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Machian effects in compact, rapidly spinning shells*

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The interior field of a thin mass shell of arbitrary angular momentum per unit mass $-a$ is examined as a power series in a parameter ($V^2 = 1 - 2m/R + a^2/R^2$) which measures the nearness of the shell to its gravitational radius. The exterior field is assumed to be the Kerr metric. Shell shape is arbitrary beyond the requirement of sphericity in the limits $a \to 0$ or $V \to 0$. It is shown that, as $V \to 0$, the interior inertial frames are dragged around rigidly at the same rate as the shell, for all a. A class of shapes is found for which (in the same limit) the interior spacetime becomes completely flat. This generalizes small-a results due to Brill and Cohen and to De La Cruz and Israel. Limits for physically acceptable a are found which correlate with related work by Smarr.

I. INTRODUCTION

In general relativity, it is expected that an accelerated mass distribution will "drag" along with it the inertial frames in its vicinity to a certain extent, so that these frames will appear accelerated to an inertial observer at infinity. Such frame dragging is associated with Mach's concept of inertia,[1] and is called a "Machian" effect. Rotating, nearly spherical mass shells provide a relatively tractable example of this dragging. Consequently, they have been studied in some detail, although sporadically.[2-7]

Here we are concerned with cases of "perfect" dragging, where the inertial frames at and interior to the shell are dragged around rigidly with the same angular velocity as the shell, so that an interior observer cannot detect the shell's rotation.[8] This phenomenon was discovered and pursued in the course of approximation schemes, in which static solutions of Einstein's equations were perturbed with a small shell angular velocity, Ω. It occurs only in the limit where the shell is at its own gravitational radius ("relativistic compactness"). Brill and Cohen[3] first found perfect dragging at order Ω, but went no further because of difficulties with shell-shape perturbations that appear at order Ω^2. De La Cruz and Israel[5] (we shall call their paper DCI hereafter) advanced to order Ω^3 by taking the Kerr field for the *exact* exterior spacetime.[9] This limitation allowed them to specify the shell shape arbitrarily, because it removed the need to do a self-consistent calculation of all shell stresses and gravitational field. In addition to the order-Ω^3 "perfection," they found a particular shell shape which made the interior spacetime flat at order Ω^2.

It is of great interest to know whether these provocative results are limited to small Ω, or whether they occur also for rapidly spinning shells. In this paper we remove the restriction to slow rotation by expanding in a "compactness parameter" $V^2 = 1 - 2m/R + a^2/R^2$ instead of in Ω. We find that perfect dragging occurs for all Ω when $V \to 0$, and that in this limit complete interior flatness is almost always obtainable. In the process we discover how to find, with arbitrary accuracy, those shells which separate the Kerr exterior from a flat interior.

Admittedly, the limiting case we wish to study is somewhat academic. One does not expect to find highly relativistic elastic shells in nature, nor could such a shell be stationary when $V = 0$ without experiencing infinite surface stresses. For V small but nonzero, however, this paper demonstrates a useful technique for studying bodies of large spin angular momentum. The extreme case illustrated here is chosen to probe Mach's conjecture that a sufficiently massive object should govern nearby inertial frames. In this regard, we mention that the proper surface density of the shell remains finite at the gravitational radius, but it is not real for large values of a. The stresses and surface density are considered in Appendix C.

Our method is outlined in Sec. II. Then we discuss the interior spacetime more carefully (Sec. III) and recast the DCI junction conditions into an exact form which is suited to two different expansion schemes (Sec. IV). Expansions in Ω and V establish even-order flatness (Sec. V). The dragging (odd order) is discussed in Sec. VI.

II. OUTLINE OF METHOD

Consider a closed shell of vanishing thickness (that is, a surface layer Σ) which divides spacetime into two regions. For the exterior region V^+ we assume the Kerr metric, written in Boyer-Lindquist[10] coordinates as

$$(ds^2)_+ = -(1-2mr/\beta)dt^2 + (4amr/\beta)\sin^2\theta d\phi dt + (r^2+a^2+2mra^2\sin^2\theta/\beta)\sin^2\theta\, d\phi^2 + \beta(dr^2/\Delta + d\theta^2)\,, \quad (1a)$$

where

$$\beta = r^2 + a^2\cos^2\theta, \quad \Delta = r^2 - 2mr + a^2\,, \quad (1b)$$

m is the mass of the shell, and $-a = J/m \propto \Omega$ is the angular momentum per unit mass.

