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Spherically symmetric charged dust distribution in the Brans-Dicke theory

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The equations governing the equilibrium of a static charged dust distribution in Einstein's theory as considered by Das, and De and Raychaudhuri is generalized to the Brans-Dicke theory. It has been observed that these generalized equations do not admit any spherically symmetric solution.

I. INTRODUCTION

In the study of the Brans-Dicke (BD) source-free electrostatic fields it has been observed[1] that the assumption that the 4-4 component of the static metric is dependent on the BD scalar ϕ and the electrostatic scalar ψ leads to a relationship

$$g_{44} = -\phi^{-1}(4\pi\psi^2 + A\psi + B)\,, \tag{1.1}$$

where A and B are arbitrary constants. Further, by a proper choice of the constants A and B, so as to reduce (1.1) to

$$g_{44} = -4\pi\phi^{-1}(\psi + \sqrt{2})^2\,, \tag{1.2}$$

and then assuming the relation (1.2) to be valid in the interior of a charged dust distribution, it has been shown[2] that there exists a relationship between the charge density and the matter density distributions given by

$$\sigma/\rho = \pm\phi^{-1/2}\,. \tag{1.3}$$

It may be verified that for $\phi = \text{const} = 1$, Eqs. (1.1), (1.2), and (1.3) reduce, respectively, to the well-known results of Einstein's theory, viz.,

$$g_{44} = -(4\pi\psi^2 + A\psi + B)\,, \tag{1.4}$$

$$g_{44} = -4\pi(\psi + \sqrt{2})^2\,, \tag{1.5}$$

and

$$\sigma/\rho = \pm 1\,. \tag{1.6}$$

We may recall that the solutions (1.4), (1.5), and (1.6) are the well-known results of Einstein's gravitational theory obtained by Majumdar,[3] Papapetrou,[4] Das,[5] and De and Raychaudhuri.[6] It may be further noted that with the validity of the relations (1.5) and (1.6), Einstein's field equations characterizing static charged dust distribution in equilibrium reduce to a single nonlinear equation

$$\frac{\partial^2 W}{\partial x^2} + \frac{\partial^2 W}{\partial y^2} + \frac{\partial^2 W}{\partial z^2} = -4\pi\rho'(1+W)^3\,, \tag{1.7}$$

while the static metric reduces to

$$ds^2 = (1+W)^2(dx^2 + dy^2 + dz^2) - (1+W)^{-2}dt^2\,, \tag{1.8}$$

where

$$(1+W)^{-2} = 4\pi(\psi \pm \sqrt{2})^2 \text{ and } \rho = \sqrt{4\pi}\rho'\,.$$

A regular solution of (1.7), which could be matched with the Reissner-Nordström solution, was obtained by Bonnor.[7] The existence of such a solution led him to conclude that the presence of charge may halt the gravitational collapse of large masses.

Thus, so far, our attempt has been to generalize certain well-known results, viz., (1.4)–(1.6) of Einstein's theory to the BD theory. In the present paper our objective is to solve the BD field equations taking the validity of the relations (1.2) and (1.3) into account and imposing the restriction of spherical symmetry on the metric tensor. In fact, in Sec. III we have made such an attempt and have arrived at the conclusion that the BD field equations do not admit any solution when the relations (1.2) and (1.3) and the condition of spherical symmetry are imposed together. However, when the relations (1.2) and (1.3) only are imposed, the BD field equations have been shown to reduce to a set of equations which reduces to (1.7) in the limit in which the BD coupling parameter $\omega \to \infty$ and $\phi \to$ const.

