# Analysis of quantum-nondemolition measurement

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The quantum-nondemolition measurement process suggested by Braginsky and co-workers is analyzed and shown not to work because of the fundamentally linear coupling between the oscillator and the measuring apparatus. A second-order coupling would work but the effect is probably too small for experimental realization.

Recently Braginsky *et al.*[1-3] (referred to hereafter as B), in examining the ultimate limits to the sensitivity of gravitational radiation detection, proposed a scheme by which it would be possible to detect the energy of the electromagnetic modes in a resonant cavity without disturbing the energy of the system. The method was claimed to be applicable even to the ground state of the cavity.

Although the claim that it is in principle possible to measure the state of the resonator without disturbing its energy significantly will be shown to be true, the technique suggested in B is in principle unable to do so. The plan of this paper will be to first describe and then analyze the B proposal, and then to describe a technique in which it is possible to achieve the above aim.

## I. ANALYSIS OF FIRST-ORDER DETECTION

Since I will be referring to some details of the B scheme, I will first describe the essential features of that scheme.

An electromagnetic resonant cavity, idealized as an $L$-$C$ circuit, is postulated in which an electromagnetic field exists in some quantum state. The resonator is assumed to have only one mode, of frequency $\omega$, and the possible states therefore differ only in the number of quanta in this mode. The capacitor is assumed to be a split capacitor. An electron beam is fired between the plates of one of the capacitors and is then focused by an electron lens system, where the detectors are located. The beam which has not been stopped by the detectors is then focused through the second capacitor (Fig. 1).

The idea behind the scheme is that in passing through the first capacitor the electrons are given some vertical momentum by the field within the capacitor. The detectors are placed at the minima of the diffraction pattern produced by the finite width of the capacitor (or of a slit within the capacitor) when the field within the capacitor is zero. The presence of a field within the capacitor will shift the diffraction pattern, and the rate at which electrons are detected in the former minima will be proportional to the squared intensity of the field (Fig. 2).

The success of this design is based on two separate claims[4]:

(a) The system can be arranged so that the electrons which are not detected will have a negligible effect on the state of the system.

(b) Those electrons which are detected have a small probability of having disturbed the detector. The arguments presented in B to support the above claims are in general non-quantum-mechanical in nature (except for the occasional use of discreteness of energy levels, etc.). Although a quantum investigation supports assertion (a), it does not support (b). The arguments for the possibility of realizing part (a) given in B are essentially classical and will be reproduced here.

In going through the capacitor, the particle feels a force in the $y$ direction due to the field, designated with the abstract symbol $q$, such that the interaction is of the form

$$m\frac{d^2y}{dt^2}=\alpha q h(x)\,, \tag{1.1}$$

where $\alpha$ represents a coupling constant (dependent on the charge of the particle and on the relation between $q$ and the electric field). $h(x)$ here gives the $x$ dependence through the capacitor of the field. Equation (1.1) is assumed to be derivable from a Lagrangian of the form

$$\tfrac{1}{2}m\left[\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2\right]+\alpha y q h(x)+\frac{1}{2}\left(\frac{dq}{dt}\right)^2-\tfrac{1}{2}\omega^2q^2\,, \tag{1.2}$$

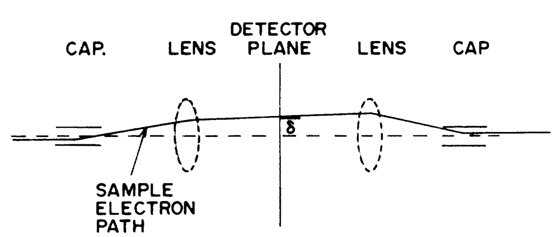


FIG. 1. The proposed Braginskii detection scheme.

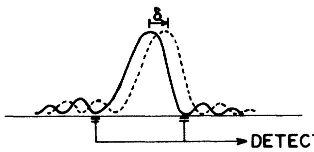


FIG. 2. Interference pattern in the detector plane.

where the relation between $q$ and $E$ is such that the Lagrangian for $q$ takes this form.

The equation for $q$ is then

$$\frac{d^2q}{dt^2} + \omega^2 q = \alpha y h(x) , \quad (1.3)$$

from which we obtain an equation for the rate of change of energy in the cavity,

$$\frac{dE}{dt} = \frac{1}{2}\frac{d}{dt}\left[\left(\frac{dq}{dt}\right)^2 + \omega^2 q^2\right] = \alpha \frac{dq}{dt} y h(x) \quad (1.4)$$

or

$$\Delta E = \int \alpha \frac{dq}{dt} y h(x)\, dt . \quad (1.5)$$

If $h(x)$ is effectively a $\delta$ function at $x=0$ and $x=L$ (of unit area), we obtain

$$\Delta E = \frac{dq(T)}{dt}\frac{y(T)}{v_x} + \frac{dq(0)}{dt}\frac{y(0)}{v_x} , \quad (1.6)$$

where $t=0$, $T$ are the times at which the electron passes through the two capacitors, and $v_x$ is the velocity in the $x$ direction.

If $T$ is chosen to be a multiple of the period of the resonator, and the lens system is designed so that $y(T) = -y(0)$, then the above expression for $\Delta E$ is zero (as the resonator is in free oscillation between $T=0$ and $t=T$, the "velocity" $dq/dt$ will be periodic).

The above is a classical argument. Will the conclusion be valid if the whole system is treated quantum mechanically? As the above system with its lenses, etc., and its two-dimensional nature is difficult to analyze, a much simpler one-dimensional system can be studied, which will display the essential features. A two-dimensional system will then be studied to show that there are no subtleties there which would invalidate the argument.