Let Σ be embedded in V^+ by requiring

$$r = R(1+\epsilon V^2 F) \equiv R\mathfrak{F}\,, \quad (2)$$

$$t = t, \quad \theta = \theta, \quad \phi = \phi. \quad (3)$$

Thus we adopt (t, θ, ϕ) as coordinates of the hypersurface Σ. In (2),

$$\epsilon \equiv a^2/R^2, \quad V^2 \equiv 1-(2m/R) + a^2/R^2\,, \quad (4)$$

and R is an arbitrary constant. DCI specified F to be the function $k\cos^2\theta$ (k = constant); the choice $k = -1$ led to interior flatness at order ϵ. Here, F will be equatorially symmetric and must be finite when $\epsilon = 0$ and/or $V = 0$, but will be otherwise arbitrary. These restrictions make Σ spherical in the absence of rotation ($\epsilon = 0$), and locate it at its own event horizon in the limit of relativistic compactness ($V \to 0$). The redundant notation (F and $\mathfrak{F}$) for the r-embedding function will be useful below.

An exact interior solution to Einstein's equations which matches onto a Kerr exterior is not known. An approximate solution must therefore be found and matched, order by order, with an appropriate expansion of the Kerr metric. For the interior region, V^-, we write the metric in the form

$$(ds^2)_- = QN^{-1}(d\rho^2 + dz^2) - NdT^2 + \rho^2 N^{-1}(d\Phi - \omega dT)^2\,, \quad (5)$$

where Q, N, and ω are functions of ρ and z, to be determined. This form of the line element is simpler to work with at higher orders than is the equivalent form used in DCI; the two forms are related by simple redefinitions of the coefficients (see Appendix A).

Local inertial frames rotate with angular velocity ω relative to the coordinate Φ. Relative to an inertial observer at infinity, they rotate at the rate

$$[\omega dT - d\Phi)/dt]_{d\phi/dt=0} \equiv \bar{\omega}\ ; \quad (6)$$

this is the "dragging" rate.

Let Σ be embedded in V^- by the parametric equations

$$\rho^2 = R^2 N f(1+\chi)\sin^2\theta\,, \quad (7a)$$

$$z^2 = R^2 N g^2(1+\Xi)^2\cos^2\theta\,, \quad (7b)$$

$$T = Ct, \quad \Phi = \phi + a\kappa t/R^2\,, \quad (8)$$

where f, g, χ, and Ξ are functions of θ, while C and κ are arbitrary constants. Inclusion of N in (7) simplifies later computations. The functions f and g are defined to be independent of V, while χ and Ξ are to vanish with V.

Now, Σ is a single, unique hypersurface whether it be viewed from V^+ or V^-. Therefore, we must require the three-metric[11] g^+_{ab}, defined on Σ via (1)–(3), to agree with g^-_{ab}, defined on Σ via (5), (7), (8). Thus

$$g^+_{ab} = g^-_{ab}, \quad a, b = 0, 2, 3\,. \quad (9)$$

In Sec. IV we will obtain an equivalent set of conditions from the four nonvanishing equations in (9). For $V = 0$, those conditions will determine f and g, and restrict C and κ. For general V, they will determine χ, Ξ, and the values of N and ω on Σ, in terms of the exterior shape function F. It will be possible to adjust F, C, and κ to make $N = 1$, $\omega = 0$, and $\bar{\omega} = \Omega$ to any desired accuracy in a, in the limit $V \to 0$. The field equations (Sec. III) then show that $Q = constant$ to the same accuracy, and this constant may be set to unity by a scale change on ρ and z.

