PHYSICAL REVIEW D VOLUME 18, NUMBER 4 15 AUGUST 1978

Use of chiral symmetry in pion-condensation calculations

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(Received 6 June 1977)

It is demonstrated that, contrary to the assumptions of a number of authors, results obtained on pion condensation in the σ model are not characteristic of general chiral models.

Several articles on pion condensation in neutron-star matter have used the chiral σ model to embody the pion-nucleon and pion-pion interactions which are of prime importance to the dynamics of the system.[1-3] A characteristic result of the model is that, in the absence of isobar contributions, nuclear correlations, and nucleon recoil, the critical density for pion condensation would approach infinity if G_A were to approach unity,

$$\rho_{\text{critical}} = G_A{}^{-1}(G_A{}^2 - 1)^{-1/2} m_\pi F_\pi{}^2 (2)^{-1} .$$

The reason for this is that favorable π-meson P-wave forces (which contain a factor of G_A) are almost balanced by unfavorable S-wave forces. Whether or not pion condensation would actually occur thus depends strongly on the correction terms, in this class of model.

In the present paper we point out that this result, and the other results obtained for pion condensation in the σ model, are dependent on the specific model and are not characteristic of general chirally symmetric models. We shall do this first by making an elementary calculation of pion condensation in another chiral model, one which still contains only nucleons and mesons. Then we locate the point in the development of Campbell, Dashen, and Manassah[2] (CDM) in which these other chiral models have been effectively discarded.

It should be emphasized that we shall be dealing with a rather idealized physical situation, in which most of the nuclear physics has been discarded. We wish here to deal only with the question of whether, even in this idealized situation, chiral symmetry determines the general features of the results.

The idealizations are the following: We take a chiral model in which the only baryons are the neutron and the proton, in which there are meson fields coupled to the nucleons, and in which there is no nucleon-nucleon force except for that implied by the mesons. The pion mass term will be the only direct chiral-symmetry-breaking term in the Lagrangian. The treatment will be at the tree-graph level. However, it is a well-known result that the true general results of chiral symmetry, namely, the soft-pion theorems, are valid for the sums of tree graphs alone, as well as for the complete amplitudes.

In the limit of infinitely massive baryons, we look for the lowest energy state of neutron matter containing an infinitesimal density of charged pions, condensed in a plane-wave mode. The density at which this energy becomes lower than that of the normal state of neutron matter is the critical density for the phase transition. The state will be constrained to be electrically neutral, so that the proton charge balances the (negative) π-meson charge. The pions are described by the classical values for the charged pion fields,

$$\begin{aligned} \varphi(\vec{x}) &= \varphi_0 e^{ikz} , \\ \Pi(\vec{x}) &= -i\omega_0 \varphi_0 e^{ikz} , \end{aligned} \tag{1}$$

where $\Pi(\vec{x})$ is the canonical momentum variable, and ω_0 is to be determined dynamically by minimizing the energy. The electric charge of the pion is given by

$$\rho_\Pi^Q(\vec{x}) = i(\Pi^*\varphi - \varphi^*\Pi) = -2\omega_0\varphi_0^*\varphi_0 , \tag{2}$$

and the equation of constraint is $\langle\rho_p\rangle + \rho_\Pi^Q = 0$, where ρ_p is the proton density operator.

Both of our models are taken from Weinberg's 1968 paper.[4] Model A is the model used by Au and Baym[1]; for the present problem it is equivalent to the σ model of CDM.[2] The Lagrangian is given by

$$\begin{aligned} \mathcal{L}_A = \mathcal{L}_A^{(0)} &- F_\pi{}^{-2}\bar\psi\gamma^\mu\vec\tau\cdot(\vec\varphi\times\partial_\mu\vec\varphi)\psi \\ &- f m_\pi{}^{-1}\bar\psi\gamma_5\gamma^\mu\vec\tau\cdot\partial_\mu\vec\varphi\psi + O(\varphi^3) , \end{aligned} \tag{3}$$

where

$$\mathcal{L}_A^{(0)} = \bar\psi(i\gamma^\mu\partial_\mu - M)\psi - \tfrac{1}{2}\partial_\mu\vec\varphi\cdot\partial^\mu\vec\varphi - \tfrac{1}{2}m_\pi{}^2\vec\varphi^2 .$$

Here ψ is the nucleon field, and φ is the pion field. We have retained terms only up to order φ^2 in the expansion of the Lagrangian in powers of the pion field, since we need to calculate the energy only to order φ^2 to determine the critical conditions for pion condensation.

