PHYSICAL REVIEW D VOLUME 18, NUMBER 4 15 AUGUST 1978

Nonlinear scalar field dynamics in the Schwarzschild geometry

B. E. Brumbaugh
Department of Physics, University of Utah, Salt Lake City, Utah 84112
(Received 10 March 1978)

The dynamics of the nonlinear massless scalar field described by $\Box\Phi = 4\lambda\Phi^3$ is numerically studied in the Schwarzschild geometry. It is found that such a scalar field is entirely radiated away during the gravitational collapse of a scalar charged star. Sawyer has argued that a quantum fluctuation can give rise to a quasistatic component of the scalar field which will trap subsequent scalar radiation and grow to a singular field. A phenomenological model used to study this situation leads to no such singular growth. All results, in fact, suggest that the dynamics of weak nonlinear fields in a black-hole background is qualitatively similar to that of linear fields.

I. INTRODUCTION

Linear fields (solutions of linear field equations) treated as first-order perturbations on a Schwarzschild background have been extensively studied and much is now known about the dynamics of such fields; in particular, it is known that during gravitational collapse such fields do not develop singularities at the event horizon $r = 2M$. (We use here units in which $c = G = 1$.) Nonlinear field equations such as $\Box\Phi = 4\lambda\Phi^3$, which in the Schwarzschild background becomes

$$\begin{aligned}\Box\Phi &\equiv (-g)^{-1/2}[(-g)^{1/2}\Phi_{,\alpha}g^{\alpha\beta}]_{,\beta}\\ &= -\frac{1}{1-2M/r}\frac{\partial^2\Phi}{\partial t^2} + \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(1-\frac{2M}{r}\right)\frac{\partial}{\partial r}\Phi\right]\\ &\quad + \frac{1}{r^2}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Phi}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\varphi^2}\Phi\right)\\ &= 4\lambda\Phi^3\,, \end{aligned}\tag{1}$$

describing a massless self-coupled scalar field may, however, describe fields having qualitatively different behavior from those described by linear equations. We use Eq. (1) here to study this possibility, especially the possibility of the evolution of singularities.

Recently, Sawyer[1] has used Eq. (1) as a model in considering the relevance of nonlinear couplings between fields in the Schwarzschild geometry, and we rely extensively here on his analysis. He has found that there is a static spherically symmetric solution to Eq. (1) which near the event horizon behaves as

$$\Phi_c = (32\lambda M)^{-1/2}(r-2M)^{-1/2} + \text{less singular terms}\,. \tag{2}$$

Sawyer has described the interesting consequences if such a singularity should develop. These have led him to raise the following questions which we attempt to answer here:

(i) Will gravitational collapse of a scalar charged star lead to the evolution of a scalar field, described by Eq. (1), which becomes singular at the event horizon?

(ii) Sawyer argues that Φ_c of Eq. (2) contributes to a potential barrier for radiation outgoing from the event horizon. Can a quantum fluctuation of Φ, singular near the event horizon, act as a trap for outgoing radiation so that subsequent radiation leads, because of the nonlinearity, to the development of the singular solution of Eq. (2)?

II. EVOLUTION DURING COLLAPSE

A. "Classical" collapse

To answer the first question we numerically study a spherically symmetric scalar field evolving according to Eq. (1) for $\psi(r,t) = r\Phi(r,t)$:

$$-\psi_{,t,t} + \psi_{,r_*,r_*} = \frac{r-2M}{r^4}(2M\psi + 4\lambda r\psi^3)\,, \tag{3}$$

where r_* is the tortoise coordinate defined by $r_* = r + 2M\ln(r/2M - 1) - 4M$. In choosing the initial values of the field from which to start the evolution, we satisfy two conditions. First, outside the star we choose ψ to be a static solution to Eq. (3) so that $\Phi_{,t,t} = \Phi_{,t} = 0$ on an initial $t = t_0$ = constant hypersurface. This guarantees that by our choice of initial field values we do not somehow imply the previous existence of a configuration which is not physically meaningful. The second condition to be satisfied is that the field be well behaved on the stellar surface. For our purposes, it is not necessary to specify the details of the collapse; it is sufficient here to require that $\partial\Phi/\partial\tau$ be finite for all τ on the stellar surface where τ is the time as measured by an observer comoving with the stellar surface. This condition is satisfied by choosing Φ such that[3]

