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Quark-antiquark interaction at all momentum transfers

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The running coupling constant of quantum chromodynamics $\alpha_s(q^2)$ is obtained for all q^2 by integrating the renormalization-group equation. The β function in the asymptotic-freedom region is given by perturbation theory through order g^5 and at large coupling by the string model with string tension related to the Regge slope α'. We incorporate these features into a Padé approximant to the β function thereby obtaining it for all g. The constant of integration Λ of the renormalization-group equation is chosen to be about 500 MeV, as required by deep-inelastic phenomenology. α_s is thus completely determined by α', Λ, and the first two terms of the β function. The resulting charmonium and Υ spectroscopy is in excellent agreement with experiment. In particular the Υ'-Υ mass difference is forced to be nearly equal to the ψ'-ψ mass difference. In addition, it is shown that the structure of $\alpha_s(q)$ signals the presence of instantons.

I. INTRODUCTION

Quantum chromodynamics (QCD) has become a serious possibility as a theory for strong interactions. A large number of studies have associated QCD with diverse theoretical and phenomenological developments. In this paper we are concerned with the unity of a number of these developments. This unity is approached by a construction of an expression for $\alpha_s(q^2)$, the strong coupling constant of QCD, as a function of the momentum transfer.

The principal theoretical ingredients in our construction of α_s are a renormalization-group method incorporating asymptotic freedom, string-model confinement, and the analytic structure of the Callan-Symanzik function $\beta(\alpha_s)$. A single expression for α_s applies to deep-inelastic scattering, Regge behavior, and heavy-quark meson spectroscopy. The structure of $\beta(\alpha_s)$ is related to instanton physics.

The construction of $\alpha_s(q^2)$, carried out in Sec. II, proceeds as follows: The short-distance behavior can be calculated in perturbation theory by virtue of asymptotic freedom. The string model, which is accommodated in lattice gauge theory, relates the large-distance behavior of α_s to the Regge slope α'. α_s at all scales is then determined by simple analyticity assumptions about $\beta(\alpha_s)$ defined by

$$\frac{dg}{d(\ln q)}=\beta(\alpha_s),$$

where $\alpha_s(q^2)=g^2/4\pi$. With those assumptions we can find a Padé approximant (to β) which is constrained by the small and large distance behavior of α_s. We finally integrate the β-function equation to find α_s for all q^2. The only remaining free parameter becomes fixed once we specify the value of the scale-fixing parameter Λ, which arises as the constant of integration and is measured in deep-inelastic scattering experiments.

We will, then, have determined $\beta(\alpha_s)$ and $\alpha_s(q^2)$. In Sec. III we examine some of the properties of those functions and among other things, discover that β has structure signaled by a pole in its Padé approximant $\alpha_s/2\pi\approx-0.08$. This pole might be expected to be of order unity, since $\alpha_s/2\pi$ is generally regarded as an expansion parameter. Our explanation of this low value is that, in fact, field theory *does* have structure at such a low value of $\alpha_s/2\pi$—namely the structure induced by instantons. Callan, Dashen, and Gross[1] have estimated that instantons become important in the range $\alpha_s/2\pi\sim\frac{1}{15}$ to $\frac{1}{10}$. Our results, then, support the idea that linear confinement (which, after all, is responsible for the small pole of β) is an effect related to the existence of instantons.

In Sec. IV we apply our results to meson spectroscopy. The charmonium and Υ systems provide important tests of the quark-antiquark interaction. Only for these dominantly nonrelativistic systems can the energy levels be calculated relatively unambiguously. Even in the charmonium system, the relativistic effects are significant (for instance, the χ states,[2] degenerate nonrelativistically, are split by about 150 MeV). The relativistic corrections are dependent on such things as an anomalous color magnetic moment and as a result relativistic effects tend to obscure the form of $\alpha_s(q^2)$. For this reason, we need as much data as possible from heavy-quark mesons, and the Υ system helps to fulfil this need. In our model of the $q\bar{q}$ interaction, we find that $m(\Upsilon')-m(\Upsilon)\sim m(\psi')-m(\psi)\simeq0.58$ GeV.[2] This is to be compared[3] with the early predictions of Eichten and Gottfried[4] that $m(\Upsilon')-m(\Upsilon)\sim0.43$ GeV. Thus our form of $\alpha_s(q^2)$ not only predicts correctly the value of $\psi'-\psi$ (within the limitations of relativistic corrections, indeterminacy of Λ, etc., all of which will be discussed in Sec. IV) but it agrees with the measured value,[5] $\Upsilon' - \Upsilon \sim 0.60$ GeV. We will conclude by showing that as the quark mass increases beyond $m_q = 8$ GeV, the quarkonium energy spacings increase, rather than remain constant as would be the case if the potential were logarithmic.[6]

