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Quantum gravity and path integrals

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The path-integral method seems to be the most suitable for the quantization of gravity. One would expect the dominant contribution to the path integral to come from metrics which are near background metrics that are solutions of classical Einstein equations. The action of these background metrics gives rise to a new phenomenon in field theory, intrinsic quantum entropy. This is shown to be related to the scaling behavior of the gravitational action and to the topology of the gravitational field. The quadratic terms in the Taylor series of the action about the background metrics give the one-loop corrections. In a supersymmetric theory the quartic and quadratic but not the so-called logarithmic divergences cancel to give a one-loop term that is finite without regularization. From the one-loop term one can obtain the effective energy-momentum tensor on the background metric. In the case of an evaporating black hole, the energy-momentum tensor will be regular on the future horizon. The usual perturbation expansion breaks down for quantum gravity because the higher (interaction) terms in the Taylor series are not bounded by the quadratic (free) ones. To overcome this I suggest that one might replace the path integrals over the terms in the Taylor series by a discrete sum of the exponentials of the actions of all complex solutions of the Einstein equations, each solution being weighted by its one-loop term. This approach seems to give a picture of the gravitational vacuum as a sea of virtual Planck-mass black holes.

I. INTRODUCTION

Although general relativity has been around for more than 60 years, it has been generally ignored by most physicists, at least until recently. There are three reasons for this. First, the differences, between general relativity and Newtonian theory were thought to be virtually unmeasurable. Second, the theory was thought to be so complicated mathematically as to prevent any general understanding of its qualitative nature being achieved or any detailed predictions being made. Third, it was a purely classical theory whereas all other theories of physics were quantum mechanical.

The first two objections to general relativity have largely been met in the last fifteen to twenty years. On the observational side we now have very accurate verifications of general-relativistic effects in the solar system and fairly convincing evidence for such strong-field predictions as black holes and the "big bang." On the theoretical side, while there are still some unproved conjectures such as cosmic censorship, the development of new mathematical techniques has given us a pretty complete qualitative understanding of the theory while the advances in computers have enabled us, at least in principle, to make quantitative predictions to any desired order of accuracy. However, the third objection still stands; despite a lot of work (and some successes) we do not yet have a satisfactory quantum theory of gravity whose classical limit is general relativity. This is probably the most important unsolved problem in theoretical physics today. I shall not attempt to review all that has been done but simply give my personal view of some of the difficulties involved and how they might be overcome.

There are three main ways of quantizing a classical field theory. The first is the operator approach in which one replaces the field variables in the classical equations by operators on some Hilbert space. This does not seem appropriate for gravity because the Einstein equations are nonpolynomial in the metric. It is difficult enough to interpret the product of two operators at the same spacetime point, let alone a nonpolynomial function. The second method is the canonical approach in which one introduces a family of spacelike surfaces and constructs a Hamiltonian. Although many people favor this, the division into space and time seems to me to be contrary to the whole spirit of relativity. Also it is not clear that the concept of a spacelike surface has any meaning in quantum gravity since one would expect that there would be large quantum fluctuations of the metric on small length scales. Further, I shall want to consider topologies of the spacetime manifold that do not permit any well-behaved families of surfaces let alone spacelike ones. For these reasons I prefer the path-integral approach though it too has problems concerning the measure and the very meaning of the integral. In what follows I shall try to describe some of these problems and the ways that one might solve them.

II. PATH INTEGRALS

The basic idea of the Feynman path integral is that the amplitude to go from a state with metric

g_1, and matter fields ϕ_1 at time t_1 to a state with metric g_2 and matter fields ϕ_2 at time t_2 is given by an integral over all field configurations which take the given values at times t_1 and t_2:

$$\langle g_2, \phi_2, t_2 | g_1, \phi_1, t_1 \rangle = \int D[g] D[\phi] \exp(iI[g, \phi]),$$

where $D[g]$ is a measure on the space of all metrics, $D[\phi]$ is a measure on the space of all matter fields, I is the action, and the integral is taken over all field configurations with the given initial and final values. (I am using units in which $c = \hbar = k = 1$.) The gravitational contribution to the action is normally taken to be

$$\frac{1}{16\pi G} \int R(-g)^{1/2} d^4x .$$

However, the Ricci scalar R contains second derivatives of the metric. In order to obtain an action which depends only on first derivatives, as is required by the path-integral method, one has to remove the second derivatives by integrating by parts. This produces a surface term which can be written in the form

$$\frac{1}{8\pi G} \int K(h)^{1/2} d^3x + C ,$$

where the integral is taken over the boundary of the region for which the action is being evaluated, K is the trace of the second fundamental form of the boundary in the metric g, h is the induced metric on the boundary, and C is a term which depends only on the boundary and not on the particular metric g.