III. PROPERTIES OF THE INTERIOR METRIC

For the metric (5), the vacuum field equations $G^{\mu\nu} = 0$ reduce to (subscripts denote partial derivatives)

$$Q_\rho = \tfrac{1}{2}\rho[(N_\rho{}^2 - N_z{}^2) - \rho^2(\omega_\rho{}^2 - \omega_z{}^2)]QN^{-2}\,, \quad (10a)$$

$$Q_z = \rho(N_\rho N_z - \rho^2\omega_\rho\omega_z)QN^{-2}\,, \quad (10b)$$

$$N_{\rho\rho} + N_\rho/\rho + N_{zz} = [(N_\rho{}^2 + N_z{}^2) + \rho^2(\omega_\rho{}^2 + \omega_z{}^2)]N^{-1}\,, \quad (11a)$$

$$\omega_{\rho\rho} + 3\omega_\rho/\rho + \omega_{zz} = 2(\omega_\rho N_\rho + \omega_z N_z)N^{-1}. \quad (11b)$$

Equations (11) are the integrability conditions for Eqs. (10).

The flat spacetime $Q = N = 1$, $\omega = 0$ is clearly a solution of (10) and (11). It is the appropriate solution when $\epsilon = 0$, for which the Kerr metric reduces to the Schwarzschild metric. Now suppose that there is a small parameter x, in terms of which Q, N, and ω may be expanded in power series,

$$Q = 1 + x^2 q_1 + x^4 q_2 + \cdots\,,$$
$$N = 1 + x^2 n_1 + x^4 n_2 + \cdots\,, \quad (12a)$$

$$\omega = x w_0 + x^3 w_1 + x^5 w_2 + \cdots\,, \quad (12b)$$

where the q_i, n_i, and w_i may be functions of ρ and z. (Solutions in this form appear in DCI for $x^2 = \epsilon$, where $w_0 = constant$. We shall also consider the case $x = V$.) Substitute (12) into (11) and separate orders of x. The left-hand sides of (11) provide the homogeneous parts of the resulting equations, while the nonlinear right-hand sides provide the inhomogeneous terms. On Σ, the homogeneous solutions for n_i and w_i are complete sets of equatorially symmetric functions (see Appendix B). Therefore, n_i and w_i may be independently adjusted to match any equatorially symmetric boundary conditions—even if the particular solutions for the inhomogeneous equations are nonzero. Furthermore, if $N=1$ exactly (a case we will study), then the inhomogeneous part of (11) vanishes at all orders of x. From (10), Q is then a constant up to the square of the order of the first nonconstant w_i. The homogeneous w_i will thus be the functions limiting the flatness of the interior space-time.

IV. THE JUNCTION CONDITIONS

From $1+g^+_{00}=1+g^-_{00}$, one has

$$\mathcal{G} \equiv (1+\epsilon - V^2)\mathcal{F}/(\mathcal{F}^2+\epsilon \cos^2\theta) = 1 - C^2N + \epsilon f(1+\chi)\kappa'^2 \sin^2\theta \,, \tag{13}$$

where

$$\kappa' \equiv \kappa - CR^2\omega/a \,. \tag{14}$$

Dividing $g^+_{33}=g^-_{33}$ by $R^2\sin^2\theta$ yields

$$\epsilon \mathcal{G} \sin^2\theta + \epsilon \mathcal{F}^2 = f(1+\chi) \,. \tag{15}$$

Dividing $g^+_{03}=g^-_{03}$ by $a\sin^2\theta$ and using (15) to remove χ gives

$$(1-\epsilon\kappa' \sin^2\theta)\mathcal{G} = (\epsilon+\mathcal{F}^2)\kappa' \,. \tag{16}$$

Equations (13)–(16) are exact. The condition $g^+_{22}=g^-_{22}$ in its exact form is somewhat lengthy and will not be quoted here. For $V=0$ it reduces to

$$g_\Theta \cos\theta - g\sin\theta = [1+\epsilon \cos^2\theta - (1+\epsilon)^4\cos^2\theta/(1+\epsilon\cos^2\theta)^3]^{1/2} \,. \tag{17}$$

For $V\neq 0$, it is solvable by quadrature for Ξ in terms of g, f, χ, $\mathcal{F}$, and N.