II. THE CONSTRUCTION OF $\alpha_s(q^2)$

Our construction of the strong coupling constant of QCD $\alpha_s(q^2)$ will rest on the assumption that quark-antiquark binding can be described by the interaction kernel

$$K(q^2) = -\frac{4}{3}\gamma^{(1)\mu}\gamma^{(2)\nu}P_{\mu\nu}\frac{\alpha_s(q^2)}{q^2}, \tag{1}$$

where $P_{\mu\nu}$ is the dimensionless propagation matrix (for instance, in Feynman gauge, $P_{\mu\nu} = g_{\mu\nu}$), and $-\frac{4}{3}$ is the SU(3)$_c$ factor. This assumption is true in the static limit through order α_s^2 (as was shown by Appelquist et al., Feinberg, and Fischler[7] (in independent calculations), the order which suffices for our calculation below. If only two-body quark forces exist, the confining potential must be octet exchange in order to have no residual confining force between color singlets. For large α_s (small q^2) it may be that some color-singlet exchange is present in the kernel, as well as anomalous magnetic moment and nonvector-vector pieces. However, we assume that these effects do not, to lowest order in $(v/c)^2$, affect the smoothness assumptions that we will make on the β function.

In developing an expression for $\alpha_s(q^2)$ we incorporate as many of the current theoretical ideas of QCD as we are able. Our purpose is to test the whole fabric of these ideas, so we include speculative features along with established ones.

Our construction is based on the renormalization-group equation[8]

$$\frac{dg}{d\ln q} = \beta(g),$$

where $\alpha_s(q^2) = g^2/4\pi$. The first two terms in the power-series expansion of β

$$\beta(g) = \beta_3 g^3 + \beta_5 g^5 + \cdots$$

have been calculated.[9] These coefficients are functions of the number of flavors. The effective number of flavors is that for which the corresponding quark masses are not heavier than the energy scale of interest.[10] For our purposes, this number is three, corresponding to up, down, and strange quarks. We neglect loops of charmed or heavier quarks. The calculated values are

$$\beta_3 = -\frac{9}{2}\left(\frac{1}{8\pi^2}\right),$$

$$\beta_5 = -16\left(\frac{1}{8\pi^2}\right).$$

It is well known[11] that these values are independent of the definition of α_s (see Appendix A). In terms of the natural expansion parameter[12] $\alpha_s/2\pi = g^2/8\pi^2$, the renormalization-group equation can be rewritten as

$$\frac{d\left(\frac{\alpha_s}{2\pi}\right)}{d\ln q^2} = \frac{\alpha_s}{2\pi}\frac{\beta}{g}$$

$$= -\frac{9}{2}\left(\frac{\alpha_s}{2\pi}\right)^2 - 16\left(\frac{\alpha_s}{2\pi}\right)^3 + \cdots .$$

Two features to notice in this expression, which become important in our later discussion, are that both of the first two terms have the same sign, and that the coefficients are significantly larger than unity.

A common procedure[9] for estimating α_s is to truncate the power series for β and integrate the renormalization-group equation. This procedure only gives reasonable results for small values of $\alpha_s/2\pi$. We wish to do better. It might be guessed that a Padé approximant is much better than a truncated power series. The validity of this guess rests on the analytic and asymptotic structure of β.

The most reasonable analytic structure of β is the familiar one from perturbation theory: a branch point in β/g at $\alpha_s/2\pi = 0$, with a cut along the negative real axis. Since an asymptotic series exists, the discontinuity of the cut starts more slowly than any power of $\alpha_s/2\pi$ at the branch point. Thus, a rational function, with the poles along or near the negative real axis, can be expected to represent the structure well. This, then, is our Padé representation. (If the analytic structure were somewhat more complicated, a Padé representation could still be good, if, for example, the singularities have negative real parts.)

In order to obtain the (large α_s) asymptotic behavior of β, we draw our cue from lattice gauge theory.[13] It has been shown[14] that (on a lattice), quarks are confined by a "string" of color flux. The energy of this $q\bar{q}$ system increases linearly with separation at large distances. Thus the interaction $K(q^2)$ of Eq. (1) implies the r-space (large-r) potential $V \propto r$. Fourier transforming, we find

$$\tilde{V}(q^2) = \frac{\alpha_s}{q^2} \propto \frac{1}{q^4}, \text{ or } \alpha_s = \frac{K_0}{q^2}.$$