II. FIELD EQUATIONS

The BD field equations characterizing charged dust distribution corresponding to the static metric

$$ds^2 = g_{\alpha\beta}dx^\alpha dx^\beta - V^2 dt^2 \tag{2.1}$$

are[2]

$$\begin{aligned} R_{\alpha\beta} \equiv \bar{R}_{\alpha\beta} + \frac{1}{V} V_{;\alpha\beta} &= -\frac{8\pi}{\phi V^2}\left(\tfrac{1}{2}g_{\alpha\beta}g^{\sigma\gamma}\psi_{,\sigma}\psi_{,\gamma} - \psi_{,\alpha}\psi_{,\beta}\right) \\ &\quad - \frac{\omega}{\phi^2}\phi_{,\alpha}\phi_{,\beta} - \frac{1}{\phi}\phi_{;\alpha\beta} \\ &\quad - \frac{\omega+1}{2\omega+3}\frac{8\pi\rho'}{\phi}g_{\alpha\beta}\,, \end{aligned} \tag{2.2}$$

$$R_{44} \equiv -V g^{\alpha\beta} V_{;\alpha\beta} = -\frac{8\pi}{\phi}\left(\frac{\omega+2}{2\omega+3}\right)\rho' V^2 - \frac{8\pi}{2\phi} g^{\sigma\gamma}\psi_{,\sigma}\psi_{,\gamma} - \frac{1}{\phi}\phi_{;44}, \quad (2.3)$$

$$g^{\alpha\beta}\phi_{;\alpha\beta} + \frac{1}{V} g^{\alpha\beta} V_{,\alpha}\phi_{,\beta} = -\frac{8\pi}{2\omega+3}\rho', \quad (2.4)$$

and

$$\frac{1}{V} g^{\alpha\beta} V_{;\alpha\beta} - \frac{1}{V^2} g^{\alpha\beta}\psi_{,\alpha} V_{,\beta} = \sigma, \quad (2.5)$$

where $\bar{R}_{\alpha\beta}$ is the Ricci tensor corresponding to the metric $g_{\alpha\beta}$. The Greek indices run from 1 to 3, and a comma or a semicolon followed by an index represents partial differentiation or covariant differentiation, respectively. With (1.2) and (1.3) being valid, (2.3) is identically satisfied and the field equations (2.2), (2.4), and (2.5) reduce, respectively, to

$$\bar{R}'_{\alpha\beta} = -(\omega + \tfrac{3}{2}) p_{,\alpha} p_{,\beta}, \quad (2.6)$$

$$g'^{\alpha\beta} p_{:\alpha\beta} = -\frac{8\pi}{2\omega+3}\rho' e^{-2p}(1+W)^2, \quad (2.7)$$

and

$$g'^{\alpha\beta} W_{:\alpha\beta} = -4\pi\rho' e^{-2p}(1+W)^3, \quad (2.8)$$

where $g'_{\alpha\beta} = [e^p/(1+W)^2] g_{\alpha\beta}$ with $\phi = e^p$ and $4\pi(\psi \pm \sqrt{2})^2 = (1+W)^{-2}$. $\bar{R}'_{\alpha\beta}$ is the Ricci tensor corresponding to the metric $g'_{\alpha\beta}$ and a colon followed by an index denotes covariant differentiation with respect to $g'_{\alpha\beta}$. It can be observed that when $\phi \to \infty$ and $p \to$ const (to be taken zero so that $\phi \to G^{-1} = 1$), (2.7) is identically satisfied while (2.6) suggests $g'_{\alpha\beta} = \eta_{\alpha\beta}$ (Minkowskian space metric). In view of this, (2.8) reduces to (1.7). Obviously, the relations (1.2) and (1.3) reduce, respectively, to (1.5) and (1.6). This agreement is in conformity with the requirement of the BD theory and suggests that the situation characterized by Eqs. (2.6)–(2.8) is a natural generalization of the corresponding situation in Einstein's theory as represented by (1.7).

III. NONEXISTENCE OF A SPHERICALLY SYMMETRIC SOLUTION

We prove in the following steps that no spherically symmetric static solution to the field equations (2.2)–(2.5) exists when the relations (1.2) and (1.3) are assumed to be valid. The BD field equations corresponding to the spherically symmetric metric,

$$ds^2 = e^{\alpha} dr^2 + r^2(d\theta^2 + \sin^2\theta\, d\Phi^2) - e^{\beta} dt^2, \quad (3.1)$$