$$\Phi = a + b\exp[(r_* - t)/4M] \tag{4}$$

on the null ray which asymptotically is arbitrarily close to the stellar surface (see Fig. 1). The constants a and b are determined by demanding that Φ and $\Phi_{,r}$ be everywhere continuous.

The result of the numerical computation (see Fig. 2) is that for late times in the region $t \gg r_*$,

$$\psi(r,t) \to f(r)/t\,,$$

where $f(r) \propto r$ is the static solution of the linear ($\lambda = 0$) problem which is well behaved at $r = 2M$. [This is to be compared with the result from the linear problem[3] $\psi \to f(r)/t^2$.] Since all r_* lie within this region for late enough times, we conclude that, like the linear field, *the nonlinearly self-coupled massless scalar field is entirely radiated away during stellar collapse*; no singularity develops.

B. Collapse with outgoing radiation

In the above calculation we have not included any effects that may be produced by the Hawking radiation that should be present. Because of the nonlinear nature of the problem, the following question arises: Can the field which is present due to the collapsing scalar charged star trap, through the nonlinear self-coupling, the outgoing radiation arising from the Hawking process to build a singular field? We now attempt to answer this question by using a classical model of Hawking radiation in the hope that it will embody the essence of the physical problem.

We model the Hawking radiation by replacing the previously chosen surface field values with

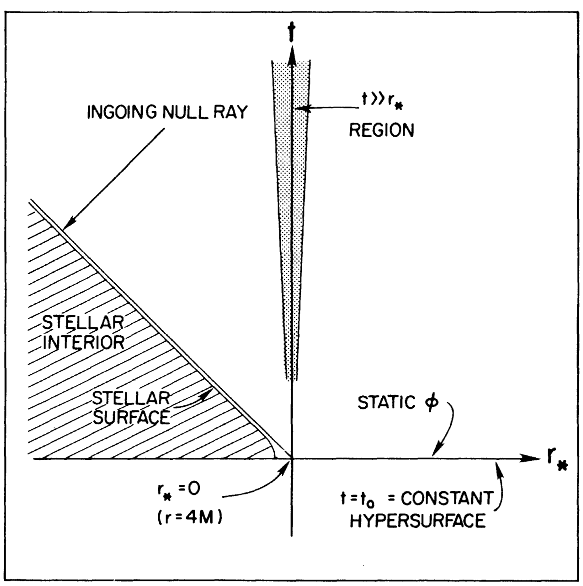

FIG. 1. The collapse problem pictured in r_*, t coordinates. The ingoing null ray is chosen to intersect the $t = t_0$ = constant hypersurface at $r_* = 0$ ($r = 4M$). On this null ray $\Phi \sim a + b\ \exp[(r_* - t)/4M)]$.

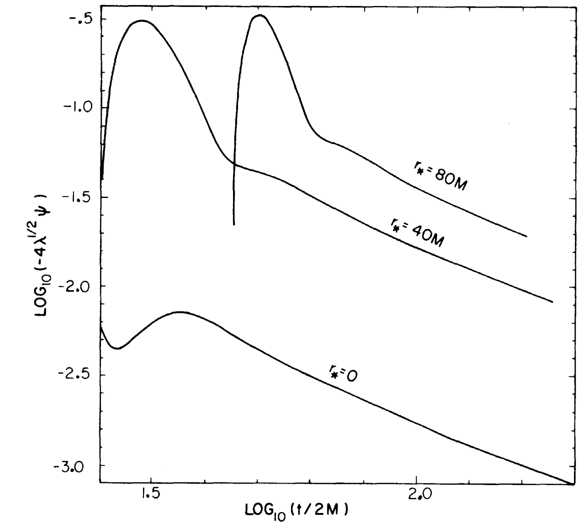

FIG. 2. Results of the collapse calculation of ψ in the region $t \gg r_*$ at various values of r_*. The static external field is chosen here such that at $r_* = 0$, $\Phi = (16M\lambda^{1/2})^{-1}$ at $t = t_0$.