For the case $x^2 \equiv \epsilon = 0$, one has from (2), (12), and (13),

$$C^2 = V^2[1+O(\epsilon)] \equiv V^2\gamma^2, \quad T = V\gamma t \,. \tag{18}$$

Equations (2) and (8) allow evaluation of (13)–(16) when $x \equiv V = 0$, with the result

$$f = (1+\epsilon)^2/(1+\epsilon\cos^2\theta) \tag{19}$$

$$\kappa = 1/(1+\epsilon) + O(V^2) \equiv 1/(1+\epsilon) + V^2\kappa_2 \,. \tag{20}$$

For a flat interior space-time, $N=1$ and the ρ-embedding reduces to $\rho^2 = R^2 f \sin^2\theta$ when $V\to 0$. Thus f is the unique, exact ρ-embedding function that allows a flat spacetime interior to a shell at its gravitational radius. Expanding f for $\epsilon<1$, one immediately recovers the order-ϵ results in DCI.

The z-embedding function g cannot be evaluated exactly, but it is a trivial matter to expand (17) to any desired order of ϵ and thus obtain g to arbitrary accuracy when $\epsilon < \frac{1}{3}$. One finds

$$g = 1 - \tfrac{1}{2}\epsilon\cos^2\theta - \epsilon^2\cos^2\theta(1-\tfrac{3}{8}\cos^2\theta) + O(\epsilon^3) \,, \tag{21}$$

which agrees with DCI's order-ϵ results and provides the order-ϵ^2 corrections. For $\epsilon = \frac{1}{3}$ the square root in (17) vanishes at the poles and the expansion used to obtain (21) diverges there. For $\epsilon > \frac{1}{3}$ there is a range around each pole for which g is imaginary; where this happens (17) has no convenient expansion and must be integrated numerically. These results concur with those of Smarr,[12] who examined rotating black holes (but not shells) of arbitrary charge. In terms of our parametrization, he found that for $\epsilon > \frac{1}{3}$ the spatial part of the Kerr horizon can be embedded in pseudo-Euclidean space near the poles, but not in Euclidean space. This situation also influences the proper surface density (see Appendix C).

For most of the discussion below, the key equation is obtained from (13) by using (15) and (16) to eliminate f and κ',

$$C^2N = V^2\gamma^2 N = 1 - \mathcal{G}(\epsilon+\mathcal{F}^2)/(\mathcal{G}\epsilon\sin^2\theta + \epsilon + \mathcal{F}^2) \,. \tag{22}$$

One may now consider cases where V and ϵ are both nonzero but at least one is small.

V. EXPANSIONS OF N

To see how (22) relates to the results in DCI, consider first the case $x^2 = \epsilon < 1$. Then (22) becomes

$$\gamma^2 N = 1 + \epsilon F^* - \epsilon^2\{V^2FF^* + (1-V^2)F[3(1-V^2) - (1-3V^2)\cos^2\theta] - (3-4V^2) + (2-5V^2)\cos^2\theta + V^2\cos^4\theta - V^4(1-\cos^2\theta)^2\} + O(\epsilon^3) \,, \tag{23}$$

where

$$F^* \equiv (1-V^2)(F+\cos^2\theta) - (2-V^2) \,.$$

If for F one were to substitute $F\equiv F_1+\epsilon F_2+\epsilon^2F_3+\ldots$, it is clear that at order ϵ^i, after regrouping terms, the only terms containing F_i would be $(1-V^2)F_i$, coming from ϵF^* in (23). F_j $(j>i)$ would not appear at order ϵ^i, hence F_i could be adjusted to cancel all other terms of order ϵ^i on the right-hand side of (23). Since this could, in principle, be done for all i, N can be made unity to arbitrarily high orders and for any value of V. Thus, for example, any F_1 of the form

$$F_1=s-\cos^2\theta\,, \tag{24a}$$

together with

$$\gamma^2=1+\epsilon[(1-V^2)s-(2-V^2)]+O(\epsilon^2)\,, \tag{24b}$$

will make $N=1+O(\epsilon^2)$ for all V. Here, s is any function of V^2 that does not diverge as $V\to 0$. The choice $s=0$ recovers the embedding used by DCI to obtain interior flatness at order ϵ. The choice of F_2 which removes the ϵ^2 terms from N can be found from (23) and (24) almost by inspection.

