where E at each point on Σ may be evaluated anywhere along the particular flowline. Comparing (113) and (115) we see that each baryon adds energy E to the black hole, where E refers to its own flowline. Thus, the question of whether energy outflow from a rotating black hole is possible hinges on the sign of E.

To determine the sign let us with the help of (78), (79), and the condition $B_{\alpha}u^{\alpha} = 0$ rewrite (92) in the form

$$E = -\chi u_0 - (B_1u^1 + B_2u^2 + B_3u^3)(B^1/u^1 + B^2/n^2) \times (8\pi nu^0)^{-1} . \tag{116}$$

Since E is conserved along a flowline, it may be evaluated far from the black hole. There we expect the flow to be nonrelativistic; thus in asymptotically flat coordinates $u^0 \approx 1$, $u_0 \approx -1$ and u^1, u^2, and u^3 are the ordinary space components of the velocity. We see that the last term in (116) is of order $B^2/8\pi n$ with indeterminate sign. Now, $B^2/8\pi n$ is the magnetic energy per baryon. In the asymptotic regime it is surely much smaller than the rest energy per baryon, which itself is smaller than χ. Thus on each flowline which comes from the asymptotic region, E must be positive. But in steady state every flowline which enters the black hole must come from the asymptotic region. It follows from (115) that $\dot{M}$ has the same sign as $\dot{b}$, namely positive; the black hole cannot lose energy. Energy extraction is not possible in the stationary state, no matter what the flow pattern is. Apparently the contrary conclusion reached by Ruffini and Wilson[8] is an artifact of the geodesic approximation.

It is still possible that extraction of energy by a magnetized plasma may be possible in a time-dependent situation; such extraction might have something to do with the short-timescale variability of quasars and Lacertid objects. It is also possible that dissipative effects due to viscosity (especially in shocks) or finite electrical conductivity may make extraction possible, though this seems less likely.

IX. A CIRCULATION CONSERVATION THEOREM IN GRM

For an unmagnetized perfect fluid one has the generalized Bernoulli theorem (83) according to which μn_0 is conserved along each flowline in stationary flow. In addition, for a perfect fluid one has the generalized Kelvin theorem (44). From it under appropriate boundary conditions follows that μu_0 is not only conserved along flowlines, but is the same number for all flowlines [see (48a)]. Now, for a magnetized plasma we have the Bernoulli-type result (92). To show further that E in (92) is the same for all flowlines, we would, by analogy with the preceding discussion, have to first find a Kelvin-type theorem for a magnetized plasma. We have not succeeded in finding such a theorem valid under all conditions, but have proved one for axisymmetric stationary flow. This is sufficient to show that E is constant from flowline to flowline.

Let us take the tensor $w_{\alpha\beta}$ of Sec. III to be the curl of the vector

$$z^{\alpha} = \chi u^{\alpha} + C(u_0 - Au_3)B^{\alpha}/4\pi , \tag{117}$$

where C is defined by (72); it is clear that the assumptions of stationary axisymmetric flow has already been made. By its very definition $w_{\alpha\beta}$ satisfies (38). We now show it also satisfies (37).

Directly from the definition

$$w_{\alpha\beta}u^{\beta} = \chi u_{\beta\,;\,\alpha}u^{\beta} - \chi_{,\alpha} - \chi u_{\alpha\,;\,\beta}u^{\beta} - \chi_{,\beta}u^{\beta}u_{\alpha} - (CD\mu^{-1}B_{\alpha})_{,\beta}u^{\beta}/4\pi + (CD\mu^{-1}B_{\beta})_{,\alpha}u^{\beta}/4\pi , \tag{118}$$

where use has been made of (82). We now proceed to rewrite each of the six terms in (118). The first term vanishes in view of the normalization of u_{β}. Differentiating χ explicitly and taking account of (25) we have for the second term

$$-\chi_{,\alpha} = -n^{-1}p - (B^2/4\pi n)_{,\alpha} . \tag{119}$$

The third term in (118) is given by (26):

$$-\chi u_{\alpha\,;\,\beta}u^{\beta} = n^{-1}(p + B^2/8\pi)_{,\alpha} + n^{-1}(p + B^2/8\pi)_{,\beta}u^{\beta}u_{\alpha} - n^{-1}(B_{\alpha\,;\,\beta}B^{\beta} + B_{\alpha}B^{\beta}{}_{;\beta} + u_{\alpha}u_{\beta}B^{\beta}{}_{;\gamma}B^{\gamma})/4\pi . \tag{120}$$