$$\Phi = a + b\ \sin[c(t - r_*)] \tag{5}$$

along the ingoing null ray approximating the world line of the stellar surface. Here a and b are again determined from the continuity of Φ and $\Phi_{,r}$. We choose the parameter c to be of the order of M^{-1} and the initial external field values to be such that the continuity conditions give a value of b to be of the order M^{-1}. These choices reflect the appearance of Hawking radiation[4] to a distant observer as determined by the linear ($\lambda = 0$) theory: Hawking radiation is outgoing radiation having a blackbody frequency spectrum peaked around $\omega \sim M^{-1}$ and having an amplitude of the order of M^{-1}.

Numerically evolving the field as before, we find that the effective potential barrier located near $r = 3M$ reflects some of the radiation which is being fed into the region of interest by Eq. (5). This reflected radiation then freely propagates without distortion into the developing black hole without accumulating anywhere and thereby without growing to a singular field (see Fig. 3). By changing the value of c in Eq. (5), we find that there is a change in the proportion of the radiation that is reflected, but there are no qualitative differences in the results.

III. QUANTUM SEEDING

Consider a component of the field Φ_Q as a small fluctuation about the static solution of Eq. (2). To first order in these small fluctuations, Eq. (1) gives us

$$-\psi^{(l)}_{Q,t,t}+\psi^{(l)}_{Q,r_*,r_*}=(1-2M/r)\left(\frac{2M}{r^3}+\frac{l(l+1)}{r^2}+12\lambda\Phi_c{}^2H(r_*-r_{0*})\right)\psi^{(l)}_Q\equiv V^{(l)}_{\rm eff}(r)\psi^{(l)}_Q\,,$$
$$\Phi_Q\equiv\sum_{l,m}r^{-1}\psi^{(l)}_Q(r,t)Y_{lm}(\theta,\varphi)\,, \tag{6}$$

where r, t, θ, and φ are the usual Schwarzschild coordinates.

Sawyer argues that nonlinear couplings between fields should serve to prevent the buildup of a singular field where a singularity is predicted by "classical" field equations which do not include these couplings. The self-coupling (the $\lambda\Phi^3$ term) in Eq. (1) does not, by itself, limit field strengths. To prevent such singularities, couplings of the Φ field to other fields must be invoked. In Eq. (6) above, the Heaviside step function $H(r_*-r_{0*})$ is used to model phenomenologically the effect of these couplings of the Φ field to other fields, couplings which presumably serve to limit the strength of the Φ field for $r_*<r_{0*}$.

It is inconvenient to use Eq. (6) as it stands due to the presence of Φ_c which must be determined numerically from the static version of Eq. (1). Instead we approach the second question posed at the end of Sec. I by numerically evolving the radiation-like field Φ_Q according to

$$-\psi_{Q,t,t}+\psi_{Q,r_*,r_*}=\left[(1-2M/r)\frac{2M}{r^3}+H(r_*-r_{0*})\frac{3}{8Mr}\right]\psi_Q\equiv V_{\rm simp}(r)\psi_Q\,. \tag{7}$$

(We consider only the spherically symmetric $l=0$ case.) The effect of $V^{(l)}_{\rm eff}(r)$ in Eq. (6) is to trap and reflect outgoing radiation; the simplified effective potential $V_{\rm simp}(r)$ in Eq. (7) differs significantly from $V^{(l)}_{\rm eff}(r)$ for large positive r_* but should embody the essence of the physical problem.