For $x^2=V^2<1$, Eq. (22) gives

$$\begin{aligned}(1+\epsilon)^2\gamma^2N=&[1+\epsilon\cos^2\theta+\epsilon F(1-\epsilon)(1+\epsilon\cos^2\theta)]\\&-V^2\{\epsilon^2F^2(1-2\epsilon-\epsilon\cos^2\theta-\hat\epsilon(1-\epsilon)\sin^2\theta[5-\epsilon-\hat\epsilon(1-\epsilon)\sin^2\theta])\\&+\epsilon F(1-\epsilon\cos^2\theta-2\hat\epsilon\sin^2\theta[3-\hat\epsilon(1-\epsilon)\sin^2\theta])-\hat\epsilon(1+\epsilon\cos^2\theta)\sin^2\theta/(1+\epsilon)\}+O(V^4)\,,\end{aligned} \tag{25}$$

where $\hat\epsilon\equiv\epsilon/(1+\epsilon)$.

By expanding F in powers of V^2 and arguing as we did for the ϵ expansion, we observe that N can (in principle) be made unity to arbitrarily high orders in V^2, for all values of ϵ. To lowest order, any F of the form

$$F=(s'-\cos^2\theta)/(1-\epsilon)(1+\epsilon\cos^2\theta)+O(V^2)\,, \tag{26a}$$

together with

$$\epsilon^2\gamma^2=(1+\epsilon s')\hat\epsilon^2+O(V^2)\,, \tag{26b}$$

will make $N=1+O(V^2)$ for all values of ϵ except $\epsilon=1$. Here, s' is any function of ϵ that does not diverge as $\epsilon\to 0$. The choice $s'=0$ yields an F which, when expanded for $\epsilon<1$, agrees with DCI's order-ϵ embedding (for $V=0$, $k=-1$) and provides all higher-order corrections. Thus, for a shell at its gravitational radius, (26) is the exact exterior embedding which makes the interior space-time flat. For V small but nonzero, one may obtain the order-V^2 corrections to (26) without difficulty, from (25).

VI. THE INTERIOR DRAGGING

To complete the proof that (26) makes the interior flat as $V\to 0$, we must show that, concurrently, the interior dragging is rigid. Then there will be a global coordinate frame in which $\omega=0$ and, by (10), the remaining interior metric function, Q, will be a constant.

As seen by a stationary observer at infinity, the shell rotates with the angular velocity[13]

$$\Omega^+\equiv(d\phi/dt)_\Sigma=-a/(R^2+a^2)+O(V^2)\,, \tag{27}$$

while from (6), (18), and (20), the interior dragging relative to infinity is

$$\bar\omega=-a/(R^2+a^2)+\omega\gamma V+O(V^2)\,. \tag{28}$$

Comparing with (27), we find that the interior inertial frames rotate rigidly with the shell when $V\to 0$, as seen from the outside. Thus "perfect" interior dragging occurs for *arbitrary* angular momentum when $V\to 0$. Remarkably, this result depends only on the assumption (i) that the shell is spherical in the absence of rotation ($a=0$), which was the crucial assumption leading to (18) and (20).

Owing to the infinite gravitational redshift when $V\to 0$, however, the situation as viewed from inside is not yet settled. An interior observer sees the shell rotating with the angular velocity

$$\begin{aligned}\Omega^-\equiv(d\Phi/dT)_\Sigma&=[(d\Phi/dt)(dt/dT)]_\Sigma\\&=(\Omega^++a\kappa/R^2)/\gamma V\,.\end{aligned} \tag{29}$$

Combining (20), (27), and (29) yields

$$\Omega^-=-(a/\gamma R^2)\times O(V)\,. \tag{30}$$

Since γ is nonzero, Ω^- vanishes with V. The key assumptions involved in this result are (i) above and that, also for $a=0$, (ii) the interior coordinate frame reduces to the nonrotating system of the standard interior Schwarzschild solution plus (arbitrary) terms that vanish with V^2. The question remaining is whether ω vanishes in this frame when $a\neq 0$.