This behavior occurs as $q^2 \to 0$, $\alpha_s \to \infty$. Differentiating to find β,

$$\frac{d(\alpha_s/2\pi)}{d\ln q^2} = \frac{\alpha_s}{2\pi}\frac{\beta}{g} = -\frac{K_0/2\pi}{q^2} = -\frac{\alpha_s}{2\pi},$$

so we find $\beta/g \to -1$ as $g \to \infty$. Not only is the asymptotic power of β found, but so is the coefficient. We obtain a further constraint by observing that the "string tension," which we have not yet used is obtained from the Regge slope α', and we derive[15] in Appendix B that

$$\lim_{r\to\infty} V(r) = r/(2\pi\alpha').$$

The constant limiting value of β/g suggests the appropriateness of an (n,n) Padé approximant. Since we know three terms in the power series

$$\beta/g = 0 - \tfrac{9}{2}(\alpha_s/2\pi) - 16(\alpha_s/2\pi)^2 + \cdots,$$

a (1,1) Padé approximant

$$\beta/g \approx -\frac{9}{2}\frac{\alpha_s}{2\pi} \bigg/ \left(1 - \frac{32}{9}\frac{\alpha_s}{2\pi}\right)$$

can be made. Such a form, however, does not approach -1 at infinity. Therefore, we extend the approximation to a (2,2) rational fraction approximation. The two remaining coefficients are determined by forcing the limit to be -1, and obtaining the correct slope value. The first of these coefficients is easily determined, but the second requires some work. We express this one remaining parameter as α_0 in the following expression:

$$\beta/g \approx \frac{-81\dfrac{\alpha_s}{2\pi}\left(\dfrac{\alpha_s}{2\pi} + \dfrac{\alpha_0}{2\pi}\right)}{81\left(\dfrac{\alpha_s}{2\pi}\right)^2 + \left(18 - 64\dfrac{\alpha_0}{2\pi}\right)\dfrac{\alpha_s}{2\pi} + 18\left(\dfrac{\alpha_0}{2\pi}\right)}. \tag{2}$$

To determine α_0 (in terms of the Regge slope) we must integrate the renormalization-group equation and express $\alpha_s(q^2)$ in terms of $V(r)$ [related to the Fourier transform of $\alpha_s(q^2)$].

The renormalization-group equation is integrated with β written as in Eq. (2) to give

$$\ln(q^2/\Lambda^2) = \frac{2}{9}\frac{2\pi}{\alpha_s} - \ln\left(\frac{\alpha_s}{\alpha_0}\right) - \frac{145}{81}\ln\left(\frac{1+\alpha_0/\alpha_s}{2}\right). \tag{3}$$

Λ is the integration constant which fixes the scale of QCD and is determined by data from deep-inelastic scattering experiments. This will be discussed later. We express α_s as a function of q^2, carry out the appropriate Fourier transform and finally determine α_0 as a function of α' (and Λ). Details of this calculation are to be found in Appendix C. The final result is

$$\alpha_0 = \frac{3\times 2^{-145/81}}{4\pi\alpha'\Lambda^2}. \tag{4}$$

With this value for α_0, we can (numerically) solve Eq. (3) to find $\alpha_s(q^2)$. The only parameters we need specify are the parameters Λ and α'.

α', the Regge slope, is taken from the ρ-f-g-h trajectory[16] to be 0.9 GeV^{-2}. Λ is chosen to be approximately 500 MeV,[17] although there is not yet sufficient data to determine Λ very accurately. Actually, it is consistent with data to choose $400 \le \Lambda \le 600$ MeV and in Sec. IV we show meson spectroscopy results using both $\Lambda \approx 550$ MeV, and $\Lambda \approx 420$ MeV (corresponding to the choices of $\alpha_0/2\pi = 0.040$ and 0.070).

We will now take, as an example, $\Lambda \approx 550$ MeV and will show that our definition of Λ is consistent with the definition used by those who do deep-inelastic phenomenology. They[17] define Λ_D by integrating the renormalization-group equation with β expanded only as far as the first nontrivial term

$$\ln(q^2/\Lambda_D) = \frac{2}{9}\frac{2\pi}{\alpha_s}. \tag{5}$$

We recognize the right-hand side as the first term of Eq. (3). We use Eq. (5) to define the function $\Lambda_D(q^2)$. From Eq. (3) we can then find $\Lambda_D(q^2)$ in terms of Λ. With our parameters $\alpha' = 0.9\ \text{GeV}^{-2}$ and $\Lambda = 550$ MeV we draw $\Lambda_D(q^2)$ in Fig. 1. We see that at large q^2 (where the deep-inelastic experiments are done), Λ_D is approximately flat with value 0.55. Our definition of Λ is thus consistent with that of deep-inelastic phenomenology. It is interesting to note that if the right-hand side of Eq. (3) were simply an expansion of β through order g^6, the resulting graph of $\Lambda_D(q^2)$ would be less flat (at high q^2) that that of Fig. 1.

With our chosen (experimental) values of α' and Λ, we now proceed to study the structure of β.