In order to make sure that one registers this surface term correctly one has to join the initial and final spacelike surfaces by a timelike tube at some large radius r_0. It is convenient to rotate the time interval on this timelike tube between the two surfaces into the complex plane so that it becomes purely imaginary. This makes the metric on the boundary positive definite so that the path integral can be taken over all positive-definite metrics g that induce the given metric for the boundary.

Suppose that one wants to find the number $n(E)dE$ of states of the gravitational and matter fields which have energy between E and $E + dE$ as measured from infinity. This will be given by

$$n = \frac{1}{2\pi i} \int_{-i\infty}^{i\infty} Z(\beta) \exp(\beta E) d\beta ,$$

where

$$Z(\beta) = \sum_n \langle g_n, \phi_n | \exp(-\beta H) | g_n, \phi_n \rangle$$

is the partition function for the system consisting of the gravitational and matter fields contained in a box of radius r_0 at a temperature $T = \beta^{-1}$. This partition function can be expressed as a path integral over all matter and gravitational fields that are periodic in imaginary time with period β, i.e.,

$$Z = \int D[g] D[\phi] \exp(-\hat{I}) ,$$

where $\hat{I} = -iI$ is the Euclidean action and the path integral is taken over all positive-definite metrics g whose boundary is a two sphere of radius r_0 times a circle of circumference β representing the periodically identified imaginary time axis.

One would expect that the dominant contribution in the path integral for Z would come from metrics g and matter fields ϕ that are near background fields g_0, ϕ_0 that extremize the action, i.e., are solutions of the classical field equations with the given periodicity and boundary conditions. Neglecting, for the moment, the question of the radius of convergence, one can expand the action in a Taylor series about the background fields

$$\hat{I}[g, \phi] = \hat{I}[g_0, \phi_0] + I_2[\tilde{g}, \tilde{\phi}] + \text{higher-order terms} ,$$

where $g = g_0 + \tilde{g}$, $\phi = \phi_0 + \tilde{\phi}$, and I_2 is quadratic in the perturbations $\tilde{g}$ and $\tilde{\phi}$. If one neglects the higher-order terms, then

$$\ln Z = -\hat{I}[g_0, \phi_0] + \ln \int D[g, \phi] \exp(-I_2[\tilde{g}, \tilde{\phi}]) .$$

One can regard the first term in the equation above as the contribution of the background field to the partition function and the second term as the contribution of thermal gravitons and matter quanta on the background geometry.[1]

III. THE BACKGROUND FIELDS

One wants to find solutions of the Einstein equations that are asymptotically flat and which at infinity are periodic in imaginary time with period β. The simplest such solution is flat Euclidean space which is periodically identified in the imaginary time direction. It is natural to choose the term C to make the action zero in this case, i.e.,

$$C = -\frac{1}{8\pi G} \int K^0 (h)^{1/2} d^3x ,$$

where $K^0 = 2r_0^{-1}$ is the trace of the second fundamental form of the boundary $S^2 \times S^1$ in the flat-space metric η. This can be regarded as a choice of the zero of energy. Thus the flat-space background metric makes no contribution to the path integral, although the fluctuations around flat space will give a contribution corresponding to thermal gravitons which will be evaluated in the next section.