Consider the Lagrangian

$$\tfrac{1}{2}\dot{x}^2 + \alpha q \theta(x)\theta(L-x) + \tfrac{1}{2}(\dot{q}^2 - \omega^2 q^2) , \quad (1.7)$$

where $\theta(x)$ is the unit step function which is 0 for $x < 0$. This system can be quantized and leads to a Schrödinger equation describing its quantum behavior. If we expand the wave function in terms of the states $\phi_j(q,t)$ of the free harmonic oscillator, we obtain a wave function of the form

$$\sum_j \psi_j(x,t)\phi_j(q,t) , \quad (1.8)$$

where the particle functions $\psi_j$ obey the equation

$$i\frac{\partial\psi_j}{\partial t} = -\frac{1}{2}\frac{\partial^2\psi_j}{\partial x^2} - \sum_l \alpha q_{jl}(t)\theta(x)\theta(L-x)\psi_l , \quad (1.9)$$

where we define

$$q_{jl}(t) = \int \phi_j^*(q,t) q \phi_l(q,t)\, dq$$

$$= \begin{cases} \left(\dfrac{n+1}{\omega}\right)^{1/2} e^{i\omega t} , & j = l+1 \\ \left(\dfrac{n}{\omega}\right)^{1/2} e^{-i\omega t} , & j = l-1 \\ 0 , & j \neq l \pm 1 . \end{cases} \quad (1.10)$$

Expand the $\psi_j$ in a perturbation series in $\alpha$, such that $\psi_j = \sum_r \psi_j^{(r)}$ where $\psi^{(r)}$ is of order $\alpha^r$. Assume that the only nonzero zeroth-order term is for $j = n$, with

$$\psi_n^{(0)}(x,t) = e^{-i(k^2/2)t}\frac{e^{ikx}}{\sqrt{2\pi}} . \quad (1.11)$$

The initial state therefore has the oscillator in the $n$th state, and the particle has velocity $k$ toward the oscillator and originates from $x = -\infty$.

The only nonzero first-order terms are $\psi_{n\pm 1}^{(1)}$ as the interaction through $q$ connects only these states to $\psi_n^{(0)}$ to first order in $\alpha$. The Green's function solutions for these are given by

$$\psi_{n\pm 1}^{(1)}(x,t) = \frac{e^{-i(\kappa^2/2)t}}{(2\pi)^{3/2} i} 2\alpha\langle q_n^{\pm}\rangle \int \frac{e^{ik'x}}{k'^2 - (\kappa^2 + i0)}\left[\int_0^L e^{i(k-k')x'} dx\right] dk'$$

$$= \frac{e^{-i(\kappa^2/2)t}\alpha\langle q_n^{\pm}\rangle}{(2\pi)^{3/2} i}\int \frac{e^{ik'x}(e^{i(k-k')L} - 1)}{[k'^2 - (\kappa^2 + i0)][i(k-k')]} dk' , \quad (1.12)$$

where

$$\langle q_n^\pm\rangle = \begin{cases} \left(\dfrac{n+1}{\omega}\right)^{1/2}, \\ \left(\dfrac{n}{\omega}\right)^{1/2}, \end{cases}$$
$$\kappa = +(k^2 \pm 2\omega)^{1/2} \tag{1.13}$$

For $x > L$, this becomes

$$\psi_{n\pm1}^{(1)}(x > L, t) = \frac{e^{-i\kappa^2 t/2}\alpha\langle q_n^\pm\rangle e^{i\kappa x}}{\sqrt{2\pi}\, i\kappa(k-\kappa)}\left(e^{i(k-\kappa)L} - 1\right), \tag{1.14}$$

while for $x < 0$

$$\psi_{n\pm1}^{(1)}(x < 0, t) = \frac{e^{-i\kappa^2 t/2}\alpha\langle q_n^\pm\rangle e^{-i\kappa x}}{\sqrt{2\pi}\, i\kappa(k+\kappa)}\left(e^{i(k+\kappa)L} - 1\right). \tag{1.15}$$

The former represents the wave exiting from the far side of the interaction region while the latter represents reflection by the region. If the velocity $k$ is chosen so that $(k-\kappa)L \approx L\omega/k = \omega T$ is an integral multiple of $2\pi$, the numerator in Eq. (1.14) will then be zero.

For the reflected wave (1.15) no such simple condition applies, as the condition that $(k+\kappa)L$ be a multiple of $2\pi$ is almost impossible to realize in practice for large $k$. However, this reflected term is very small when compared with the transmitted wave anyway [i.e., is down by a factor of order $\omega/(\kappa^2/2)$ which is assumed to be small].

Classically, the oscillator obeys the equation

$$\frac{d^2q}{dt^2} + \omega^2 q = \alpha\theta(x)\theta(L-x) \tag{1.16}$$

and

$$\frac{dE}{dt} = \dot{q}\alpha\theta(x)\theta(L-x).$$

Therefore,

$$\Delta E = \int_0^T \alpha\dot{q}\,dt = \alpha[q(t) - q(0)]. \tag{1.17}$$

When the particle is within the oscillator (i.e., $0 < x < L$) the equilibrium position of the oscillator is shifted to the point $q = \alpha$, but during this time $q$ is still a periodic function with period $2\pi/\omega$. Therefore, if $T$ is chosen to be a multiple of this period, the change in the oscillator energy caused by the particle's traversing the oscillator is zero. The classical condition for the oscillator to remain in its state after passage of the particle is the same as the quantum condition—namely, that $\omega T$ be an integral multiple of $2\pi$. There is no classical analog to the reflected wave.

This result gives us confidence that a quantum analysis of the setup in B would duplicate the classical insofaras satisfying the requirement (a) to reasonable precision. (Even classically perfect compensation is not possible as the length of the particle's path and the velocity of the particle depend in detail on the unknown value of the field as the particle passes the first capacitor.)