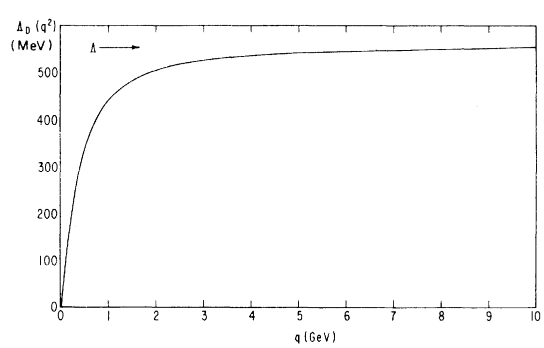

FIG. 1. The effective value of Λ, defined by $\alpha_s(q^2) = 4\pi/9 \ln[q^2/\Lambda_D{}^2(q^2)]$. $\Lambda_D(q^2)$ is nearly constant and equal to our integration constant Λ over the deep-inelastic range.

III. THE STRUCTURE OF β

We have seen that

$$\beta(\alpha) = -\tfrac{9}{2}\,(\alpha_s/2\pi) - 16(\alpha_s/2\pi)^2 + \cdots .$$

This expression has the following properties: (1) the coefficients in β/g of powers of $\alpha_s/2\pi$ are considerably larger than unity, (2) the first two coefficients are both negative, and (3) the Padé approximant to β has a pole at $\alpha_s/2\pi = -\alpha_0/2\pi \approx -0.08$, implying structure on this scale in the real β. This might be considered a very small value of $\alpha_s/2\pi$ for the β function to have structure (notice that the first zero of β is at -0.04, also of very small structure). We consider these three features to be closely related to each other and to the existence of instantons.

The large values of the power-series coefficients suggest structure on a small scale. Each of the first two terms becomes unity at $\alpha_s/2\pi \approx 0.25$, only a factor of ~3 larger than the pole of β. Thus the first two terms already seem to be indicating the structure.

The terms in perturbation theory, especially the high orders, generally oscillate in sign. Their having the same sign (in the series for the energy levels, at least) is an indication[12] of the possibility of "tunnelling." It is reasonable that the convergence properties of the series for the β function are the same as for the energy; and especially, since the first two terms seem to indicate the structure, the pattern of signs may start at the beginning of the series. This suggests that the observed structure be associated with tunnelling.

If the second term had been positive, the Padé approximation including the linear confinement would have led to a β similar to that which we actually obtained, but without the pole (and zero) at small negative $\alpha_s/2\pi$.

There is, in fact, tunnelling in QCD, as is well known.[1] This tunnelling is related to the instanton configurations of the classical Euclidean QCD field theory. Callan, Dashen, and Gross[1] have estimated that instantons become important in the range $\alpha_s/2\pi \approx \frac{1}{15}$ to $\frac{1}{10}$, exactly the typical size over which we find structure in the β function.

Since they use a different approximation to $\alpha_s/2\pi$ than our form, it is important to check the compatibility of our expression and their estimate. We do this by calculating the most likely scale for an instanton. The amplitude for an instanton is proportional to[18]

$$W\frac{dr}{r}\,d^4x = \frac{1}{r^4\alpha_s{}^6}\,e^{-2\pi/\alpha_s}\,\frac{dr}{r}\,d^4x\ ,$$

where r is the size of the instanton. Since we express α_s in momentum space, we maximize the expression

$$\frac{q^4}{\alpha_s{}^6}\,e^{-2\pi/\alpha_s}\ ,$$

with $\alpha_s = \alpha_s(q^2)$, where q is the typical momentum of the instanton. Using our expression for α_s, we find the maximum occurs at $\alpha_s/2\pi = 0.08$, on the same scale as our structure, and the same scale as the estimate of Callan *et al.* The smallness of this value has to do with the large amount of phase space available to the instanton. [The same calculation using $\alpha_s/2\pi = \frac{2}{9}\ln(q^2/\Lambda^2)$, as Callan *et al.* do, leads to a maximum at $\alpha_s(\mathrm{CDG})/2\pi = 0.09$. For smaller α_s both our definitions of α_s are in close agreement, hence their calculation of the *onset* of instantons should not be much different if they were to use our α_s.]

Thus the various aspects of the structure in our expression are consistent with the interpretation of that structure as reflecting instantons. Since this structure is most important for charmonium and Υ spectroscopy, we conclude that instantons are important in the structure of these states.

IV. MESON SPECTROSCOPY

Before we can apply our results to meson spectroscopy we must convert our expression for α_s to a potential V. To do this we make an expansion of $K(q^2)$ [from Eq. (1)] in powers of v^2/c^2, where v is the quark velocity. To lowest order this interaction is instantaneous—that is, it can be expressed as a function of the three-momentum $\vec{q}^{\,2}$. For a special choice of gauge,[19] the first-order term [of Eq. (1)] is also instantaneous. This expansion of K will be called $\tilde{V}(\vec{q}^{\,2})$.