Solve (16) for κ' and substitute (22); there results

$$\kappa' = 1/(1+\epsilon) + V^2(\kappa_2 - \omega\gamma R^2/aV)$$
$$= (1 - V^2\gamma^2 N)/(\epsilon + \mathfrak{F}^2)\,. \qquad (31)$$

Equation (31) provides a solution for ω on Σ in terms of the arbitrary constant κ_2, and the embedding parameter γ and function $\mathfrak{F}$ which are fixed by placing requirements on N.

If one expands (31) for small ϵ and uses (24) and (25), then for proper choice of κ_2 one obtains results equivalent to the order-a^3 results in DCI. For all s, ω is seen to vanish as a^3V.

It is more interesting to expand for small V. Then one uses (14) and (20) in (31) and finds for *arbitrary* ϵ,

$$\kappa_2 - \omega\gamma R^2/aV = -(\gamma^2 N + 2\mathcal{E}F)/(1+\epsilon) + O(V^2)\,. \qquad (32)$$

For general κ_2, N, and F, (32) shows $\omega \sim O(aV)$. In special cases one may make $\omega \sim O(a^3V)$. For example, if (26a) is used in (32) and one sets $\kappa_2 = -(1+\epsilon s')/(1+\epsilon)^3$, then

$$\omega\gamma R^4 = 2a^3V(s' - \cos^2\theta)/[(1+\epsilon)^2(1-\epsilon)(1+\epsilon\cos^2\theta)] + O(V^3)\,. \qquad (33)$$

Because κ_2 is a constant, it is not possible to remove the order-V dependence from ω.

Since (see Appendix B) there are readily available global solutions only for those ω which are polynomials in $\cos^2\theta$ on Σ, a global form for ω follows from (32) only approximately. Taking $s' = 0$ in (33), for example, an expansion for small ϵ leads to

$$\omega(1-\epsilon^2) = -\tfrac{2}{5}V\epsilon(a/R^2)[(1 - 3\epsilon/7) + (\tfrac{2}{3} - 4\epsilon/9)(\rho^2 - 4z^2) - (8\epsilon/63)(\rho^4 - 12\rho^2z^2 + 8z^4) + O(\epsilon^2)] + O(V^3)\,, \qquad (34)$$

where γ was defined by (26b). The constant terms may be removed from the right-hand side of (34), and the scale of the remainder adjusted at will, by changing the scale of γ; this does not affect the line element (5) if a scale change is also made in ρ and z.

VII. DISCUSSION

To a large extent, the key results at even vs odd orders in a are independent of each other. The odd-order function ω, and therefore the interior dragging $\bar{\omega}$, vanish like $O(V)$ for all a regardless of the behavior of the even-order function N. In turn, N can be made unity to any desired accuracy in either one of V or a, for arbitrary values of the other, without considering ω. One's control over N and ω can be used to make $Q=1$ as $V \to 0$ for any a, and the interior spacetime flat in that limit.

The one exception to these results occurs for $\epsilon = 1$, the relativistic limit of rapid rotation.[14] Then for any finite F, Eq. (25) gives

$$\gamma^2 N = \tfrac{1}{4}(1 + \cos^2\theta) + O(V^2)\,, \qquad (35)$$

so it is impossible to make N and Q constants. Perfect dragging persists, however, since (32) is still valid. In fact we now have *exactly*

$$\kappa_2 - \omega\gamma R^2/aV$$
$$= -\tfrac{1}{2}(F + \tfrac{1}{2}V^2F^2 + \gamma^2 N)/(1 + V^2F + \tfrac{1}{2}V^4F^2)\,, \qquad (32')$$

with F still arbitrary. For the special choice $\kappa_2 = 0$ and $F = [(1 - 2\gamma^2V^2N)^{1/2} - 1]/V^2$, ω vanishes *identically* for all V. Therefore, an observer at infinity sees perfect interior dragging for all V, while an interior observer sees rigid dragging which becomes perfect as $V \to 0$.