This leaves part (b) to worry about, and the above example demonstrates the problem. To first order in $\alpha$, one has either a wave which has not been affected by the oscillator at all ($\psi_n^{(0)}$) or one has a wave ($\psi_{n\pm1}^{(1)}$) which has been affected, but which has in the process changed the state of the oscillator with certainty [i.e., there are no first-order terms of the form $\psi_n^{(1)}(x,t)$]. Any experimental technique must be designed so as to reject those particles which would have been there even if the interaction had never taken place, and these are exactly the particles represented the terms $\psi_n^{(0)}$. If this rejection is perfect (e.g., by looking in the minima of the diffraction pattern of the unaffected particles), the only particles one can detect are those which have with certainty affected the state of the oscillator. Therefore, although it is true that a particle which passes only through the left capacitor has a low probability of exciting the capacitor, those which are *detected* have a probability of nearly unity of affecting the state of the oscillator. As the first-order terms will in general dominate those particles which are detected, when compared with higher-order terms, the probability of altering the state by the measurement process is very high.

The reason for this behavior lies in the property of the harmonic oscillator that the expectation value of the coordinate $q$ in any energy eigenstate is zero. Another way of looking at the problem is to write $q$ in terms of annihilation and creation operators for quanta of the oscillator

$$q = \frac{ae^{-i\omega t}}{\sqrt{\omega}} + \frac{a^\dagger e^{i\omega t}}{\sqrt{\omega}}. \tag{1.18}$$

Any interaction that proceeds via $q$ must do so by annihilating or creating a quantum in the oscillator. Since the two processes will be about equal *a priori*, the expectation value for the energy may change little, but the fluctuations in the energy will be increased. This accounts for the classical arguments which would imply very little change in the average energy.

The essential result obtained from the analysis so far is that for the one-dimensional scheme proposed, the nondemolition detection will not work because the interaction is first order in the coordinate $q$ of the oscillator. The $\vec{B}$ scheme proposed, however, is a two-dimensional scheme. In one dimension the state of the detecting particle can change only when accompanied by a change in its energy. In two dimensions the direction of the beam can change without necessarily being accompanied by a change in energy. One might expect that the electron could bounce off of the electric field without changing the energy of the oscillator, in analogy with recoilless scattering in solid state physics.

Furthermore, the above analysis has examined only the first-order terms in the perturbation series. In the two-dimensional case, it may be that the higher-order terms are responsible for the "recoilless scattering" and their effect will dominate the first-order terms.

## II. ANALYSIS OF TWO-DIMENSIONAL SCHEME

To show that this does not happen, I will now analyze a two-dimensional model of the experiment proposed by Braginsky.

The model is defined by the Lagrangian

$$\tfrac{1}{2} m(\dot{x}^2+\dot{y}^2)+\tfrac{1}{2}(\dot{q}^2-\omega^2 q^2)-\epsilon q y h(x)\,, \tag{2.1}$$

where $h(x)$ will be taken to be given by

$$h(x)=\begin{cases}1, & |x|<L\\ 0, & |x|>L\,.\end{cases} \tag{2.2}$$

To connect with the actual system proposed by Braginsky,

$$\epsilon=\frac{1}{2}\left(\frac{e^2}{C}\right)^{1/2}\frac{\omega}{d}\,,\qquad q=\sqrt{C}\,V/\omega\,, \tag{2.3}$$

where $C$ is the capacitance (0.3 pF), $\omega$ is the angular frequency of the oscillator ($2\times10^{10}$/sec), $2d$ is the separation of the plates ($10^{-3}$ m), $V$ is the voltage across the capacitor, and $m$ is the mass of the electron. In the analytic expressions, $m$ and $\hbar$ will be taken to be unity and will be reinserted into expressions when necessary for numerical evaluation.

The above model seems to me to be a reasonable model for the actual physical setup. There are a number of terms neglected here. The electric field inside the capacitor is assumed to be "rigid", i.e., the shape of the modes between the plates of the capacitor is assumed to be independent of the position of the electron going through the capacitor. The force on the electron due to its own image charges is neglected. (These would be independent of the state of the field in the cavity anyway, and would thus not affect the measurability of that field.) The mode structure within the cavity is assumed to have the form $h(x)y$, which neglects the effects of fringing fields at the edges of the capacitor. Using a more complicated expression for the shape of the mode would not alter the conclusions reached here, but would complicate the mathematics. All relativistic effects are neglected, including the effects of any magnetic fields within the capacitor.

A quantity which operates as an effective coupling constant is

$$\alpha=\left(\frac{e^2}{C\hbar\omega}\right)^{1/2}\approx 0.2\,. \tag{2.4}$$

The second-order terms will be found to be of order $\alpha^2$ less than the first-order terms. This quantity must be kept less than unity for a perturbation analysis to apply at all. For larger values, all orders in a perturbation expansion would become equally likely, *a priori*, and would almost certainly not result in a "nondemolition" measurement.

I will assume that the incident momentum of the electron is $p$ with a spread $\delta p$ (such that $\delta p/p \sim 10^{-3}$) and such that $p^2 \gg 2\omega$ (i.e., the energy of the electron is much greater than the energy of a quantum of the oscillator).