Then we get the potential V to lowest order, by simply Fourier-transforming $\tilde{V}$,

$$V(r) \propto \int d^3q\, e^{i\vec{q}\cdot\vec{r}}\,\tilde{V}(\vec{q}^{\,2}).$$

In Appendix B we discuss the details of this transformation and in particular we point out that there is an ambiguity created by the singularity of $\tilde{V}$ at $\vec{q}^{\,2} = 0$. This ambiguity is reflected in an arbitrary constant V_0, which is to be added to V and is chosen, for instance, to set the mass of ψ. (Note that although V_0 is present in all potential models, it is not generally regarded as a parameter. A change of δV_0 accompanied by a change in all heavy-quark masses of $-\delta V_0/2$ has little effect on the spectrum.)

The order v^2/c^2 term in the potential is taken to be the generalization of the Fermi-Breit perturbation. We calculate[19, 20]

$$\delta V = \frac{3}{2m^2}\frac{1}{r}\frac{dV}{dr}(\vec{s}_1+\vec{s}_2)\cdot\vec{L} + \frac{2}{3m^2}\vec{s}_1\cdot\vec{s}_2\nabla^2 V - \frac{1}{3m^2}[3(\vec{s}_1\cdot\hat{r})(\vec{s}_2\cdot\hat{r}) - (\vec{s}_1\cdot\vec{s}_2)]\left(\frac{d^2V}{dr^2}-\frac{1}{r}\frac{dV}{dr}\right)$$

$$+\frac{1}{4m^2}\left[2p_i\left(V-r\frac{dV}{dr}\right)p_i\right]+\frac{1}{2}\nabla^2\left(3V+r\frac{dV}{dr}\right)+2\frac{dV}{dr}\vec{L}^2-\frac{p^4}{4m^3},$$

where V is the potential and m is the quark mass. We now use our potential to calculate the nonrelativistic quark bound state.

As mentioned before, the value of Λ measured in deep-inelastic scattering is quoted in various references as being anywhere between 400 and 600 MeV (in fact, the range of values is even greater than that, but the values we mention seem to dominate the literature). In this section we will see that the value of Λ that we need in order to agree with charmonium spectroscopy (i.e., $\psi'-\psi = 0.586^2$) must lie in the above range. This result is very important. It shows a unity between two apparently unrelated pieces of phenomenology—deep-inelastic scattering on the one hand, and meson spectroscopy on the other.

Regarding the precise fitting of data, a word of caution is necessary at this point. Unless we know the precise structure of the vertex function [γ^μ in Eq. (1)] we cannot accurately fix any parameters (that is, if we allow Λ to become a free parameter, not constrained by deep-inelastic phenomenology). The reason is, of course, that v^2/c^2 corrections to the potential can cause energy shifts of as much as 200 MeV. Without a model for the vertex we can make at best a crude fit of Λ. Furthermore, as we will mention shortly, even if we assume, as we have done, that γ^μ is the vertex, the v^4/c^4 corrections are not likely to be sufficiently small to trust the perturbation. These remarks will be illustrated in what follows by giving the results for the two cases where (a) the zeroth-order potential (no v^2/c^2 corrections) is used, and (b) an expansion is made to order v^2/c^2 (so we use the potential $V+\delta V$). As discussed below, the spectrum is insensitive to m_c; we fix it to be $m_c=2.0$ GeV. For this example, we choose $\alpha_0/2\pi=0.040$ ($\Lambda\cong 550$ MeV), and $m_b=5.35$ GeV. V_0 is chosen separately for the two cases [(a) and (b)] so that $\psi=3.100$ GeV. The potential is shown in Fig. 2.[21] (In what follows we use particle names and masses interchangeably.) For case (a) (zeroth order),

$$\psi=3.100,\quad \Upsilon=9.359,$$
$$\psi'=3.785,\quad \Upsilon'=10.039,$$
$$\psi''=4.141,\quad \Upsilon''=10.340$$

(in GeV). In case (b) [$O(v^2/c^2)$] the values are

$$\psi=3.100,\quad \Upsilon=9.400,$$
$$\psi'=3.689,\quad \Upsilon'=10.068,$$
$$\psi''=4.081,\quad \Upsilon''=10.452.$$

Those are to be compared with the experimental values[2,5]

	Three-peak fit[5]	Two-peak fit[5]
$\psi=3.098\pm0.003$	$\Upsilon=9.40\pm0.02$	$\Upsilon=9.41\pm0.02$
$\psi'=3.684\pm0.004$	$\Upsilon'=9.99\pm0.05$	$\Upsilon'=10.06\pm0.04$
	$\Upsilon''=10.41\pm0.12$	