As in the case considered above in Sec. II B, we choose initial field values $\Phi_Q=1-\cos[(r_*-t)/8M]$ on the ingoing null ray and $\Phi_Q=0$ outside of the null ray on a $t=t_0=$ constant hypersurface. The result of the numerical computation is that ψ_Q *is everywhere sinusoidal with no singular evolution at late times*. Figure 4 shows a plot of the locus of

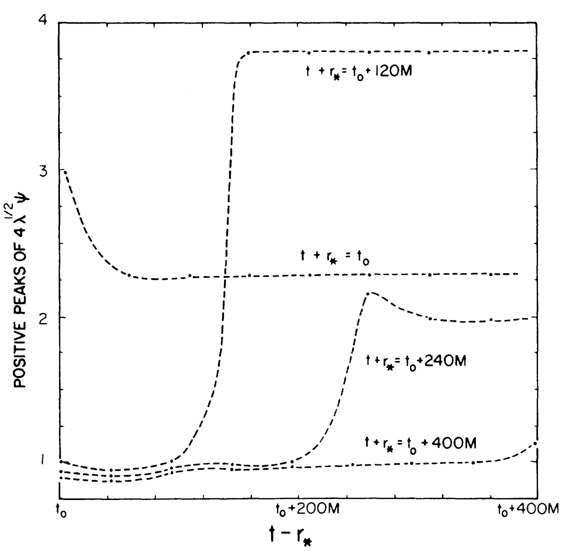

FIG. 3. Results of the collapse-with-outgoing-radiation calculation. Here the frequency is chosen as $c=(8M)^{-1}$. Plotted is the locus of positive peaks of ψ, as determined along $t+r_*=$ constant characteristics, as a function of $t=r_*$. Large positive values of $t-r_*$ correspond to the region of the event horizon.

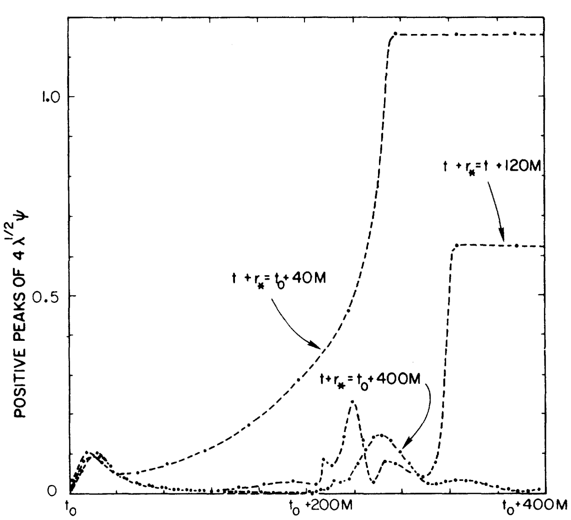

FIG. 4. Results of the linearized "quantum seeding" problem described by Eq. (7). This plot is the locus of positive peaks of ψ, as determined along $t+r_*=$ constant characteristics, as a function of $t-r_*$. The frequency of the radiation on the ingoing null ray is $(8M)^{-1}$ and the value of r_{0*} in Eq. (7) is chosen to be $-100M$.

positive peaks of ψ_Q, as specified along a curve where $t+r_*=$ constant, as a function of $t-r_*$. This figure shows that ψ_Q not only does not grow to a singularity, it does not even become large enough to violate the conditions of validity of the linearization procedure which led to Eq. (6).

In order to gain some insight into this, we have studied analytically the very similar problem in which $V_{\text{eff}}^{(l)}(r)$ in Eq. (6) is replaced by the square potential barrier defined by

$$V_{\text{sq}}(r_*)=\begin{cases}0, & r_*<r_{0*}\\ a^2, & r_*>r_{0*} \text{ but } r(r_*)\lesssim 3M\\ 0, & r(r_*)\gtrsim 3M\,.\end{cases} \tag{8}$$

The evolution problem now is just the familar barrier penetration problem of elementary quantum mechanics with different boundary conditions. If one considers the Fourier transform of ψ_Q defined by

$$\psi_Q=\int_{-\infty}^{+\infty} f(r,\omega)e^{i\omega t}\,d\omega\,, \tag{9}$$

one can show that, for physically meaningful boundary data, the poles of $f(r,\omega)$ all lie in the upper half of the complex ω plane. The consequence of this is that ψ_Q can be expressed as a sum of terms each of which dies off exponentially in time plus, if one chooses the boundary values properly, terms which are pure sinusoids.