The wave function for the electron oscillator system is written as

$$\sum_j \Psi_j(x,y,t)\,\phi_j(q,t)\,, \tag{2.5}$$

where $\phi_j$ is the $j$-quanta eigenstate of the free oscillator. The electron wave functions obey

$$\left[i\frac{\partial}{\partial t}+\frac{1}{2}\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)\Psi_j\right]=\epsilon y h(x)\sum_l q_{jl}\Psi_l e^{i(j-l)\omega t}\,,$$

$$q_{jl}=\left(\frac{l}{\omega}\right)^{1/2}\delta_{j+1,l}+\left(\frac{l+1}{\omega}\right)^{1/2}\delta_{j-1,l}\,. \tag{2.6}$$

The solutions for $\Psi_j$ can be expanded in a perturbation series in $\epsilon$,

$$\Psi_j=\sum \epsilon^r \Psi_j^{(r)}\,. \tag{2.7}$$

I will be interested in the Fourier transform of $\Psi_j$ in the region $x>L$. The Fourier transform of $\Psi_j^{(r)}$ in this region will be denoted by $\Psi_j^{(r)}$, where

$$\Psi_j^{(r)}(k_1,k_2)=\int\Psi_j^{(r)}(x,y)\,\frac{e^{-i(k_1x+k_2y)}}{2\pi}\times e^{+iEt}\,dx\,dy\,,\qquad E=\tfrac{1}{2}(k_1{}^2+k_2{}^2) \tag{2.8}$$

[the integral is defined for $x<L$ by taking the analytic extension of $\psi_j^{(r)}(x,y)$ for values of $x>L$ to values of $x<L$].

The zeroth-order solution for the electron wave function is taken to be

$$\Psi_j^{(0)} = \delta_{nj}\int \psi^{(0)}(k_1,k_2)\,\frac{e^{-iEt}e^{i(k_1x+k_2y)}}{2\pi}\,dk_1dk_2\,. \tag{2.9}$$

In order to mimic the effect of the finite width of the capacitor, $\psi^{(0)}$ is taken to be given by

$$\psi^{(0)}(k_1,k_2) = S(k_1)\,\frac{\sin k_2 d}{\sqrt{\pi d}\,k_2}\,, \tag{2.10}$$

where $S(k_1)$ is some unit norm function strongly peaked at $k_1 \simeq p$ and of width $\delta p$ (e.g., a normalized Gaussian).

The detection scheme envisaged is to measure the number of electrons which are scattered into the first minimum of the diffraction pattern. (The lens serves to focus electrons with the same wave number in the $y$ direction onto a single spot on the detector plane.) I will in the following therefore calculate the probability of an electron scattering with $y$-wave-number $\pi/d$ (i.e., into the first minimum of the unperturbed diffraction pattern).

The terms which will interest me are the lowest-order terms which correspond to scattering into that direction by having altered the state of the oscillator and the lowest-order terms which have not altered the state of the oscillator. The probability densities for these two processes are, respectively,

$$P^{(1)} = \sum_{+,-} \epsilon^2 \int |\psi_\pm^{(1)}(k_1,\pi/d)|^2 dk_1\,,$$
$$P^{(2)} = \epsilon^4 \int |\psi_n^{(2)}(k_1,\pi/d)|^2 dk_1\,, \tag{2.11}$$

where $\pm$ stands for $n\pm 1$. [The lower-order term $2\,\mathrm{Re}(\psi_n^{(2)}\psi^{(0)*})$, for nondisturbative scattering is zero at the $y$-wave number of interest since $\psi^{(0)}$ is zero there.]

The first-order amplitudes $\psi_\pm^{(1)}$ are given by the equation

$$\psi_\pm^{(1)}(\mu,\lambda) = \int dk_1dk_2 \left[\psi^{(0)}(k_1,k_2)q_\pm\,\delta'(k_2-\lambda) \times \frac{-\sin(k_\pm-k_1)L}{k_\pm(k_\pm-k_1)}\,\delta(\mu-k_\pm)\right], \tag{2.12}$$

$$k_\pm = (k_1^2+k_2^2-\lambda^2 \mp 2\omega)^{1/2}\,.$$

After some manipulation this becomes

$$\psi_\pm^{(1)}(\mu,\lambda) = \left(\frac{\partial}{\partial\lambda} + \frac{\lambda}{\bar{k}}\,\frac{\partial}{\partial\bar{k}}\right)\left(q_\pm\,\psi^{(0)}(\bar{k},\lambda)\,\frac{\sin(\mu-\bar{k})L}{\bar{k}(\mu-\bar{k})}\right)\,,$$
$$\bar{k} = (\mu^2 \mp 2\omega)^{1/2}\,. \tag{2.13}$$

The only term which contributes to the scattering into $\lambda = \pi/d$ is the term proportional to $\partial\psi^{(0)}/\partial\lambda$. One finally obtains

$$P^{(1)} \simeq \frac{d^3\epsilon^2(2n+1)}{\pi^3\omega^3}\,\sin^2(\omega L/p)\,. \tag{2.14}$$

The argument of the sin can be rewritten as $\pi T/\tau$ where $T$ is the transit time through the capacitor and $\tau$ is the period. If $T$ is chosen so as to minimize $P^{(1)}$ (i.e., $T=\tau$), the spread $\delta p$ in $p$ also becomes important and one obtains

$$P^{(1)} \simeq \frac{d^3\epsilon^2(2n+1)}{\pi\omega^3}\left(\frac{\delta p}{p}\right)^2\,. \tag{2.15}$$