Extreme Machian effects thus appear to characterize stationary shells near their gravitational radii. The price one pays for the *interior* effects is the divergence of the surface stresses as $V \to 0$ (Appendix C), which is required to support the shell in this limit. It is apparently just this divergence which "locks in" the interior inertial frames with the rotating shell, since Lindblom and Brill[6] have shown that a (slowly rotating) freely falling shell leads to perfect dragging only as seen from the exterior. Although we have simplified matters by taking the shell to be vanishingly thin, there is no obvious reason to expect substantive changes in any of these results if the shell is allowed to have a small nonzero thickness. The question of what would happen with a thick shell or a solid ball (as of nuclear matter) remains open.

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APPENDIX A: TRANSFORMATION ON THE INTERIOR LINE ELEMENT

For purposes of comparison, the form of the interior line element used in DCI is due to Lewis[15]:

$$ds^2 = e^{2(\nu-\lambda)}(d\rho^2 + dz^2) + \rho^2 e^{-2\lambda} d\Phi^2 - e^{2\lambda}(dT - \Psi d\Phi)^2 . \tag{A1}$$

The form used in this paper, Eq. (5), follows from (A1) if one defines new coefficients Q, N, and ω to replace ν, λ, and Ψ,

$$e^{2\lambda} \equiv NW, \quad e^{2\nu} \equiv QW, \quad \Psi \equiv -\rho^2\omega/(N^2 W), \tag{A2}$$

where

$$W = 1 - \rho^2\omega^2/N^2 . \tag{A3}$$

APPENDIX B: HOMOGENEOUS SOLUTIONS FOR N AND ω

At each order i there is an infinite set of independent solutions to the homogeneous part of (11a). These solutions are

$$n_{(L)i} = \sum_{j=0}^{L} c_{(L)j}\, \rho^{2(L-j)} z^{2j}, \quad L = 0, 1, 2, \ldots, \infty \tag{B1}$$

where

$$c_{(L)j} = -2(L-j+1)^2 c_{(L)j-1}/[i(2i-1)] . \tag{B2}$$

The homogeneous solutions of (11b) for $w_{(L)i}$ have the same form except that the constants c are replaced by c' where

$$c'_{(L)j} = -2(L-j+1)(L-j+2)c'_{(L)j-1}/[i(2i-1)] . \tag{B3}$$

The solutions (B1)–(B3) have the property that, on a nearly spherical shell where $\rho = R\sin\theta + O(x^2)$ and $z = R\cos\theta + O(x^2)$ (x = any small parameter), $n_{(L)i} \propto P_{2L}(\cos\theta) + O(x^2)$ and $w_{(L)i} \propto dP_{2L+1}(\cos\theta)/d(\cos\theta) + O(x^2)$. Thus, $n_{(L)i}$ and $w_{(L)i}$ constitute complete sets of equatorially symmetric functions on Σ.

APPENDIX C: THE STRESSES AND SURFACE DENSITY

The surface stress energy tensor[16] S_a^b is defined by

$$-(8\pi G/c^2)S_a^b = \gamma_a^b - \delta_a^b \gamma_c^c , \tag{C1}$$

where $\gamma_a^b = K_a^{+b} - K_a^{-b}$ is the jump in extrinsic curvature K_a^b at the shell. We will set $G = c^2 = 1$ below. The K_a^b depend on the embedding

$$K_{ab}^{\pm} = a_{\mu\nu} e_{(a)}^{\mu} \left[\frac{\partial n^\nu}{\partial X^b} + n^\lambda \{{}^{\nu}_{\lambda\tau}\} e_{(b)}^{\tau}\right]\Bigg|^{\pm} ,$$

where $a_{\mu\nu}$ is the metric in the embedding spacetime, $e_{(a)}$ are coordinate basis vectors for the hypersurface, n is the unit outward normal, and $\{{}^{\nu}_{\lambda\tau}\}$ are the Christoffel symbols formed from the $a_{\mu\nu}$. Indices on K_{ab} are raised with the inverse of the hypersurface metric g_{ab}. The X^b are the hypersurface coordinates (θ, ϕ, t).