We point out some interesting features. In the unperturbed case we find that $\Upsilon'-\Upsilon\approx\psi'-\psi$. As we shall see shortly, this relationship (the equality of the mass difference between ground state and first radially excited state) holds over a large range of quark masses and, in particular, implies that the value of $\psi'-\psi$ is not sensitive to our choice of m_c. Hence Λ also does not depend on the choice of m_c. Furthermore, we have demonstrated the result $\Upsilon-\Upsilon\simeq\psi'-\psi$ without recourse to the ansatz logarithmic potential proposed by Quigg and Rosner.[6] Our result differs from the early prediction of Eichten and Gottfried[4] which gives $\Upsilon'-\Upsilon\simeq0.42$; $\Upsilon''-\Upsilon\sim0.33$. In Fig. 2 we show our potential and superpose the Υ and

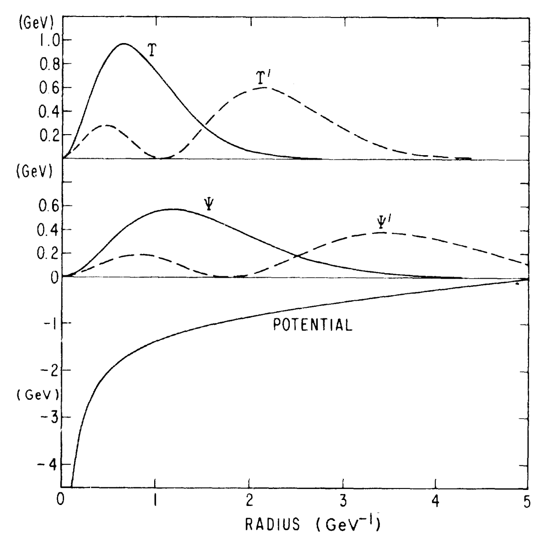

FIG. 2. The potential $V(r)$ and $4\pi r^2|\Psi(r)|^2$ for the ψ, ψ', Υ, and Υ'. Λ is chosen to be ~ 550 MeV ($\alpha_0/2\pi = 0.04$).

charmonium wave functions. It is seen that the wave functions are mostly supported in the "intermediate" region of the potential.

The results quoted are not particularly sensitive to the value of Λ. To see this we compare the energy levels for charmonium and Υ in the case that $\alpha_0/2\pi = 0.070$ ($\Lambda \cong 420$ MeV). In that case (for the zeroth-order potential with the same masses as before), $\psi' - \psi = 0.624$ and $\Upsilon' - \Upsilon = 0.600$. Once again, the approximate equality in energy spacings is seen to hold true.

We must still address the question of the "perturbed" values for the mass differences [case (b)]. In that case it appears that $\Upsilon' - \Upsilon$ is slightly larger than $\psi' - \psi$ (and, in fact, tends to support the two-bump fit[5]). However, there is good reason to suspect that the difference is due to an overestimate of the effect of the perturbation. In fact, the "Fermi-Breit" perturbation of ψ involves highly singular terms (such as a δ-function). This suggests that higher-order v^2/c^2 corrections are large and act towards suppressing the perturbation. It turns out that we compute $\psi - \eta_c$ (using our expression for δV) to be 220 MeV, whereas it is likely that the experimental value is less than 100 MeV (we prefer to reject the possibility that $X(2.83)$[22] is the η_c. There are a number of well-known difficulties with that interpretation.[23]) Furthermore, we compute ${}^3P_2 - {}^3P_0 = 270$ MeV as compared to the experimental difference of corresponding χ states[2] (~135 MeV). This too makes us believe that the computed relativistic corrections are a large overestimation.

We return to the mass differences of states A, A', A'', etc., where A's are $L=0$ $q\bar{q}$ mesons, and the primes denote modes of radial excitation. As we have seen, in the model of Eichten and Gottfried, $A' - A$ decreases with quark mass (at least up to $m_q = 6$ GeV) whereas, in a logarithmic potential,[6] $A' - A$ remains constant. We list, in Table I our values of $A' - A$ and $A'' - A'$ as functions of quark mass. We find that although for $2.0 \lesssim m_q \lesssim 8.0$ the difference remains almost constant, it begins to rise for heavier quarks. This behavior can be attributed to the fact that, as the quark mass increases, the wave function "spends more time" in the Coulomb-type region of the potential. For a pure Coulomb potential, $A' - A$ is proportional to m_q, so (up to logarithmic modifications) that is the behavior we are to expect for our potential at large quark masses.

V. SUMMARY

From theoretical considerations, we have found a rational function which we believe is a good approximation to the QCD β function. From the β function we find $\alpha_s(q^2)$, which is parametrized by two quantities taken from phenomenology other than heavy-quark meson spectroscopy, one from the Regge slope and the other from deep-inelastic scattering data.