It is quite easy to see from scaling arguments that any vacuum solution of the Einstein equations has zero action if its topology is $R^3 \times S^1$, i.e., the same as periodically identified flat space. However, one can obtain solutions with nonzero action by going to other topologies. The simplest example is the Schwarzschild solution. This is normally given in the form

$$ds^2 = -\left(1 - \frac{2M}{r}\right)dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 .$$

Putting $t = -i\tau$ converts this into a positive-definite metric for $r > 2M$. There is an apparent singularity at $r = 2M$, but this is like the apparent singularity at the origin of polar coordinates, as can be seen by defining a new radial coordinate $x = 4M(1 - 2Mr^{-1})^{1/2}$. Then the metric becomes

$$ds^2 = \left(\frac{x}{4M}\right)^2 d\tau^2 + \left(\frac{r^2}{4M^2}\right)^2 dx^2 + r^2 d\Omega^2 .$$

This will be regular at $x = 0$, $r = 2M$ if τ is regarded as an angular variable and is identified with period $8\pi M$ (using units in which $G = 1$). The manifold defined by $x \geq 0$, $0 \leq \tau \leq 8\pi M$ is called the Euclidean section of the Schwarzschild solution. On it the metric is positive definite, asymptotically flat, and nonsingular (the curvature singularity at $r = 0$ does not lie on the Euclidean section).

Because the Schwarzschild solution is periodic in imaginary time with period $8\pi M$ at infinity, it will contribute to the partition function for $\beta = 8\pi M$ or $T = (8\pi M)^{-1}$. Because $R = 0$, the action will come from the surface term only. This gives $\hat{I} = M\beta/2 = (1/16\pi)\beta^2$. Thus the background metric contributes $-\beta^2/16\pi$ to $\ln Z$. Now

$$Z = \sum_n \langle n | \exp(-\beta E_n) | n \rangle ,$$

where E_n is the energy of the nth eigenstate. Thus the expectation value of the energy is

$$\langle E \rangle = -\frac{d}{d\beta} \ln Z = M ,$$

as one might expect. The entropy S is defined to be

$$S = -\sum_n p_n \ln p_n ,$$

where p_n is the probability of being in the nth state. Thus

$$S = \beta\langle E \rangle + \ln Z = 4\pi M^2 = A/4 ,$$

where A is the area of the event horizon. This is a quantum-field-theory derivation of the entropy that was assigned to black holes on the basis of particle-creation calculations done on a fixed spacetime background.[2] It is a most surprising result since classical solutions in other field theories do not contribute to the entropy. The reason the classical solutions in gravity have intrinsic entropy whereas those in Yang-Mills or scalar field theories do not, is closely connected to the facts that the gravitational action is not scale invariant and that the gravitational field can have different topologies.

Under a scale transformation $g \to k^2 g$, k constant, $I \to k^2 I$. This implies that the action of an asymptotically flat metric with period β must be of the form

$$\hat{I} = B\beta^2 ,$$

where B is independent of β, since β determines the scale of the solution. Thus

$$\langle E \rangle = -\frac{d}{d\beta} \ln Z = 2B\beta$$

and

$$\ln Z = -\tfrac{1}{2}\langle E \rangle \beta$$

and not

$$\ln Z = -\langle E \rangle \beta ,$$

as would be expected if there were only a single state with energy $\langle E \rangle$ contributing to the sum that defines the partition function. Because $\ln Z$ is only $-\beta\langle E \rangle/2$ it does not cancel out the term $\beta\langle E \rangle$ in the formula for the entropy S and so

