Time-dependent property of the Prasad-Sommerfield monopole

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We show that the Prasad-Sommerfield solution for the 't Hooft monopole can be transformed to an exact time-dependent solution (which is singular on the light cone) of the SU(2) Yang-Mills-Higgs system.

The purpose of this note is to describe a hitherto undiscussed feature of the Prasad-Sommerfield[1] solution for the equations of motion for the 't Hooft monopole,[2] namely, a transformation of this solution to an exact time-dependent solution for the SU(2) Yang-Mills-Higgs system, which is singular on the light cone.

For convenience we use the notations and conventions of Ref. 1. Specifically, the Lagrangian under consideration is

$$\mathfrak{L} = -\tfrac{1}{4} F^{\mu\nu a} F^{a}_{\mu\nu} - \tfrac{1}{2}\Pi^{\mu a}\Pi^{a}_{\mu} + \tfrac{1}{2}\mu^2(\phi^a\phi^a) - \tfrac{1}{4}\lambda(\phi^a\phi^a)^2, \tag{1}$$

where

$$\Pi^a_\mu = \partial_\mu \phi^a + e\epsilon^{abc} A^b_\mu \phi^c \tag{2}$$

and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + e\epsilon^{abc} A^b_\mu A^c_\nu . \tag{3}$$

The equations of motion for the Lagrangian (1) read

$$\partial_\mu \Pi^{\mu a} + e\epsilon^{abc} A^b_\mu \Pi^{\mu c} - \frac{\partial V(\phi)}{\partial \phi^a} = 0 \tag{4}$$

and

$$\partial_\mu F^{\mu\nu a} - e\epsilon^{abc} F^{\mu\nu b} A^c_\mu + e\epsilon^{abc}\Pi^{\nu b}\phi^c = 0 , \tag{5}$$

where

$$V(\phi) = -\tfrac{1}{2}\mu^2(\phi^a\phi^a) + \tfrac{1}{4}\lambda(\phi^a\phi^a)^2. \tag{6}$$

In order to solve the system (4), (5) we start by making the ansatz

$$\phi^a = \hat{r}^a H(r,t)/(er) , \tag{7}$$

$$A^a_0 = \hat{r}^a J(r,t)/(er) , \tag{8}$$

$$A^a_i = \epsilon_{aij}\hat{r}_j [1 - K(r,t)]/(er) , \tag{9}$$

where

$$\hat{r}^a = r^a/r . \tag{10}$$

The ansatz (7)–(9) goes beyond the ansatz of Ref. 1 insofar as we allow the functions H, J, and K to be time dependent. Inserting (7)–(9) into (4), (5) we get the following:
From Eq. (4),

$$r^2(H,_{rr} - H,_{tt}) = 2HK^2 + \frac{\lambda}{e^2}(H^3 - C^2 r^2 H) , \tag{11}$$

where

$$C = \mu e/\sqrt{\lambda} \tag{12}$$

and

$$H,_r = \frac{\partial}{\partial r} H(r,t), \text{ etc.} \tag{13}$$

For $\nu = 0$ from Eq. (5),

$$r^2 J,_{rr} = 2JK^2 . \tag{14}$$

For $\nu = 1, 2, 3$ from Eq. (5),

$$r^2(K,_{rr} - K,_{tt}) = K(K^2 - 1) + K(H^2 - J^2) , \tag{15}$$

$$rJ,_{tr} = J,_t , \tag{16}$$

$$J,_t \cdot K + 2K,_t \cdot J = 0 . \tag{17}$$

Since Prasad and Sommerfield look for static solutions, in their paper (16) and (17) reduce to identities and thus do not appear.

Let us consider the subsystem (16), (17) first. From Eq. (16),

$$J(r,t) = rf(t) + g(r) , \tag{18}$$

where f and g are arbitrary functions of the respective variables (sufficiently differentiable, of course). Now since the boundary condition for J is[1]

$$J(r,t) \xrightarrow[r\to\infty]{} \text{const}, \tag{19}$$

we have

$$f(t) \equiv 0 \tag{20}$$

and thus are left with

$$J(r,t) = g(r) . \tag{21}$$

Using (21), Eq. (17) reduces to

$$K,_t \cdot J = 0 , \tag{22}$$

whence either

$$K,_t = 0 \tag{23}$$

or

$$J = 0. \tag{24}$$

We choose (24) since we are interested in time-dependent solutions for $K(r, t)$.