This construction of β has led to the following successes: (1) the constants Λ and α' suffice to describe the ψ system, (2) we are forced to obtain $\Upsilon' - \Upsilon \simeq \psi' - \psi$, (3) we have succeeded in relating the structure of the β function to instanton physics.

Finally we have made predictions of the energy spacings to be expected from very-heavy-quark bound states.

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APPENDIX A. UNIVERSALITY OF α_s

The question arises as to whether the calculated values of the β function are applicable to α_s as defined in our Eq. (1). What we show here is that β_3 and β_5 are independent of the definition of α_s.

Let

$$g^A = g^B + a(g^B)^3 + \cdots$$

TABLE I. Energy spacings as a function of quark mass. A denotes $q\bar{q}$ ($l=0$) ground state. A' and A'' are the first and second radial excitations of A. The calculations are done with $\alpha_0/2\pi = 0.040$ and energy levels are calculated using (i) V (zeroth order in v^2/c^2) and (ii) $V + \delta V$ (first order in v^2/c^2). All quantities are quoted in units of GeV.

m_q	$A'-A$ (zeroth order)	$A'-A$ (first order)	$A''-A'$ (zeroth order)	$A''-A'$ (first order)
2.0	0.685	0.589	0.456	0.392
5.35	0.680	0.668	0.391	0.384
8.0	0.708	0.712	0.383	0.379
12.0	0.760	0.774	0.377	0.381
16.0	0.812	0.832	0.381	0.387
25.0	0.928	$\cdots$	$\cdots$	$\cdots$

and $t=\ln q$. Both g^A and g^B are defined as coupling constants of QCD but may differ in their definition.

$$\frac{dg^A}{dt}=\frac{dg^B}{dt}+3a(g^B)^2\frac{dg^B}{dt}+\cdots$$
$$=\beta_3^B(g^B)^3+\beta_5^B(g^B)^5+3a\beta_3^B(g^B)^5+O((g^B)^7),$$

but the left-hand side is $\mathrm{LHS}=\beta_3^A(g^A)^3+\beta_5^A(g^A)^5$. Upon expanding g^A in terms of g^B, this becomes

$$\mathrm{LHS}=\beta_3^A(g^B)^3+3a\beta_3^A(g^B)^5+\beta_5^A(g^B)^5+O((g^B)^7)\ .$$

Equating coefficients of g^B we find that $\beta_3^A=\beta_3^B$ and $\beta_5^A=\beta_5^B$.

Carrying out this procedure to one more order it is easy to see that, in general, $\beta_7^A\neq\beta_7^B$.

APPENDIX B. THE "STRING PICTURE" AND THE SLOPE OF THE REGGE TRAJECTORY

We consider a classical massless relativistic string whose ends travel at the speed of light (set $c=1$). If k is the energy per unit length (string tension) then we show that $E^2=2\pi kJ$.

Let the string be of length $2L$ and consider a piece of string a distance l from the center and having length dl. Its motion is perpendicular to its length, hence the effective energy of this piece, whose rest mass id $dm=kdl$, becomes $dm'=k\gamma dl$, where $\gamma=[1/(1-\omega^2l^2)]^{1/2}$ (ω is the angular velocity). Thus

$$E=k\int_{-L}^{L}dl\left(\frac{1}{1-\omega^2l^2}\right)^{1/2}=\frac{k}{\omega}\pi\ .$$

Similarly,

$$J=\int_{-L}^{L}(lp)dl=\omega k\int_{-L}^{L}\frac{l^2}{(1-\omega^2l^2)^{1/2}}\,dl$$
$$=\frac{k}{\omega^2}\,\frac{\pi}{2}\,.$$

From this, $E^2/J=2\pi k$.

The connection between this string model and quantum mechanics is made in the following way. In light mesons, the high-angular-momentum states have large velocities and so can be expected to be relativistic. Rather than simply alter the kinetic energy (which is the natural temptation) we consider the classical relativistic string and assume that for high n (radial excitation number) and l the correspondence principle holds. If so, experiment dictates that $2\pi k=1/\alpha'$, where α' is the Regge slope. Hence k is known. The final step is to apply the correspondence principle to the nonrelativistic string. Since k is known, so is the potential.