Model B incorporates a ρ meson, with field $\vec{V}_\mu$ in a Yang-Mills structure with coupled pion fields. Defining

$$\vec{\mathfrak{F}}_{\mu\nu} = \partial_\mu \vec{V}_\nu - \partial_\nu \vec{V}_\mu + m_\rho F_\pi^{-1} \vec{V}_\mu \times \vec{V}_\nu \,, \tag{4}$$

we write the Lagrangian density of the model as

$$\begin{aligned}\mathfrak{L}_B = \mathfrak{L}_B^{(0)} - f m_\pi^{-1} \bar{\psi} \gamma_5 \gamma^\mu \vec{\tau} \psi \cdot \partial_\mu \vec{\varphi} \\ - \frac{i}{\sqrt{2}} m_\rho F_\pi^{-1} \bar{\psi} \gamma^\mu \vec{\tau} \cdot \vec{V}_\mu \psi \\ - \sqrt{2}\, m_\rho F_\pi^{-1} \vec{V}_\mu \cdot \vec{\varphi} \times \partial^\mu \vec{\varphi} + O(\varphi^3) \,,\end{aligned} \tag{5}$$

where

$$\mathfrak{L}_B^{(0)} = \mathfrak{L}_A^{(0)} - \tfrac{1}{4} \vec{\mathfrak{F}}^{\mu\nu} \cdot \vec{\mathfrak{F}}_{\mu\nu} - \tfrac{1}{2} m_\rho^2 \vec{V}_\mu \cdot \vec{V}^\mu \,.$$

Here we have fixed Weinberg's constants to give universal coupling of the ρ to pions and nucleons. The essential difference in the two Lagrangians is that in $\mathfrak{L}_B$ the ρ meson is coupled to the nucleon isospin current, whereas in $\mathfrak{L}_A$ there is no ρ meson, and there is a direct coupling (S-wave interaction) between the two currents.

In the limit of large nucleon mass we can write these Lagrangians as

$$\begin{aligned}\mathfrak{L}_A = \mathfrak{L}_A^{(0)} - F_\pi^{-2} \psi^\dagger \vec{\tau} \psi \cdot \vec{\varphi} \times \frac{\partial \vec{\varphi}}{\partial t} \\ - i f m_\pi^{-1} \psi^\dagger \sigma_i \vec{\tau} \psi \cdot \partial_i \vec{\varphi} + O(\varphi^3) \,,\end{aligned} \tag{6a}$$

$$\begin{aligned}\mathfrak{L}_B = \mathfrak{L}_B^{(0)} - i f m_\pi^{-1} \psi^\dagger \sigma_i \vec{\tau} \psi \cdot \partial_i \vec{\varphi} \\ - \frac{1}{\sqrt{2}} m_\rho F_\pi^{-1} \psi^\dagger \vec{\tau} \psi \cdot \vec{V} \\ - \sqrt{2}\, m_\rho F_\pi^{-1} \vec{V}_\mu \cdot \vec{\varphi} \times \partial^\mu \vec{\varphi} + O(\varphi^3) \,.\end{aligned} \tag{6b}$$

The most direct way of extracting the consequences of the Lagrangian (6b) for the threshold of pion condensation is to construct the meson inverse propagator in the nuclear medium, consisting now of a pure neutron gas,

$$\Delta(\vec{k}, \omega)^{-1} = -\omega^2 + \vec{k}^2 + m_\pi^2 + \Pi(k, \omega) \,, \tag{7}$$

and use the condition for the critical point,[5]

$$\Delta^{-1}(\vec{k}, \omega) = 0, \quad \frac{\partial}{\partial \omega} \Delta^{-1}(\vec{k}, \omega) = 0 \,. \tag{8}$$