$$S = \beta\langle E \rangle/2 = B\beta^2 .$$

Yet we have only a single background metric. So how does this give rise to entropy or uncertainty about the quantum state and why is it that the action of the background metric is only $\beta\langle E \rangle/2$ and not $\beta\langle E \rangle$? To answer the second question, consider two surfaces of constant imaginary time τ_1 and τ_2 in the Euclidean section of the Schwarzschild solution. They will have boundaries at the surface of the box at radius r_0. However, they will also have a boundary at $r = 2M$ when they intersect each other. The amplitude to propagate from the surface τ_1 to the surface τ_2 will be given by a path integral of all metric configurations bounded by the two surfaces and the walls of the box at the radius r_0. The dominant contribution to the log of the amplitude will be the action of the classical solution of the Einstein equations. This is just the portion of the Schwarzschild solution between these surfaces. Again $R = 0$ so that the action is given by the surface terms. There is a contribution of $\frac{1}{2}M(\tau_2 - \tau_1)$ from the boundary at radius r_0 but there is also a contribution from the angle between the two surfaces at $r = 2M$. This is also equal to $\frac{1}{2}M(\tau_2-\tau_1)$ so t at the total action is $M(\tau_2-\tau_1)$ i.e., mass times imaginary time interval, as one might expect for a single state and the entropy would be zero. However, when one considers the Euclidean Schwarzschild metric simply as a metric which fills in the boundary at radius r_0, one does not have a boundary at $r=2M$ and so one does not include a contribution to the action from there of $M\beta/2$. Neglecting this contribution can be regarded in some sense as summing over all the states of the metric for $r<2M$ which were not included on the Euclidean section. Similar results hold for charged and rotating black holes.[1] In each case the background metric contributes an entropy equal to a quarter of the area of the event horizon.

IV. THE ONE-LOOP TERMS

I now come to the question of evaluating the path integrals over the quadratic terms in the fluctuations about the background fields. These are often referred to as one-loop corrections because, in Feynman diagram terms, they are represented by a graph with any number of external lines joined to a single closed loop. Consider first the case of a scalar field ϕ obeying (say) the conformally invariant wave equation. The quadratic term of the action will be of the form

$$I_2=\frac{1}{2}\int \phi A\phi(g_0)^{1/2}d^4x\,,$$

where A is a second-order differential operator. With the condition that ϕ be zero on the boundary, the operator A will have a discrete spectrum of eigenvalues λ_n and eigenfunctions ϕ_n,

$$A\phi_n=\lambda_n\phi_n\,.$$

The eigenfunctions can be normalized so that

$$\int \phi_n\phi_m(g_0)^{1/2}d^4x=\delta_{nm}\,.$$

Any field ϕ which is zero on the boundary can be expanded in terms of the eigenfunctions

$$\phi=\sum_n a_n\phi_n\,.$$

The measure $D[\phi]$ on the space of all fields ϕ can be expressed in the terms of the eigenfunction expansion

$$D[\phi]=\prod \mu da_n\,,$$

where μ is some normalization constant with dimensions of mass or $(\text{length})^{-1}$. Using these formulas, the path integral over field ϕ becomes

$$\begin{aligned}\int\prod\mu da_n\exp(-\tfrac{1}{2}\lambda_n a_n{}^2)&=\prod 2^{1/2}\pi^{1/2}\mu\,\lambda_n{}^{-1/2}\\&=[\det(\tfrac{1}{2}\mu^{-2}\pi^{-1}A)]^{-1/2}\,.\end{aligned}$$

The number $N(\lambda)$ of eigenvalues whose value is less than λ has an asymptotic expansion of the form[3]

$$N(\lambda)=\sum_{n=0}^{\infty}P_n\lambda^{2-n}\,,\quad \lambda\to\infty$$

where

$$P_0=\frac{1}{32\pi^2}\int (g_0)^{1/2}d^4x\,,$$

$$P_1=\frac{1}{16\pi^2}\int[(\tfrac{1}{6}-\xi)R-m^2](g_0)^{1/2}d^4x\,,$$

$$\begin{aligned}P_2=\frac{1}{2880\pi^2}\int[&R^{abcd}R_{abcd}-R_{ab}R^{ab}\\&+(6-30\xi)\Box R+\tfrac{5}{2}(6\xi-1)^2R^2\\&+30m^2(1-6\xi)R+90m^4]d^4x\,,\end{aligned}$$