APPENDIX C. TAKING THE FOURIER TRANSFORM OF $\alpha_s(q^2)$–DETAILS OF CALCULATIONS AND CONSIDERATION OF V_0

The Fourier transform is

$$V(r)\propto\int d^3q\,e^{i\vec{q}\cdot\vec{r}}\,\tilde{V}(\vec{q}^{\,2})$$
$$\propto\int dq\,\frac{\sin qr}{qr}\left[\frac{4}{3}\,\alpha_s(q^2)\right].$$

With the correct normalization,

$$V(r)=-\frac{8}{3\pi}\int dq\,\alpha_s(q^2)\,\frac{\sin qr}{qr}\ . \tag{C1}$$

The potential corresponding to a pure power $\alpha_s=a_nq^n$ is

$$V(r)=-\frac{4}{3}\frac{a_n}{r^{n+1}}\,\frac{1}{\Gamma(1-n)\cos\frac{n\pi}{2}}\ . \tag{C2}$$

As an example (to check normalization for instance), let $n=0$, the Coulomb case. Then

$$V(r)=-\frac{4}{3}\frac{a_0}{r}\ .$$

For $n=-2$, the confining case, the result is

$$V(r)=\tfrac{2}{3}a_{-2}r\ .$$

Note that, with the same sign of $\alpha_s(q^2)$, the Coulomb and confining potentials have opposite signs.

We used the above form of the linear potential when we found, in the text, the relationship between the parameters α_0, α', and Λ. The argument is this. Equation (3) of the text is

$$\ln\left(\frac{q^2}{\Lambda^2}\right)=\frac{2}{9}\frac{2\pi}{\alpha_s}-\ln\left(\frac{\alpha_s}{\alpha_0}\right)-\frac{145}{81}\ln\left(\frac{1+\alpha_0/\alpha_s}{2}\right).$$

At large α_s, $\alpha_s=2^{145/81}\,\alpha_0\,\Lambda^2/q^2$. The potential due to the tension of a string giving Regge slope α' is $V=r/2\pi\alpha'$. Then, by the above Fourier transform of the linear potential, this gives $\alpha_s=3/4\pi\alpha'q^2$. Identifying this with the previous expression, we obtain

$$\alpha_0=\frac{3\times2^{-145/81}}{4\pi\alpha'\Lambda^2}\ ,$$

as quoted in the text.

Let us return to the Fourier transform and Eq. (C2). For $n=-1$, the expression for $V(r)$ has a pole. Therefore, a term $\lim_{\epsilon\to0}(\epsilon/q^{1-\epsilon})$ gives a nonvanishing constant part of the potential. This limiting form actually corresponds to $\alpha_s(q^2)\propto q^2\delta^3(\vec{q})$. Such a singularity is weaker than the confining singularity $\alpha_s(\vec{q}^{\,2})\propto1/q^2$. Therefore, in doing a q integral, for example, in the Fourier transform, there is an ambiguity due to the possibility of adding an arbitrary amount of the weaker singularity. In position space this corresponds to the potential $V(r)$ being defined only up to a constant. This constant has physical effects even though it is arbitrary (for instance, it contributes to the binding energy) and its value is determined by fitting the charmonium spectroscopy.

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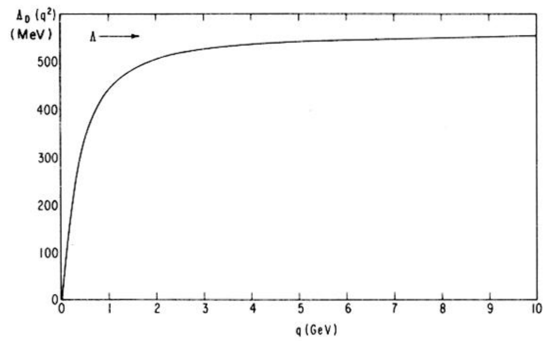

FIG. 1. The effective value of Λ, defined by $\alpha_s(q^2) = 4\pi/9 \ln[q^2/\Lambda_D{}^2(q^2)]$. $\Lambda_D(q^2)$ is nearly constant and equal to our integration constant Λ over the deep-inelastic range.

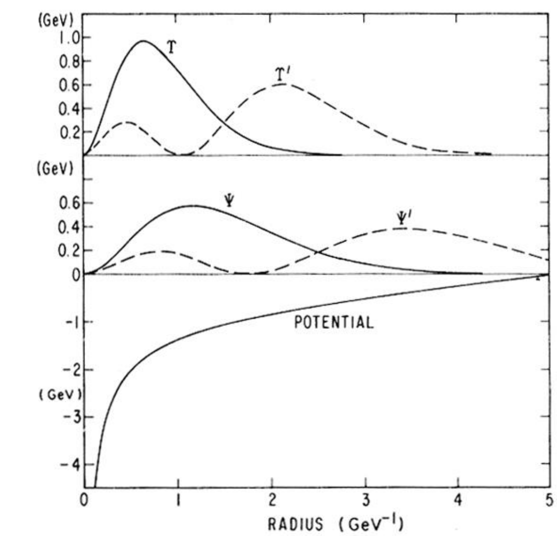

FIG. 2. The potential $V(r)$ and $4\pi r^2|\Psi(r)|^2$ for the ψ, ψ', Υ, and Υ'. Λ is chosen to be ~ 550 MeV $(\alpha_0/2\pi = 0.04)$.