are given by

$$\frac{1}{2\phi}\phi_{,11} + (\omega + \tfrac{3}{4})\left(\frac{\phi_{,1}}{\phi}\right)^2 = -\left(\frac{\psi_{,11}}{\psi \pm \sqrt{2}}\right) + \left(\frac{\psi_{,1}}{\psi \pm \sqrt{2}}\right)^2 + \frac{\phi_{,1}\psi_{,1}}{\phi(\psi \pm \sqrt{2})} + \frac{1}{2}\frac{\alpha_{,1}\psi_{,1}}{\psi \pm \sqrt{2}} + \frac{1}{4}\frac{\alpha_{,1}\phi_{,1}}{\phi} + \frac{\alpha_{,1}}{r} - \frac{\omega+1}{2\omega+3}\frac{8\pi}{\phi}\rho' e^{\alpha}, \quad (3.2)$$

$$\frac{1}{r^2}(1 - e^{\alpha}) + \frac{1}{2r}\left(\frac{\phi_{,1}}{\phi}\right) = -\left(\frac{\psi_{,1}}{\psi \pm \sqrt{2}}\right)^2 - \frac{1}{r}\left(\frac{\psi_{,1}}{\psi \pm \sqrt{2}}\right) + \frac{\alpha_{,1}}{2r} - \frac{\omega+1}{2\omega+3}\frac{8\pi}{\phi}\rho' e^{\alpha}, \quad (3.3)$$

$$\frac{\phi_{,11}}{\phi} - \frac{1}{2}\frac{\alpha_{,1}\phi_{,1}}{\phi} + \frac{2}{r}\frac{\phi_{,1}}{\phi} - \frac{1}{2}\left(\frac{\phi_{,1}}{\phi}\right)^2 + \frac{\phi_{,1}\psi_{,1}}{\phi(\psi \pm \sqrt{2})} = -\frac{8\pi}{2\omega+3}\rho' e^{\alpha}, \quad (3.4)$$

and

$$\frac{\psi_{,11}}{\psi \pm \sqrt{2}} - \frac{1}{2}\frac{\alpha_{,1}\psi_{,1}}{\psi \pm \sqrt{2}} + \frac{2}{r}\frac{\psi_{,1}}{\psi \pm \sqrt{2}} + \frac{1}{2}\frac{\phi_{,1}\psi_{,1}}{\phi(\psi \pm \sqrt{2})} - \left(\frac{\psi_{,1}}{\psi \pm \sqrt{2}}\right)^2 = \frac{4\pi\rho'}{\phi} e^{\alpha}. \quad (3.5)$$

From these equations, after a series of mathematical operations, it can be shown that

$$e^{\alpha} = r^2\left[\left(\frac{1}{r} + \frac{\psi_{,1}}{\psi \pm \sqrt{2}}\right)^2 + \frac{\phi_{,1}}{\phi}\left(\frac{1}{r} + \frac{\psi_{,1}}{\psi \pm \sqrt{2}}\right) - \frac{\omega+1}{2}\left(\frac{\phi_{,1}}{\phi}\right)^2\right] = r^2 K L, \quad (3.6)$$

where K and L are two auxiliary functions given as

$$K = \left(\frac{1}{r} + \frac{\psi_{,1}}{\psi \pm \sqrt{2}}\right) + \frac{1}{2}\left(1 + \frac{1}{k}\right)\frac{\phi_{,1}}{\phi},$$

$$L = \left(\frac{1}{r} + \frac{\psi_{,1}}{\psi \pm \sqrt{2}}\right) + \frac{1}{2}\left(1 - \frac{1}{k}\right)\frac{\phi_{,1}}{\phi},$$

with

$$k = \frac{1}{(2\omega+3)^{1/2}}.$$

In terms of the new variables K and L, the field equations reduce to the following set of independent equations:

$$(K+L)\left[\left(\frac{K_{,1}}{K}-\frac{L_{,1}}{L}\right)+(K-L)\right]=-\frac{16\pi k\rho'}{\phi}r^2KL\,, \tag{3.7}$$

$$\left(\frac{1+k^2}{2}\right)(K-L)^2+\frac{2k}{r}(K-L)-k(K^2-L^2)-\frac{1}{r}(K+L)=\frac{1}{r}\left(\frac{K_{,1}}{K}+\frac{L_{,1}}{L}\right)-(1-k^2)\frac{8\pi\rho'}{\phi}r^2KL\,, \tag{3.8}$$

and

$$\tfrac{1}{4}(K-L)\left[\left(\frac{K_{,1}}{K}-\frac{L_{,1}}{L}\right)+(K-L)\right]=0\,. \tag{3.9}$$