for an operator A of the form

$$A=-\Box+\xi R+m^2\,.$$

For the conformally invariant wave operator, $\xi=\frac{1}{6}$ and $m=0$. Thus $P_1=0$. However, P_0 is nonzero and is proportional to the volume of the space. Thus the determinant of A diverges badly. To regularize the determinant, that is to get a finite value, one has to divide out by the numbers of eigenvalues that correspond to the first two terms P_0 and P_1 in the asymptotic expansion. There are various ways of doing this such as dimensional regularization or ζ-function regularization[4] but they all amount to the rather arbitrary removal of an infinite number of eigenvalues. However, there is one possible way in which a finite answer can be achieved without regularization. If fermion fields are present in the path integral, they can be handled in a rather similar way except that they have to be treated as anticommuting Grassmann variables.[5] Because of this, the path integral gives determinants of operators in the numerator rather than in the denominator as for boson fields. If there are equal numbers of fermion and boson spin states, leading divergences will cancel because P_0 is always proportional to the volume of the background metric. Such a correspondence in the number of boson and fermion fields is a feature of supersymmetric theories,[6] in particular supergravity. These divergences arising from the P_1 terms will cancel if the masses of the fields obey some relation, in particular, if they are all zero (as in supergravity) and the background metric has vanishing Ricci scalar. In this case the quadratic path integrals will be finite without any regularization or infinite factors.

Whether the divergences cancel or are removed by regularization, the term P_2 will in general be nonzero, even in the supergravity if the topology of the background metric is not trivial.[7] This is often said to correspond to a logarithmic divergence but this is misleading because it does not give rise to any divergence at all. What it means is that after cancellation or regularization of the terms arising from P_0 and P_1, one is left with some finite number P_2 (not necessarily an integer) of eigenvalues in the denominator (or in the numerator if P_2 is negative). Because the eigenvalues have dimension $(\text{length})^{-2}$, they have to be divided by the normalization constant μ^2 to get a dimensionless answer. Thus the path integral will depend on μ if P_2 is nonzero.

In Yang-Mills theory or quantum electrodynamics (QED) the quantity corresponding to P_2 is proportional to the action of the field. This means that one can absorb the μ dependence into an effective coupling constant $g(\kappa)$ which depends on the scale κ under consideration. If P_2 is positive, $g(\kappa)$ tends to zero logarithmically for short-length scales or high energies. This is known as asymptotic freedom.

In gravity, on the other hand, the μ dependence cannot be absorbed because P_2 is quadratic in the curvature whereas the usual action is linear. For this reason some people have suggested adding quadratic terms in the curvature to the action. However, such an action seems to have a number of undesirable properties and to have a classical limit which is not general relativity but a theory with fourth-order equations, negative energy and propagation outside the light cone.[8] Thus it seems that the μ dependence of the path integral cannot be removed. This may not be a disaster because, unlike Yang-Mills theory, gravity has a natural length scale, the Planck length $G^{1/2}$. It might therefore seem natural to take μ^{-1} to be some multiple of this length.

One can obtain the energy-momentum tensor for the ϕ field by functionally differentiating the regularized path integral over ϕ with respect to the background metric,

$$T^{ab}=2(g_0)^{-1/2}\frac{\delta \ln Z}{\delta g_{0ab}}\,.$$

This energy-momentum tensor will obey the conservation equations if and only if the normalization quantity μ is held fixed under the variation of the metric.

In the case where the background metric is the Euclidean section of the Schwarzschild solution the energy tensor can be regarded as representing thermal radiation at a temperature $T=\beta^{-1}$ confined to a box of radius r_0 and in equilibrium with the black hole at the same temperature. The energy-momentum tensor will be regular even at the horizon $r=2M$ despite the fact that the local temperature will be infinite because of an infinite blue-shift. Near the walls of the box one can decompose the energy-momentum tensor into an outgoing part and an ingoing part reflected off the walls of the box. To obtain the energy-momentum tensor appropriate to a black hole radiating into empty space without any box, one merely subtracts out the energy-momentum of the ingoing, reflected part. This will be regular on the future horizon so the energy-momentum tensor will also be regular there and will have a negative-energy flux into the black hole which balances the positive-energy flux of the thermal radiation at infinity, showing that a black hole will indeed lose mass as it radiates and that there is no reason to believe, as some have claimed, that the radiation prevents the formation of an event horizon in the gravitational collapse.