In the scheme envisaged by Braginsky, however, $T \sim \frac{1}{2}\tau$, and one obtains

$$P^{(1)} \simeq \frac{d^3\epsilon^2(2n+1)}{\pi^3\omega^3}\,. \tag{2.16}$$

One must now calculate the second-order term $\psi_n^{(2)}$ in order to estimate the probability $P^{(2)}$ of an electron scattering into $y$-wave number $\pi/d$ without altering the state of the oscillator. Only if $P^{(2)} \gg P^{(1)}$ can one be said to have performed a quantum nondemolition measurement. If $P^{(1)} \gg P^{(2)}$, any electron detected will have almost certainly altered the state of the oscillator. (In calculating $\psi_n^{(2)}$ only those terms which depend on $n$ are of importance as the other terms cannot give any information as to the state of the oscillator.) The full expression for the second-order term is

$$\psi_n^{(2)}(\nu,\sigma) = \sum_{+,-} \frac{-4i\,|q_\pm|^2}{\pi}\int dk_1dk_2\,d\lambda\,d\mu \times \left\{\psi^{(0)}(k_1,k_2)\,\frac{\delta'(k_2-\lambda)\delta'(\lambda-\sigma)\sin(\mu-k_1)L\sin(\mu-\tilde{k})L}{(\mu^2-k_\pm^2-i0)(\mu-k_1)(\mu-\tilde{k})\tilde{k}}\,\delta(\nu-\tilde{k})\right\}, \tag{2.17}$$

$$k_\pm = (k_1^2+k_2^2-\lambda^2\mp 2\omega)^{1/2}\,,\quad \tilde{k} = (k_1^2+k_2^2-\sigma^2)^{1/2}\,.$$

Because $\psi^{(0)}(\nu,\pi/d)=0$, only those terms in the above expression which depend on derivatives of $\psi^{(0)}$ with respect to its second argument will contribute to $P^{(2)}$. The terms proportional to $\partial\psi^{(0)}/\partial\lambda$ in general have an amplitude which is smaller than the amplitude of $\psi_{\pm}^{(1)}$ by a factor of the order of

$$\epsilon\left(\frac{2n+1}{\omega}\right)^{1/2}\frac{\sigma L}{p\omega}\simeq\left(\frac{e^2}{C\hbar\omega}\right)^{1/2}\frac{\hbar T}{md^2}(2n+1)^{1/2}$$
$$\simeq 10^{-7} \tag{2.18}$$

These cannot significantly contribute to $P^{(2)}$. (Except for one term, they all have a similar dependence on $L$ as do $\psi_{\pm}^{(1)}$. Therefore chosing $L$ so as to minimize $P_{(1)}$ will also minimize these terms.)

The only significant term is therefore the term

$$\psi_n^{(2)}\simeq-4i\,\frac{(2n+1)}{\omega}\left(\frac{\partial^2}{\partial\sigma^2}\,\psi^0(\nu,\sigma)\right)\frac{(\sin\omega L/\nu)^2}{2\omega^2}\,. \tag{2.19}$$

Using the assumed form of $\psi^{(0)}$ this term gives a contribution to $P^{(2)}$ of

$$P^{(2)}\simeq\frac{\epsilon^4(2n+1)^2d^5}{\omega^6\pi^5}\,[\sin(\omega L/\nu)]^2 \tag{2.20}$$

(note that choosing $L$ so as to minimize $P^{(1)}$ would also minimize $P^{(2)}$). The ratio of $P^{(2)}$ to $P^{(1)}$ is given by

$$\frac{P^{(2)}}{P^{(1)}}\simeq\frac{\epsilon^2(2n+1)d^2}{\pi^2\omega^3}$$
$$\simeq\frac{e^2}{C\hbar\omega}\,\frac{2n+1}{\pi^2}$$
$$\simeq 1\times10^{-3}(2n+1)\ll 1 \text{ for low } n\,. \tag{2.21}$$

This implies that only of the order of 0.1% of the electrons detected at the diffraction minima will be electrons which have not altered the state of the oscillator.

The quantum-mechanical analysis presented above is confusing when one attempts to think classically. After all no energy is needed to alter the direction of the electron. A rather more qualitative argument can show how this effect arises quantum mechanically.

Let us write the beam after it passes through the first capacitor (with zero field in the capacitor) in the form

$$e^{-i(k^2/2)t}\int H(\lambda)e^{i\kappa x}e^{i\lambda y}d\lambda\,, \tag{2.22}$$

where

$$\kappa^2=k^2-\lambda^2$$

and

$$\int H(\lambda)e^{i\lambda y}d\lambda=\theta(y+a)\theta(a-y)$$

(i.e., the above represents the a wave of energy $k^2/2$ diffracted through a slit of width $2a$ in the $y$ direction). The effect of an electric field in the slit is to lowest order to change value of the $\lambda$ at which the maximum occurs. With a time varying field, the resultant wave becomes something like

$$e^{-i(k^2/2)t}\int H(\lambda-\alpha\cos\omega t)e^{i\kappa x}e^{i\lambda y}d\lambda\,,$$

which represents an oscillating diffraction pattern. However, expanding $H(\lambda-\alpha\cos\omega t)$ in powers of $\alpha$, the above becomes

$$e^{-i(k^2/2)t}\int H(\lambda)e^{i\kappa x}e^{i\lambda y}d\lambda$$
$$+\alpha e^{-i(k^2/2)t}\cos\omega t\int\frac{dH(\lambda)}{d\lambda}\,e^{i\kappa x}e^{i\lambda y}\,d\lambda\,. \tag{2.23}$$

The first term is the diffraction pattern of the undisturbed wave, while the second represents to the lowest order the oscillating pattern. Note that it is made of particles not with energy $k^2/2$ but of $(k^2/2\pm\omega)$. The particle must have absorbed or given up an energy of $\omega$ to produce an oscillating diffraction pattern. [Note that at $x=0$, the $y$ dependence of the first-order term is

$$\alpha\int\frac{dH}{d\lambda}\,e^{i\lambda y}\,d\lambda=-\alpha iy\int H(\lambda)e^{i\lambda y}d\lambda$$
$$=-\alpha iy\theta(y+a)\theta(a-y)\,, \tag{2.24}$$

which is just the type of term expected for the form of first-order perturbation at the capacitor with an interaction of the form $\alpha\delta(x)yq$.]