From (3.9), it can be concluded that either

$$K-L=0 \tag{3.10}$$

or

$$\left(\frac{K_{,1}}{K}-\frac{L_{,1}}{L}\right)+(K-L)=0\,. \tag{3.11}$$

The possibility (3.10), however, is ruled out so long as the BD scalar ϕ is a variable and the BD coupling parameter ω is positive. Therefore, in view of (3.11), which is the only alternative left, Eq. (3.7) suggests that the matter density should necessarily be zero, i.e., $\rho'=0$. This leads to the conclusion that the spherically symmetric charged dust distribution cannot hold itself in equilibrium if the scalar ϕ couples with the electrostatic scalar ψ and the charge density as given in (1.2) and (1.3). However, in the limit $\phi \to$ const and $\omega\to\infty$, such that $K-L\to 0$ and $k\to 0$, we find that Eqs. (3.7) and (3.9) are identically satisfied for any finite ρ' whereas Eq. (3.8) reduces to

$$L_{,1}+L^2=4\pi\rho' r^3L^3\,, \tag{3.12}$$

where

$$L=\frac{1}{r}+\frac{\psi_{,1}}{\psi\pm\sqrt{2}}\,.$$

The metric (3.1) for this case is

$$ds^2=r^2L^2dr^2+r^2(d\theta^2+\sin^2\theta\, d\Phi^2)-4\pi(\psi\pm\sqrt{2})^2dt^2\,. \tag{3.13}$$

Defining a new radial coordinate R as

$$r=\frac{R}{\sqrt{4\pi}\,(\psi\pm\sqrt{2})}\,, \tag{3.14}$$

the metric (3.13) reduces to

$$ds^2=[4\pi(\psi\pm\sqrt{2})]^{-1}[dR^2+R^2(d\theta^2+\sin^2\theta\, d\Phi^2)]-4\pi(\psi\pm\sqrt{2})^2dt^2\,. \tag{3.15}$$

In terms of the function W (as introduced in Sec. II) the metric (3.15) reduces to (1.8). Equation (3.12), in terms of the new independent variable R and the dependent variable W, reduces to

$$\frac{d^2W}{dR^2}+\frac{2}{R}\frac{dW}{dR}=-4\pi\rho'(1+W)^3\,, \tag{3.16}$$

which is exactly the same as (1.7) written in spherical polar coordinates, where W is dependent on R only. As mentioned in the introduction Bonnor[7] has obtained a physically meaningful explicit solution to this equation. Thus we find that Bonnor's solution is recoverable from the field equations (3.7)–(3.9) when the BD coupling parameter ω approaches infinity and the BD scalar $\phi\to$ const ($G^{-1}=1$). And in this extreme case ($\omega\to\infty$ and $\phi\to$ const) only the solution characterizing spherically symmetric static charged dust distribution exists.

IV. CONCLUSION

The static charged dust distribution considered here, as shown in the preceding sections, is a generalization of the situation considered by Das and De and Raychaudhuri in Einstein's theory. This is due to the finite coupling of the scalar ϕ with the charged matter. The nonexistence of any spherically symmetric solution of this generalized situation and the recovery of Bonnor's solution of Einstein's theory in the limit $\omega\to\infty$ and $\phi\to$ const, leads to the conclusion that the finite coupling of the ϕ field with the charged matter in accordance with the relations (1.2) and (1.3) disturbs the equilibrium of spherically symmetric charged dust distribution. This does not, however, rule out the possibility of static models, other than spherically symmetric, to be compatible with the class of solutions admitted by (1.2) and (1.3). We propose to examine these cases further before we arrive at any conclusion. It may be added, however, that by relaxing conditions (1.2) and (1.3), we have observed that the BD Maxwell-field equations admit static spherically symmetric solutions. This observation will be reported in another paper very shortly.

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