One might expect that the energy-momentum tensor of a conformally invariant field would have a zero trace. However, that cannot be true if P_2 is nonzero as can be seen by the following simple argument. Under the scale transformation $g_0 \to k^2 g_0$, the eigenvalues λ_n of the operator A will transform as $\lambda_n \to k^{-2}\lambda_n$. Because Z contains P_2 excess eigenvalues in the denominator, $\ln Z$ will increase by $P_2 \ln k$. But from the definition of the energy-momentum tensor,

$$\int T^a_{\ a}(g_0)^{1/2}d^4x=\frac{d}{dk}\ln Z\,.$$

Thus the integral of the trace of the energy-momentum tensor is equal to P_2. A more detailed calculation shows that it is pointwise equal to the integrand in the equation for P_2.[9-11]

V. BEYOND ONE LOOP

In a renormalizable theory such as Yang-Mills or ϕ^4 theory one can expand the action about the background field ϕ_0 in the form

$$\hat{I}[\phi]=\hat{I}[\phi_0]+I_2[\tilde{\phi}]+\lambda I_{\text{int}}[\tilde{\phi}]\,,$$

where λ is a coupling constant. For example, in ϕ^4 theory

$$I_2=\tfrac{1}{2}\int(\nabla\phi)^2d^4x$$

and

$$I_{\text{int}}=\int \phi^4 d^4x\,.$$

The path integral takes the form

$$Z = \exp(-\tilde{I}[\phi_0]) \int D[\phi] \exp(-I_2) \times \left(\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \lambda^n I_{\text{int}}{}^n \right).$$

In effect this means that one is evaluating path integrals of $(I_{\text{int}})^n$ with the measure $D[\phi]\exp(-I_2)$. This can be done because, for ϕ^4 theory, there is some constant C such that $I_{\text{int}} < C(I_2)^2$. In other words I_{int} is a measurable functional on the space of all fields ϕ with the Gaussian measure defined by $\exp(-I_2)$. Similarly, in Yang-Mills theory, the interaction part of the action is bounded by the square of the quadratic "free" part of the action. One would not have such a bound, in say, ϕ^6 theory. This is the reason why this theory is not renormalizable.

In gravity the interaction part of the action is an infinite power series in the metric and its derivatives. Thus it is not bounded by the quadratic term. One therefore cannot use the usual form of perturbation theory in quantum gravity. This is not surprising because in the classical theory we have always known that perturbation theory has only a limited range of validity: One cannot describe black holes as a perturbation of flat space. This means that the perturbation expansion has a zero radius of convergence in the quantum theory because one can always add small "virtual" black holes to any metric with an arbitrarily small increase in the action.

By considering conformal transformations $g' = \Omega^2 g$ one can see in detail at least one way in which perturbation theory breaks down. Under such a transformation the action $\hat{I}$ becomes

$$\hat{I}[g'] = -\frac{1}{16\pi G} \int_M (\Omega^2 R + 6\Omega_{,a}\Omega^{,a})(g)^{1/2} d^4x - \frac{1}{8\pi G} \int_{\partial M} [\Omega^2 K](h)^{1/2} d^3x .$$

One can decompose the space of all metrics which satisfy the boundary conditions into equivalence classes under conformal transformations where the conformal factor Ω is required to be one on the boundary. In each conformal equivalence class one can pick a metric g^* for which $R = 0$. One can then perform a path integration over the conformal factor about the metric g^*. Because the eigenvalues of these conformal transformations are negative, i.e., they reduce $\hat{I}$, one has to rotate the contour of integration so that one integrates over conformal factors of the form $\Omega = 1 + iy$, y real and $y = 0$ on the boundary. One then performs an integration over all metrics with $R = 0$.

Consider a one-parameter family $g(v)$ of metrics with $g(0) = g_0$, a solution of the Einstein equations. For small values of v the conformally invariant scalar wave operator $A = -\Box + \frac{1}{6}R$ will have no negative eigenvalues. This means that there will be a positive function ω with $\omega = 1$ on the boundary such that the metric $g^*(v) = \omega^2 g(v)$ has $R = 0$. It seems that, in asymptotically Euclidean metrics the action $\hat{I}[g^*]$ of these metrics will be positive and will increase away from the background metric g_0 (Ref. 12). Thus the contribution of such metrics will be damped.

As v increases, one or more of the positive eigenvalues of the operator A may pass through zero and become negative. As a function of v the action $\hat{I}[g^*]$ will have poles at the values $v_1, v_2, \ldots$ at which eigenvalues pass through zero. Beyond $v = v_1$, the conformal factor ω will pass through zero so that the metric g^* will be singular. However its action will still be well defined.