The problem arises because the interaction is of first order in $q$. If these first-order effects could be arranged to all cancel out there could be a hope for perturbing the particle without also perturbing the energy of the oscillator. Alternatively, it would be much simpler to design the experiment so that all first-order effects are automatically zero—namely, by making the interaction second-order in $q$. An interaction term of the form $q^2$ can be written in terms of annihilation and creation operators as

$$q^2=\frac{1}{\omega}\,[a^\dagger a+aa^\dagger+a^2e^{-2i\omega t}+(a^\dagger)^2e^{2i\omega t}]\,. \tag{2.25}$$

The first two terms represent an interaction which can measurably alter the state of the particle while leaving the energy of the oscillator unchanged. If the experiment can be designed so that the time-dependent terms have very little effect, one will succeed in measuring the state of the oscillator without altering its energy.

## III. SECOND-ORDER DETECTION

The construction of an example of such a system is straightforward. Consider the Lagrangian

$$L=\tfrac{1}{2}(\dot{x})^2+\alpha q^2 f(x)+\tfrac{1}{2}(\dot{q}^2-\omega^2 q^2)\,, \tag{3.1}$$

where $q$ represents the abstract coordinate of the oscillator, $\alpha$ is a coupling constant, and $f(x)$ gives the spatial dependance of the interaction of the particle with the oscillator. A physical realization of the above system would be obtained by letting the oscillator be an electromagnetic mode in a cavity, where $q$ will be related to the maximum electric field in the cavity, and $q^2 f(x)$ is the squared electric field along the path of the particle. The coupling constant $\alpha$ will be related to the polarizability of the particle in question. Such a specific physical realization is not necessary for the matter of principle being discussed, however.

For simplicity I will assume in any specific calculations that $f(x)$ is of the form

$$f(x)=\left(\frac{2}{L}\,x\theta(x)\theta(L/2-x)\right.\\ \left.+(L-x)\theta(L-x)\theta(x-L/2)\right), \tag{3.2}$$

i.e., $f(x)$ has a triangular shape. (Classically this gives a force on the particle which is constant in the regions $0<x<L/2$ and $L/2<x<L$.)

The quantization of this system is straightforward. If we expand the wave function for the combined particle-oscillator system in terms of the wave functions for the free oscillator with frequency $\omega$ we obtain a wave function of the form

$$\sum_j \psi_j(x,t)\phi_j(q,t)\,, \tag{3.3}$$

where $\phi_j(q,t)$ is the normalized wave function for the $j$th state of the free harmonic oscillator. The wave equation for the system can now be written as

$$\frac{i\partial}{\partial t}\psi_j=-\frac{1}{2}\frac{\partial^2}{\partial x^2}\psi_j-\sum_l \alpha q^2{}_{jl}(t)f(x)\psi_l(x,t)\,, \tag{3.4}$$

where I define

$$q^2{}_{jl}(t)=\int\phi_j^*(q,t)q^2\phi_l(q,t)dq\,. \tag{3.5}$$

Let us define the initial conditions so that the oscillator is in the pure state $j=n$, and the particle is coming into the interaction region with velocity $k$ from $x=-\infty$. The boundary conditions are therefore that near $x=-\infty$, only $\psi_n(x,t)$ has an ingoing part which I will assume is of the form

$$[\psi_n(x,t)]_{\rm in}=e^{-i(k^2/2)t}\,\frac{e^{ikx}}{\sqrt{2\pi}}\,. \tag{3.6}$$

The equation for $\psi_n$ is

$$+\left(i\frac{\partial}{\partial t}+\frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\psi_n\\ =-\alpha\left[q^2{}_{nn}f(x)\psi_n+\sum_{j\neq n}q^2{}_{nj}(t)f(x)\psi_j\right]. \tag{3.7}$$

To lowest order in $\alpha$ we can neglect the second set of terms on the right-hand side. As $q^2{}_{nn}$ is independent of $t$, the problem reduces in this order to the one-dimensional scattering problem for $\chi_n=e^{ik^2t/2}\psi_n$. One obtains

$$\frac{1}{2}\frac{\partial^2}{\partial x^2}\chi_n=-[k^2/2+\alpha q^2{}_{nn}f(x)]\chi_n\,. \tag{3.8}$$

If the initial velocity is large enough, this can be solved by a WKB approximation to give (for $x>L$)

$$\psi_n=\frac{A}{\sqrt{2\pi}}\,e^{-ik^2t/2}e^{i(kx+\delta)}\,, \tag{3.9}$$

where $A$ is approximately one, and

$$\begin{aligned}\delta&=\int_{-\infty}^{\infty}\{[k^2+2\alpha q^2{}_{nn}f(x)]^{1/2}-k\}dx\\ &\approx\int_{-\infty}^{\infty}\frac{q^2{}_{nn}f(x)dx}{k}\\ &=\frac{\alpha}{k}q^2{}_{nn}\,\frac{L}{2}\\ &=\tfrac{1}{2}\alpha q^2{}_{nn}T\,.\end{aligned} \tag{3.10}$$

Here $T$ is the time for the particle to traverse the distance $L$ in which the interaction occurs. Such a phase shift is measurable by interferometry techniques. Subtracting from $\psi_n$ the form it would have had if the interaction had been zero, namely, $\psi_n^{(0)}=e^{-ik^2t/2}e^{ikx}/\sqrt{2\pi}$, we obtain

$$(e^{i\delta}-1)\psi_n^{(0)} \tag{3.11}$$

in the interference region.