The proper surface density σ is defined by the eigenvalue equation

$$S_a^b u^a = -\sigma u^b . \tag{C2}$$

The eigenvectors (velocities) are not needed to find σ, beyond noting that $u^\theta = 0$ and $u^\phi = \Omega u^t$.

One may factor out the divergent terms in Eqs. (C1), yielding at leading order in V,

$$S_\Theta^\Theta = (16\pi RV)^{-1}[(1+\epsilon)(1-\epsilon\cos^2\theta) - 2\epsilon\sin^2\theta + O(V)]/(1+\epsilon\cos^2\theta) ,$$

$$S_\phi^\phi = (16\pi RV)^{-1}[(1+\epsilon)^2(1-\epsilon\cos^2\theta) + O(V)]/(1+\epsilon\cos^2\theta)^2 ,$$

$$S_t^t = -(8\pi RV)^{-1}\epsilon\sin^2\theta[1+\epsilon\cos^2\theta + \tfrac{1}{2}(1+\epsilon)(1-\epsilon\cos^2\theta) + O(V)]/(1+\epsilon\cos^2\theta)^2 ,$$

$$S_\phi^t = -S_t^t a(1+\epsilon)/\epsilon , \quad S_t^\phi = S_\phi^\phi a/(1+\epsilon) .$$

These expressions are exact in ϵ. The S_a^b obtained from the order-a^3 expansions of $(K_a^b)^\pm$, given in DCI, agree in the double limit $\epsilon, V \ll 1$.

These S_a^b all diverge when $V \to 0$, as expected. The terms shown come from the $(K_a^b)^+$. The $(K_a^b)^-$ are all nondivergent; their exact forms are rather long and are not reproduced here.

The divergence of S_a^b as $V \to 0$ makes the interpretation of the other results of this paper somewhat a matter of taste. However, nowhere in the computation of $\bar{\omega}$, the embedding functions, and the interior metric do divergences appear. The results obtained are therefore the correct limiting properties of a sequence of stationary, ever more compact shells of given mass and angular momentum. The sequence limit is not physically attainable; this fact need not impair the implications of the limiting results for Mach's conjecture.

From (C2) and the S_a^b, one quickly finds that the leading (divergent) terms in the proper surface density σ cancel, and σ is determined by the (finite) interior embedding. Thus, σ is finite for all ϵ, a result consistent with the behavior DCI obtained for small ϵ in the limit $V\to 0$. The surviving form of σ in our case (general ϵ) is too lengthy to quote here. It should be noted, however, that σ does become imaginary in precisely those regions where the embedding space is pseudo-Euclidean. This occurs because, as $V\to 0$, every nonvanishing term of σ contains a factor arising from the unit normal vector to the horizon. Thus, $\epsilon \simeq \frac{1}{3}$ represents an upper limit for the rotation of any acceptable shell with nearly flat interior, near its gravitational radius. Mach's conjecture has therefore been extended over the entire range of physically achievable angular momenta in such compact shells.

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*This work is based in part on a portion of the doctoral dissertation of the author.

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[9]This is a reasonable limitation for the study of relativistically compact shells, since all uncharged black holes in nature are expected to be of the Kerr family. See D. C. Robinson, Phys. Rev. Lett. 34, 905 (1975).

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[11]Latin indices have the range 0,2,3 and apply to three-tensors defined on Σ. Greek indices have the range 0,1,2,3. Repeated indices imply summation.

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[13]This very general result is independent of the details of the problem. See, for example, C. Misner, K. Thorne, and J. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), pp. 893–895.

[14]The equatorial rotation velocity of the shell is c in this limit. See Ref. 12 for a discussion of this limit when one is dealing with a rotating black hole without a shell.

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