To perform the path integration over the metrics $g^*(v)$, one has to displace the contour of integration into the complex v plane to avoid the poles at $v = v_1, v_2, \ldots$. The path integral over the conformal factor $\Omega = 1 + iy$ about each metric $g^*(v)$ will contribute a factor of $(\det A)^{-1/2}$. As the number of negative eigenvalues of A increases, one would expect this to oscillate in sign and decrease. Thus one could hope that the path integral would converge.

With a family of metrics $g(v)$ that corresponded to a long-wavelength perturbation of the metric, a reasonable approximation to the integral of $\exp(-\hat{I})$ over v would be obtained by taking just the value of I and its second derivative at the background metric g_0. However, for perturbations on length scales shorter than the Planck length, the poles in $\hat{I}[g^*]$ will approach the background metric and will invalidate the stationary-phase approximation. Indeed one might expect that for very short length scales, the integral over v might be independent of the length scale and so provided a cutoff at less than the Planck length.

VI. THE GRAVITATIONAL VACUUM

What can one do about the fact that perturbation theory breaks down in quantum gravity? One possibility that I would like to suggest is that one replace the path integrals over the Taylor series about a single background metric by a discrete sum of the exponentials of the actions of all complex metrics that satisfy the Einstein equations with the given boundary conditions, each metric being weighted by its one-loop term. This procedure is closely analogous to that adopted in the statistical bootstrap model of elementary particles[13] where one takes into account the interactions between particles by introducing new species of particles (resonances) which are then treated as free particles.

There are, probably, an infinite number of complex solutions of the Einstein equations. However, one might hope that the dominant contribution came from just a finite number of them. To illustrate how this might happen I shall consider the gravitational vacuum. This is not a pure quantum state but a density matrix for the microcanonical ensemble at zero energy. One obtains $n(0)$, the density of states at zero energy, by integrating the partition function $Z(\beta)$ over all β.

A single black hole will contribute $W_1 \exp(-\beta^2/16\pi)$ to the partition function where W_1 is the one-loop term for the Schwarzschild metric. There are probably no real positive-definite metrics which represent two or more black holes because they would attract each other and merge into a single black hole. However, one might be able to find a slightly complex solution which corresponded to the possibility of having two black holes in the box. Alternatively, one might represent several black holes by the self-dual multi Taub–Newman-Unti-Tamburino (NUT) solution.[14] In this the attraction between the ordinary "electric"-type mass of the black holes is balanced by the repulsion between the imaginary "magnetic" or NUT type mass.

One would expect the action of an N-black-hole solution to be something like $-N\beta^2/16\pi$, indeed it is exactly that in the multi Taub-NUT case.[15] An N-black-hole metric would be expected to have $3N$ zero eigenvalues corresponding to the possibility of putting the black holes anywhere in the box. These eigenvalues will give a factor proportional to $(\mu^3 V)^N$ where V is the volume of the box. One will have to divide this by $N!$ because the black holes are identical. Thus the dominant contribution will come from N of the order of $\mu^3 V$. Taking μ^{-1} to be of the order of the Planck length, one sees that one gets one black hole per Planck volume.

To estimate the mass of the black holes that give the dominant contribution one has to find the maximum of $W \exp(-\hat{I})$ as a function of β.

From the scaling behavior one finds that if P_2 is positive, the maximum occurs for a β of order one or a mass of the order of a Planck mass. Thus one has a picture of the gravitational vacuum as a sea of Planck-mass black holes. Particles such as baryons or muons could fall into these black holes and come out as different particles, thus providing a gravitational violation of baryon- and muon-number conservation. However, it seems that the rate would be very low.

On a larger scale, one can think of the gravitational collapse of a star as merely enlarging one of the Planck-mass black holes already present in the vacuum. This large black hole would radiate thermally and would eventually shrink back to a Planck-mass black hole indistinguishable from the others in the vacuum. This picture avoids the difficulties that would arise from the singularities that would necessarily occur on the Euclidean section if black holes were created or destroyed.

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