Following Ref. 1, we consider the limit $\lambda \to 0$ ("vanishing potential"), C fixed so that altogether we are left with a system of two nonlinear coupled partial differential equations:

$$r^2(K,_{rr} - K,_{tt}) = K(K^2 - 1) + KH^2, \tag{25}$$

$$r^2(H,_{rr} - H,_{tt}) = 2HK^2 . \tag{26}$$

To obtain solutions for these equations we seek a variable $y = y(r, t)$ such that H and K depend on this variable only and

$$r^2(K,_{rr} - K,_{tt}) = y^2 K,_{yy}, \tag{27}$$

$$r^2(H,_{rr} - H,_{tt}) = y^2 H,_{yy}, \tag{28}$$

since then (25) and (26) assume the form of the (static) Prasad-Sommerfield equations with $r \to y$. Therefore a solution of (25), (26) is given by

$$K(r, t) = K(y) = (Cy)/\sinh(Cy), \tag{29}$$

$$H(r, t) = H(y) = (Cy)\coth(Cy) - 1 . \tag{30}$$

To find y we note that because of (27), (28) y is a solution of

$$y,_{x_1 x_2} = 0 , \tag{31}$$

$$(x_1 + x_2)^2 y,_{x_1} y,_{x_2} = y^2 , \tag{32}$$

where $x_2 = r + t$ and $x_2 = r - t$. The system (31), (32) is solvable and the most general solution is

$$y(x_1, x_2) = \alpha \left(\frac{x_1}{1 + \beta x_1} + \frac{x_2}{1 - \beta x_2} \right) , \tag{33}$$

where α and β are arbitrary constants. In terms of r and t, y is

$$y = y(r, t) = 2\alpha r[(1 + \beta t)^2 - \beta^2 r^2]^{-1} . \tag{34}$$

If $\beta = 0$ this reduces to

$$y = 2\alpha r . \tag{35}$$

If $\beta \neq 0$, then since Eqs. (25), (26) are invariant under time translations, we may substitute

$$t \to t - 1/\beta \tag{36}$$

thus getting from (34)

$$y(r, t) = \frac{2\alpha}{\beta^2} \left(\frac{r}{t^2 - r^2} \right) . \tag{37}$$

The scale invariance of Eqs. (25), (26) allows the substitutions $y \to \gamma y$, where γ is some constant. Thus we have only two essentially different solutions:

$$\beta = 0, \quad y = r; \tag{38}$$

$$\beta \neq 0, \quad y = r(r^2 - t^2)^{-1} . \tag{39}$$

When substituted back into (29), (30), Eq. (38) just gives the (static) Prasad-Sommerfield solution which is regular everywhere and yields finite energy. Equation (39), on the other hand, gives a time-dependent exact solution for the system (25), (26) but is no longer regular. In particular, y exhibits a singularity on the light cone, and so does H (K goes to zero exponentially at this point); furthermore, the fields ϕ and $\vec{A}$ exhibit additional singularities at $r = 0$. Owing to these singularities the total energy of the system,

$$E = \int d^3x \, T^{00} , \tag{40}$$

diverges.

This is easily checked from the explicit form of T^{00},

$$T^{00} = \tfrac{1}{4} F^{ija} F^{ija} + \tfrac{1}{2} F^{0ia} F^{0ia} + \tfrac{1}{2} \Pi^{0a} \Pi^{0a} + \tfrac{1}{2} \Pi^{ia} \Pi^{ia} - \tfrac{1}{2} \mu^2 (\phi^a \phi^a) + \tfrac{1}{4} \lambda (\phi^a \phi^a)^2 \tag{41}$$

or, in terms of H and K (remember $J = 0$),

$$T^{00} = \frac{1}{e^2 r^2} \left(K,_r{}^2 + K,_t{}^2 \right) + \frac{1}{2e^2 r^4} (K^2 - 1)^2 + \frac{1}{2} \left[\frac{2K^2 H^2}{e^2 r^4} + \frac{1}{e^2 r^2} \left(\frac{H}{r} - H,_r \right)^2 \right] + \frac{1}{2} \frac{(H,_t)^2}{e^2 r^2} - \tfrac{1}{2} \mu^2 \frac{H^2}{e^2 r^2} + \frac{1}{4} \lambda \frac{H^4}{e^4 r^4} \tag{42}$$

and the explicit form of H and K.

Intuitively, the solution corresponds to a radially symmetric object with infinite energy density at $r = 0$ and on the surface $r^2 = t^2$ this surface moving our as t^2 increases.

We conclude this note by two remarks.

(i) We stress that the transition from $y = r$ to $y = r(r^2 - t^2)^{-1}$ works only in the limit of vanishing potential, since otherwise r would appear explicitly on the right-hand side of Eq. (26).

(ii) The solution given here cannot be used to construct a pseudoparticle solution, because the Higgs field cannot be identified as the fourth component of the gauge field. In any case the four-dimensional Euclidean action integral is divergent.

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[1] M. K. Prasad and C. M. Sommerfield, Phys. Rev. Lett. 35, 760 (1975).

[2] G.'t Hooft, Nucl. Phys. B79, 276 (1974).