We note that due to the fact that there is a nonvanishing source for the ρ-meson field (even in the absence of a pion field), we can solve the Lagrangian equation for a classical ρ-meson field. If we were to have attacked the problem as a ground-state energy problem and computed perturbation graphs, this classical approximation for the V field would correspond to keeping only tree graphs. The solution for the time component of the neutral field is

$$V_0^{(3)} = 2^{-1/2} m_\rho^{-1} F_\pi^{-1} \rho \,, \tag{9}$$

where ρ is the density of neutrons. The remaining components of the ρ field vanish. In this model the pion polarization part $\Pi(\vec{k}, \omega)$ would be given by

$$\Pi_p(\vec{k}, \omega) = -\frac{2 f^2 k^2 \rho}{m_\pi^2 \omega} \,, \tag{10}$$

if there were no vector-meson interactions. In the presence of the $V_0^{(3)}$ field it is given instead by

$$\Pi(\vec{k}, \omega) = -\frac{2 f^2 k^2 \rho}{m_\pi^2 (\omega - F_\pi^{-2} \rho)} + 2\omega F_\pi^{-2} \rho \,. \tag{11}$$

The second term on the right-hand side of (11) comes from the direct ρ, π interaction in (6b). The modification of the denominator in the first term comes from taking into account the difference between neutron and proton energies in the ρ-meson field, (9). Substitution of (11) into the critical condition (8) yields an equation for the critical density,

$$-3(G_A^2 k^2 F_\pi^{-2} \rho_c)^{2/3} + k^2 + m_\pi^2 + F_\pi^{-4} \rho_c^2 = 0 \,. \tag{12}$$

Here we have used the relation $f = F_\pi^{-1} m_\pi G_A$. The minimum value of ρ_c occurs for $k = 2^{3/2} G_A^2 F^{-2} \rho_c$ and is given by

$$\rho_c^B = \frac{m_\pi F_\pi^2}{(4 G_A^4 - 1)^{1/2}} \,, \tag{13}$$

in contrast to the result of model A, worked out in Refs. 1–3:

$$\rho_c^A = \frac{m_\pi F_\pi^2}{2 G_A (G_A^2 - 1)^{1/2}} \,. \tag{14}$$

For values of G_A near unity the difference in the two results is dramatic, to say the least. Yet, model B is just as chirally symmetric as is model A; in both cases we have broken the symmetry only in the pion-mass term. And the approximation used in both cases was the tree approximation.

If we look back at paper II of CDM, we can find the cause of the confusion. It is in the approximation which led to Eq. (3.15) of this paper,

$$S^{-1} = i \not\partial - M - \cos\theta \frac{\tau_3}{2} \not k - G_A \sin\theta \frac{\tau_2}{2} \gamma_5 \not k + \gamma^0 \mu_B + O(k^2, \mathfrak{L}_{SB}) \qquad [(3.15) \text{ of CDM II}] \,,$$

where all of the interaction terms of $\mathfrak{L}_{SYM}$ in the (chiral rotated) Lagrangian of Eq. (3.6) of this paper,

$$\mathfrak{L}_{eff} = \mathfrak{L}_{SYM} - \cos\theta\, k_\nu V_3^\nu - \sin\theta\, k_\nu A_2^\nu + k^\nu k^\lambda S^{\nu\lambda}(\theta) + \mathfrak{L}_{SB}(\theta) \qquad [(3.6) \text{ of CDM II}] \,,$$

have been dropped. Here $\mathfrak{L}_{SB}$ is the chiral symmetry-breaking part of the Lagrangian. Without explicitly stating it the authors have given the impression that the effect of $\mathcal{L}_{SYM}$ is unimportant, so far as the θ dependence of the ground-state energy is concerned. They then proceed to analyze the symmetry-breaking terms in $\mathcal{L}_{SB}$ at some length. However, the evidence of our example is that the detailed structure of $\mathcal{L}_{SYM}$ is very important in determining the θ dependence of the energy, even though $\mathcal{L}_{SYM}$ is independent of θ. It is certainly more important than $\mathcal{L}_{SB}$. The reason is that the explicit θ-dependent terms in Eq. (3.6) of CDM act in concert with any spin- or isotopic-spin-dependent forces (such as a ρ-exchange force) in $\mathcal{L}_{SYM}$ to give an effective nuclear potential for the baryons which depends as strongly on θ as do the explicit terms in $\mathcal{L}$.