IV. CONCLUSIONS

The study described above, of the evolution of a nonlinear field in a black-hole background, cannot be considered complete. Three shortcomings especially should be noted: (i) Hawking radiation is a quantum phenomenon and a classical analysis may not be adequate to answer the physical questions; (ii) the phenomenological model used in Sec. III for quantum seeding might not be, even qualitatively, correct; and (iii) only one particular nonlinear field theory [that of Eq. (1)] has been considered. The results, however, do seem to suggest that singularities of nonlinear fields do not develop at the event horizon, and that nonlinear fields which are initially weak evolve in a manner very similar to that of linear fields.

ACKNOWLEDGMENTS

I wish to thank Dr. R. H. Price for guidance throughout the execution of the problem. I also wish to thank Dr. R. F. Sawyer for discussions of the above material and Dr. C. T. Cunningham for suggestions on numerical details. This work was supported in part by the National Science Foundation under Grant No. PHY74-16311.

---

[1]R. F. Sawyer, Phys. Rev. D 15, 1427 (1977); 16, 1979(*E*) (1977).

[2]This particular scenario is the one which Sawyer believes to be the most likely to lead to the evolution of a singular field. He shows that if near the event horizon

$$\Phi(r,t)=b(t)(r-2M)^{-1/2}+\text{ less singular terms,}$$

and if there exists an energy dissipation mechanism which is proportional to $(\partial\Phi/\partial t)^2$, then for late times $b(t)\to\pm 4\lambda^{-1/2}$ as $t\to\infty$. Thus, for an arbitrarily small initial value of b, Φ may grown to a value characteristic of the static solution $\Phi_c(r)$ of Eq. (2).

[3]For a more detailed account, see R. H. Price, Phys. Rev. D 5, 2419 (1972).

[4]The "temperature" of a mass-M black hole is $T\sim\hbar/kM$, which implies that the frequency spectrum of the radiation be peaked around $\omega\sim M^{-1}$; and in order for the energy flux of the scalar field $\hbar(\nabla\Phi)^2$ to be consistent with σT^4 the amplitude of the radiation must be of the order of M^{-1}.

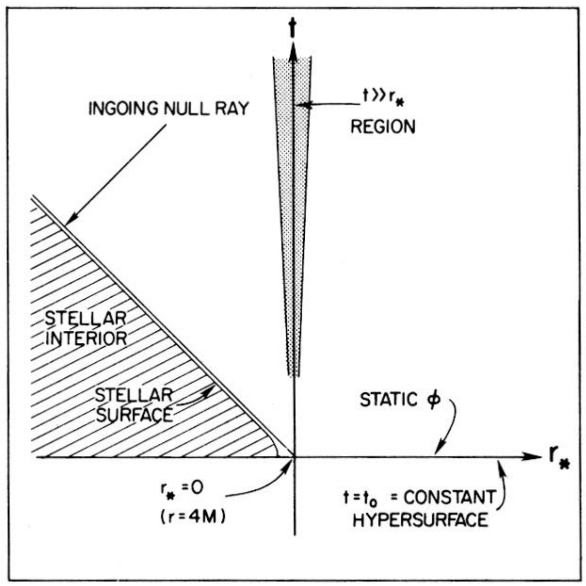

FIG. 1. The collapse problem pictured in r_*, t coordinates. The ingoing null ray is chosen to intersect the $t = t_o$ = constant hypersurface at $r_* = 0$ $(r = 4M)$. On this null ray $\Phi \sim a + b \exp[(r_* - t)/4M)]$.