The fourth term may be rewritten by use of the expression $u_{\alpha}T^{\alpha\beta}{}_{;\beta} = 0$, which follows from (1). Substituting (21) into it and making use of (23) gives

$$-\chi_{,\beta}u^{\beta}u_{\alpha} = -n^{-1}(p + B^2/8\pi)_{,\beta}u^{\beta}u_{\alpha} + n^{-1}B^{\beta}{}_{;\gamma}B^{\gamma}u_{\beta}u_{\alpha}/4\pi . \tag{121}$$

By the conservation of CD along flowlines, the fifth term in (120) reduces to

$$-(CD\mu^{-1}B_\alpha)_{,\beta}u^\beta/4\pi = -CD(B_{\alpha,\beta}u^\beta - \mu^{-1}\mu_{,\beta}u^\beta B_\alpha)/4\pi\mu\,, \tag{122}$$

while by the orthogonality of u^β and B_β, the sixth term reduces to

$$(CD\mu^{-1}B_\beta)_{,\alpha}u^\beta/4\pi = CD\mu^{-1}B_{\beta,\alpha}u^\beta/4\pi\,. \tag{123}$$

Substitution of (119)–(123) into (118) gives

$$w_{\alpha\beta}u^\beta = (\tfrac{1}{2}B^2{}_{,\alpha} - B_{\alpha;\beta}B^\beta - B_\alpha B^\beta{}_{;\beta})/4\pi n - (B^2/4\pi n)_{,\alpha} - CD(B_{\alpha,\beta} - B_{\beta,\alpha} - \mu^{-1}\mu_{,\beta}B_\alpha)/4\pi\mu\,. \tag{124}$$

This expression is further simplified as follows. From (76), (78), (79), and (82) we have

$$B^\beta{}_{;\beta} = -\mu^{-1}\mu_{,\beta}B^\beta = CDn\mu^{-2}\mu_{,\beta}u^\beta\,, \tag{125}$$

where we have used the symmetries. Next we note that

$$\tfrac{1}{2}B^2{}_{,\alpha} - B_{\alpha;\beta}B^\beta = (B_{\beta;\alpha} - B_{\alpha;\beta})B^\beta = (B_{\beta,\alpha} - B_{\alpha,\beta})B^\beta\,. \tag{126}$$

Thus, due to cancellations,

$$w_{\alpha\beta}u^\beta = (B_{\beta,\alpha} - B_{\alpha,\beta})(B^\beta + CDn\mu^{-1}u^\beta)/4\pi n - (B^2/4\pi n)_{,\alpha}\,. \tag{127}$$

Now from (82), (78), (79), and (104)–(106) it follows that

$$B^\beta + CDn\mu^{-1}u^\beta = Cn(-\delta^\beta{}_0 + A\delta^\beta{}_3)\,. \tag{128}$$

Therefore, in view of the symmetries,

$$w_{\alpha\beta}u^\beta = [B^2/n + C(B_0 - AB_3)]_{,\alpha}/4\pi\,. \tag{129}$$

But from (108) and the conclusion that $F=0$ it follows that the contents of the square brackets in (129) vanish. Thus condition (37) is satisfied and there is a conservation law of the form (32) or (30).

We have thus obtained a GRM generalization of Kelvin's theorem for the case of stationary axisymmetric flow. One application of this result concerns flow for which asymptotically $w_{\alpha\beta}\to 0$. This would be the case for asymptotically nonrelativistic and uniform flow with a uniform magnetic field. By our theorem $w_{\alpha\beta}$ is zero everywhere. Thus z_α has to be the gradient of some scalar Φ. For the z_α to be independent of x^0 and x^3, Φ must be of the form (47), where E and L are constants. From $z_0 = \Phi_{,0}$ and $z_3 = \Phi_{,3}$ we recover our previous results (92) and (93), except this time E and L are the same for all flowlines.