Returning to the models A and B discussed above, we think that the representation B, containing the ρ field explicitly, is more realistic than that of A (or equivalently that of the σ model). Therefore we conclude that the positive S-wave energies have probably been overestimated in Refs. 1–3.

However, in a more realistic theory, in which nuclear correlation effects, hitherto ignored, are taken into account, the energy will not be reduced as much by ρ effects as in model B. To see this it is more convenient to return to the method of calculating the critical condition by calculating the energy of the constrained ground state in the external field (1), a method which can be used to arrive at the expression (14) for the critical density. The modification $F_\pi^{-2}\rho$ in the denominator of the P-wave term in (11) comes from the nucleon-nucleon force due to ρ exchange,

$$E_{NN}=\tfrac{1}{4}m_\rho^2F_\pi^{-2}(4\pi)^{-1}\int d^3x\,d^3x'\,\frac{\exp(-m_\rho|\vec{x}-\vec{x}'|)}{|\vec{x}-\vec{x}'|}\langle\rho_n(\vec{x})-\rho_p(\vec{x})\rangle\langle\rho_n(\vec{x}')-\rho_p(\vec{x}')\rangle\,, \tag{15}$$

where ρ_n and ρ_p are the neutron and proton density operators. We then replace $\langle\rho_n(x)-\rho_p(x)\rangle=\rho-4\omega_0\varphi_0^*\varphi_0$ to obtain

$$E_{NN}/\mathrm{vol}=\tfrac{1}{4}F_\pi^{-2}(\rho-4\omega_0\varphi_0^*\varphi_0)^2\,. \tag{16}$$

The term $-2\omega_0F_\pi^{-2}\varphi_0^*\varphi_0$ cancels the S-wave π^-n energy of repulsion.

In the presence of correlations we should make the replacement in (15)

$$\langle\rho_n(\vec{x})-\rho_p(\vec{x})\rangle\langle\rho_n(\vec{x}')-\rho_p(\vec{x}')\rangle\rightarrow\langle\Psi_0|[\rho_n(\vec{x})-\rho_p(\vec{x})][\rho_n(\vec{x}')-\rho_p(\vec{x}')]|\Psi_0\rangle\,, \tag{17}$$

and this latter correlation function should be reduced at small distances by short-range repulsive forces.

To return to the simplest model, A, (with no ρ particle), let us ask whether there is some limit, if not the chiral one, in which the results for pion condensation should remain nearly the same when selected nuclear forces are turned on. There is one, and it was discussed in the original article by Sawyer and Scalapino.[6] This is the limit in which the nuclear forces are spin and isospin independent. The correction terms due to nuclear forces of this kind are of order $p_F^2k^2M_N^{-2}\omega^{-2}$ times the result with nuclear forces turned off.[7] The large change in results which we saw as a result of the introduction of ρ mesons is a typical effect of an isospin-*dependent* force. However, the correlations in the nuclear wave functions are thought to be largely due to a universal (spin and isospin independent) short-range repulsion. Therefore there is reason to have some confidence in an approach in which, e.g., the ρ contribution must be evaluated taking into account nuclear correlations, but in which the basic P-wave energy is unchanged by the introduction of these correlations.

When Δ particles are included in the models, the situation becomes more complicated. The remarks on the independence of results on nuclear forces given above and in Refs. 6 and 7 obtain only if the Δ-nucleon forces, about which very little is known, are the same as nucleon-nucleon forces.

Our general conclusion is that there are necessarily very great uncertainties in pion condensation calculations, and that chiral symmetry is of little or no use in reducing the uncertainties. Pion condensation has no aspect which is of the nature of a soft-pion theorem, independent of the particular chiral model chosen.

This work was supported in part by the National Science Foundation.

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