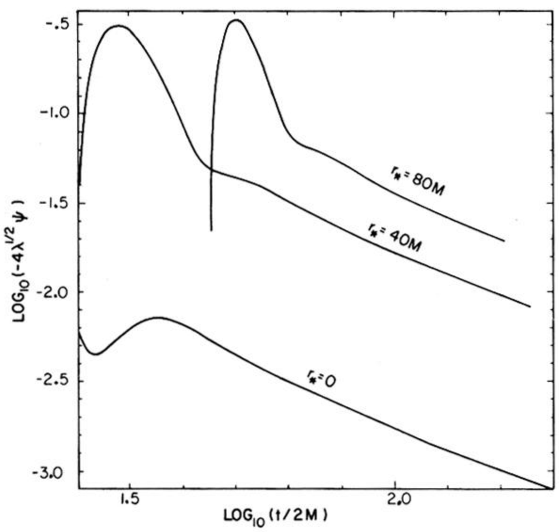

FIG. 2. Results of the collapse calculation of ψ in the region $t \gg r_*$ at various values of r_*. The static external field is chosen here such that at $r_* = 0$, $\Phi = (16M\lambda^{1/2})^{-1}$ at $t = t_o$.

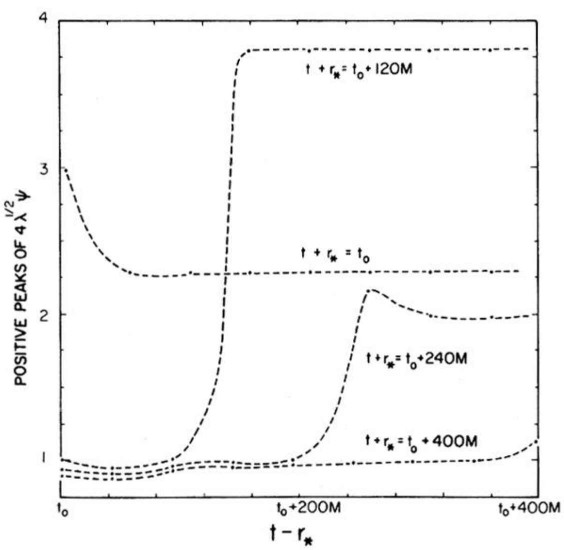

FIG. 3. Results of the collapse-with-outgoing-radiation calculation. Here the frequency is chosen as $c = (8M)^{-1}$. Plotted is the locus of positive peaks of ψ, as determined along $t + r_* =$ constant characteristics, as a function of $t = r_*$. Large positive values of $t - r_*$ correspond to the region of the event horizon.

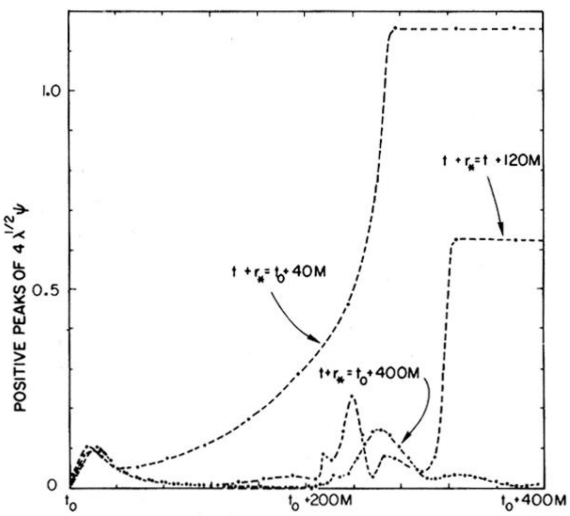

FIG. 4. Results of the linearized "quantum seeding" problem described by Eq. (7). This plot is the locus of positive peaks of ψ, as determined along $t + r_* =$ constant characteristics, as a function of $t - r_*$. The frequency of the radiation on the ingoing null ray is $(8M)^{-1}$ and the value of r_{o*} in Eq. (7) is chosen to be $-100M$.