By squaring $z_\alpha = \Phi_{,\alpha}$ and using (82) we obtain

$$\nabla_\alpha\Phi\nabla^\alpha\Phi = -(\chi^2 - C^2D^2\mu^{-2}B^2/16\pi^2)\,, \tag{130}$$

which is a Hamilton-Jacobi equation for a particle with variable mass. In solving (130) it may be essential to eliminate B^2 and leave only fluid variables. This is done as follows. We write (128) as

$$B_\alpha = -CDn\mu^{-1}u_\alpha - Cn(g_{0\alpha} - Ag_{3\alpha})\,. \tag{131}$$

Then by forming B^2 and using (82) again we have

$$B^2 = C^2n^2(D^2\mu^{-2} + g_{00} - 2Ag_{03} + A^2g_{33})\,, \tag{132}$$

which is the desired relation. Once Φ is solved for, u_α follows from $z_\alpha = \Phi_{,\alpha}$, from which B_α is eliminated by means of (131). Then B_α is recovered. Thus the problem of determining the flow field and the magnetic field reduces to solving two equations, (130), and (4) supplemented by an appropriate equation of state.

Incidentally, (132) is the law of growth of the magnetic field. We notice the tendency of B to increase as n/μ. Since μ is expected to increase slower than $n^{1/3}$ (this is the law for extreme relativistic gas), B will always grow with n, and B^2/n will grow faster than μ until the extreme relativistic regime is reached. Thus the role of the magnetic field increases during the inflow process and there is some indication that the magnetic energy per baryon reaches some sort of equipartition with the fluid energy per baryon.

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APPENDIX

In nonrelativistic magnetohydrodynamics the law of magnetic-flux conservation can also be expressed by saying that if two nearby fluid elements lie on a magnetic line at one instant, they always do so, while B divided by the density varies as the distance between them.[9] The relativistic form of this statement is obtained as follows.

Consider a vector η^μ between two nearby fluid elements. It is frozen into the fluid and thus it must be Lie-transported along the velocity:

$$D\eta^\mu/d\tau = u^\mu{}_{;\alpha}\eta^\alpha\,. \tag{A1}$$

The spacelike part of η^μ with respect to the fluid velocity $h^{\mu\alpha}\eta_\alpha$ is orthogonal to u_μ. We call it $\tilde{\eta}^\mu$, and write

$$\eta^\mu = \tilde{\eta}^\mu - \eta_\beta u^\beta u^\mu\,. \tag{A2}$$

Substituting this into (A1), making use of $\tilde{\eta}_\alpha u^\alpha = a_\alpha u^\alpha = 0$, and simplifying, we get

$$D\tilde{\eta}^\mu/d\tau = u^\mu{}_{;\alpha}\tilde{\eta}^\alpha + u^\mu\tilde{\eta}_\beta a^\beta\,. \tag{A3}$$

Now for a highly conducting plasma[7]

$$DB^\mu/d\tau = u^\mu{}_{;\alpha}B^\alpha + a_\alpha B^\alpha u^\mu - u^\alpha{}_{;\alpha}B^\mu\,. \tag{A4}$$

Substituting $u^\alpha{}_{;\alpha}$ from (4) we find

$$\frac{D}{d\tau}\left(\frac{B^\mu}{n}\right)=u^\mu{}_{;\alpha}\frac{B^\alpha}{n}+u^\mu a_\alpha\frac{B^\alpha}{n}. \qquad \text{(A5)}$$

We now see that B^μ/n and $\bar{\eta}^\mu$ obey the same type of equation. If η^μ connects two fluid elements which lie on a magnetic field line at a given instant (as seen by a comoving observer), then $\bar{\eta}^\mu$ will be proportional to B^μ/n at that instant. By (A3) and (A5) $\bar{\eta}^\mu$ and B^μ/n will always remain proportional with the same proportionality factor. Thus the fluid elements remain on the magnetic line, and B/n varies as the distance between them.

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