The flux of incident particles represented by $\psi_n$ is $k/2\pi$ while the flux of particles in the interference region is

$$|e^{i\delta}-1|^2k/2\pi\,. \tag{3.12}$$

The ratio of particles detected in the interference region to the number of particles sent through the interaction region is the ratio of these fluxes or

$$\begin{aligned}\frac{\langle N_d\rangle}{N}&=|e^{i\delta}-1|^2\\ &\approx\delta^2\approx\frac{\alpha^2(q^2{}_{nn})^2}{2}\,T^2=\frac{\alpha^2n^2T^2}{2\omega^2}\,,\end{aligned} \tag{3.13}$$

where the value of $q^2{}_{nn}$, i.e., $(n/\omega)$, has been used.

In order to determine the state (i.e., $n$) accurately, the error in $n$ introduced by the discrete

Poisson nature of the detection process should be less than $\frac{1}{2}$:

$$\tfrac{1}{2} \gtrsim \delta n \approx \frac{\delta N_d}{2\langle N_d\rangle} n \approx \frac{n}{2\sqrt{N_d}} \tag{3.14a}$$

or

$$\langle N_d\rangle \gtrsim n^2 . \tag{3.14b}$$

The total number of particles which must be sent through the oscillator should therefore obey

$$N \gtrsim \frac{2\omega^2}{\alpha^2 T^2} \tag{3.15}$$

(note that $N$ is not dependent on which state the oscillator is in). This experiment differs from that of Braginsky *et al.* in that the detection of one of the particles $N_d$ does not imply that the oscillator has changed state.

The above analysis has been concerned only with those particles for which the state of the oscillator remains unchanged in the $n$th state. One must now show that the total probability that the state is changed by sending through $N$ particles is very small, and that the flux of particles which have changed the state of the oscillator and which enter the interference region is small. (If most of the flux in the interference region were due to particles which had changed the state of the detector, then the conditional probability that the detector had changed state given that a particle had been detected in the interference region would be very large.)

To demonstrate that such a situation is possible, I will calculate the form of the functions with $j \neq n$ to first order in $\alpha$. To lowest order in $\alpha$, the equation for $\psi_j$ is

$$\left(\frac{i\partial}{\partial t} + \frac{1}{2}\frac{\partial^2}{\partial x^2}\right)\psi_j = -\alpha q^2{}_{jn}(t) f(x)\psi_n(x,t)$$
$$\approx -\alpha q^2{}_{jn}(t) f(x) e^{-(k^2/2)t}\frac{e^{ikx}}{\sqrt{2\pi}} ; \tag{3.16}$$

$q^2{}_{jn}$ is nonzero only for $j = n\pm 2$, and for these terms it takes the form

$$q^2{}_{jn}(t) = \begin{cases} e^{i2\omega t}\dfrac{[(n+1)(n+2)]^{1/2}}{\omega}, & j=n+2 \\ e^{-i2\omega t}\dfrac{[n(n-1)]^{1/2}}{\omega}, & j=n-2 . \end{cases} \tag{3.17}$$

For order of magnitude estimates the square roots will be replaced by their approximate value $n$.

These equations for $\psi_{n\pm 2}$ may be solved by Green's function techniques. If we define for $j=n\pm 2$

$$\begin{aligned} &\epsilon = \pm 2\omega, \quad \kappa = (k^2 \pm 4\omega)^{1/2} \\ &A = \frac{2\alpha\langle q^2{}_{jn}\rangle(t)e^{-ik^2/2)t}}{i} \\ &\approx 2\alpha n e^{-i(\kappa^2/2)t}/(i\omega). \end{aligned} \tag{3.18}$$

We can write the solutions for either of $\psi_{n\pm 2}$ as

$$\psi_j = \frac{A}{(2\pi)^{3/2}}\int \frac{dk'\, e^{ik'x}}{k'^2-(\kappa^2+i0)}\left[\int_{-\infty}^{\infty} f(x')e^{i(k-k')x'}\,dx'\right]$$
$$= + \frac{A}{(2\pi)^{3/2}}\int \frac{dk'\, e^{ik'x}}{k'^2-(\kappa^2+i0)}\,\frac{L}{2}\left(\frac{e^{i(k-k')L/2}-1}{(k-k')L/2}\right)^2$$
$$= \begin{cases} \dfrac{Ai}{(2\pi)^{1/2}}\,\dfrac{e^{i\kappa x}L}{4\kappa}\left(\dfrac{e^{i(k-\kappa)L/2}-1}{(k-\kappa)(L/2)}\right)^2, & x>L \\ \dfrac{Ai}{(2\pi)^{1/2}}\,\dfrac{e^{-i\kappa x}L}{4\kappa}\left(\dfrac{e^{i(k+\kappa)L/2}-1}{(k+\kappa)L/2}\right)^2, & x<0. \end{cases} \tag{3.19}$$

The total flux in each of these modes leaving the oscillator divided by the incident flux is thus

$$F_j = \frac{\alpha n}{\kappa\omega}\,\frac{L}{2}^2\left|\frac{e^{i(k-\kappa)L/2}-1}{(k-\kappa)L/2}\right|^4 + \left(\frac{e^{i(k+\kappa)L/2}-1}{(k+\kappa)L/2}\right)^4$$
$$\approx T^2\frac{\alpha^2 n^2}{\omega^2}\left(\frac{e^{i(k-\kappa)L/2}}{(k-\kappa)L/2}\right)^4 + \left(\frac{1}{kL}\right)^4 , \tag{3.20}$$

where I have used $\kappa \approx k$ and have assumed that over the spread $\delta k$ of a wave packet the exponential minus one in the second term averages out to of order unity. The second term represents particles reflected from the oscillator region and one thus need not worry about detecting them in the interference region, but only need consider whether the total probability of transition to an $n\pm 2$ quantum state be small for $N$ particles passing through the oscillator. As $N \approx 2\omega^2/\alpha^2T^2$, the probability of transition to the $n\pm 2$ state due to the second term in Eq. (2.20) is of order $n^2(kL)^{-4}$. For large enough $k$ or $L$, this can be made as small as desired. Note, however, that for given $k$, the probability of excitation increases with increasing $n$. The first term is more interesting.

We can rewrite

$$k-\kappa = \omega/(k+\kappa) \approx 2\omega/k . \tag{3.21}$$

The probability that the oscillator will be excited by some one of those of $N$ particle which goes through the oscillator is then given by

$$n^2\left(\left|\frac{e^{i(k-\kappa)L/2}-1}{\omega T}\right|\right)^4 . \tag{3.22}$$

(This also is the probability that one of the $N_d$ particles detected is a particle which has disturbed the oscillator. The two needed conditions are therefore equivalent.) By making $\omega T$ very large (i.e., by making $L$ very large) this term can be made as small as desired. However, there is a more efficient procedure. If we let $(k-\kappa)L/2 \approx \omega T$ be an integer multiple of $2\pi$, then the term in the numerator goes to zero. This can strictly be true only for one value of $k$ since $T$ depends inversely on $k$. However, if $k$ differs from the required value $k_0$ by $\delta k$, then

$$(k-\kappa)L/T \approx \omega T_0(1-\delta k/k)\,, \tag{3.23}$$

and the probability of exciting the oscillator becomes

$$P_j \approx n^2\left[\left(\frac{\delta k}{k}\right)^4 + \left(\frac{1}{kL}\right)^4\right]. \tag{3.24}$$

(Because $k-\kappa$ differs for the transitions to $n+2$ and $n-2$, one can never choose $L$ and $k$ so as to satisfy the required relation for both transitions. One can take this into account by saying that $\delta k/k$ is at least $2\omega/k^2$. One can therefore make the probability of transition extremely small.)

Furthermore, the exact expression for the transition probability depends on the form assumed for $f(x)$. If we define a function $f_\mu(x)$ such that

$$\frac{d^\mu f_\mu(x)}{dx^\mu} = \left(\frac{2}{L}\right)^{\mu-1}\left[\sum_{r=0}^{\mu}(-1)^r\binom{r}{\mu}\delta(x-rL/2)\right], \tag{3.25}$$

the power 4 which occurs in the various transition probabilities is replaced by $2\mu$. The phase shift $\delta$ is, however, left unchanged. By choosing $\mu$ sufficiently large, the probabilities can be further suppressed.

This behavior can be clarified by examining the classical equation for $q$

$$\ddot{q} + \omega^2 q = \alpha f(x) q\,. \tag{3.26}$$

The particle therefore acts on the oscillator by changing the frequency of the oscillator. Since $f(x)\approx f(t/v)$, by designing $f$ so that the effective oscillator frequency $(\omega^2-\alpha f)^{1/2}$ changes slowly with time, the state of the quantum oscillator will adiabatically track the frequency shift rather than jumping to another state. By either increasing $L$ for a fixed $k$, or by increasing $\mu$ for a fixed $k$ and $L$, one brings the system nearer this adiabatic limit.

This suggests other possible techniques for measuring the state of an electromagnetic mode in a cavity. For example, a small needle conductor can be introduced into the cavity on a torsion fiber such that the natural frequency $\Omega$ of this needle is much less than the frequency of the cavity. The dipole moment induced in this needle will couple the needle to the square of the electric field. The interaction will of the form $\alpha\theta^2 q^2$ (for small angles $\theta$ from the direction of the $E$ field). This represents a frequency-frequency coupling in that the presence of energy in either system alters the frequency of the other. Since the needle's period is much longer than the field's, the field will respond quasiadiabatically to the motion of the needle, while the needle will respond to the average energy in the cavity. If the frequency of the needle can be monitored, it will give an indication of the energy in the electromagnetic field.

A useful physical realization of the above schemes is extremely difficult, especially because of the second-order nature of the interactions necessary. Also, if effective interference experiments were to be carried out, one would like relatively long coherence lengths for the particles being used. This suggests the use of photons (which are also useful because of their zero mass and of the constancy of their velocity). Unfortunately, the direct photon-photon coupling in vacuum is *extremely* weak. However, if a material could be placed within the cavity whose optical properties depended on the square of the applied field, it could be used to provide a coupling between the optical photons used as detectors and the modes in the cavity. (Alternatively by using photons directly on a transparent bar in such a way as to induce a second-order coupling between the photons and the cavity mode of interest one could dispense with parametric amplifiers, etc.)


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[1]V. B. Braginsky and Yu. I. Vorontsov, Usp Fiz. Nauk. 114, 41 (1974) [Sov. Phys.-Usp. 17, 644 (1975)].

[2]V. B. Braginsky, Yu. I. Vorontsov, and V. D. Krivchenkov, Zh. Eksp. Teor. Fiz. 68, 55 (1975) [Sov. Phys.-JETP 41, 28 (1975)].

[3]V. B. Braginsky, in *Topics in Theoretical and Experimental Gravitation Physics*, edited by V. De Sabbata and J. Weber (Plenum, New York, 1977).

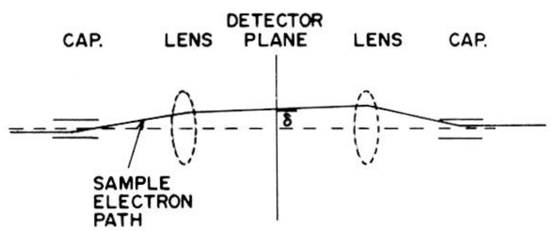


FIG. 1. The proposed Braginskii detection scheme.

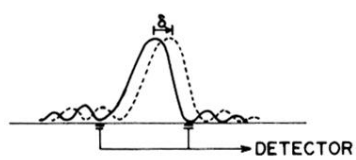


FIG. 2. Interference pattern in the detector plane.