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Quark confinement by infrared singularities*

Arthur R. Swift and Francesc S. Roig
Department of Physics and Astronomy, University of Massachusetts, Amherst, Massachusetts 01003
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Confinement of spin-1/2 quarks by an infrared-singular effective gluon propagator is shown to work in three space dimensions. The exact infrared behavior of the quark propagator is isolated. In the infrared limit the propagator vanishes as the pole moves to infinite energy. The same singular interaction is used as the kernel of the Bethe-Salpeter equation for bound states. The divergence from the kernel exactly cancels the zeros of the propagator to produce an infrared-finite bound-state equation. The cancellation occurs only when the kernel has the form of a generalized instantaneous Coulomb interaction. The existence of confined solutions is demonstrated.

I. INTRODUCTION

The ultimate success of the quark model of hadrons depends upon a convincing explanation of the failure to observe free quarks easily. At present the theoretically favored solution to this problem is the statement that quarks are permanently confined. Models in which quark confinement is an essential ingredient have been successful in explaining static properties of hadrons.[1] However, the connection between the confinement mechanisms and fundamental quark interactions has not yet been established in these models.

The analysis of two-dimensional field theories[2] has led several authors[3] to express the belief that quarks are confined as a result of the zero-momentum, or infrared, limit of a theory in which quarks interact with a massless multiplet of vector mesons in a non-Abelian, color SU(3), gauge theory. Small momentum translates to large distance in configuration space. The idea is that the quark-vector-meson (= gluon) interaction becomes very large, even infinite, at zero momentum or large distance. The complete separation of a quark-antiquark pair requires infinite energy. Support for this idea comes from the growth of the effective coupling constant in a gauge theory as the momentum is decreased from infinity.[3] The success of "infrared slavery"[3] as a confinement mechanism depends upon an affirmative answer to two questions: Do gauge theories develop the appropriate infrared behavior? Calculations in low-order perturbation theory have produced only infrared-finite amplitudes.[4] However, perturbation theory is not a reliable guide to the behavior of the full theory.[5] The second question is: Can infrared singularities produce confinement in theories with three space dimensions? Confinement has been observed only in theories with one space dimension.[2] This latter question is addressed in this paper.

Working in close analogy with two-dimensional field theories which display confinement, we postulate the existence of an effective gluon propagator function $D(\vec{k}^2)$ which is infrared singular.[6,7] Our quarks have spin $\frac{1}{2}$; spin is an essential ingredient. The interaction we use is a function of three-momentum only and has the spin structure of the fourth component of a four-vector. It is a generalized, instantaneous, Coulomb interaction. We present a detailed analysis demonstrating that this interaction is the only one in a large class which exhibits quark confinement. The effective interaction is used both in the Dyson equation for the quark propagator and as the kernel of the Bethe-Salpeter equation for bound states. This dual use of the infrared-singular interaction is justified either as a weak-coupling approximation or by identification of $D(\vec{k}^2)$ as an effective quark-quark interaction. The Dyson equation for the quark propagator is solved exactly in the infrared limit. The "exact" quark propagator vanishes in the infrared limit as the quark pole shifts to infinite energy. The quark propagator is used in a bound-state Bethe-Salpeter equation whose kernel is the singular function $D(\vec{k}^2)$. There is a miraculous cancellation between the vanishing propagators and the diverging kernel. The result is a set of bound-state equations which are finite in the infrared limit. Finally, we demonstrate the existence of confined solutions. Our conclusion is that infrared confinement can work under special conditions.

In the absence of a fundamental theory we have no Ward identities to constrain our postulated interaction. In a more fundamental approach these Ward identities are part of a test of consistency of infrared singularities and the structure of a gauge theory.[7] We are interested here in the question of how an infrared singularity produces quark confinement. Although we appeal to gauge theories to justify our model, we neglect all internal-symmetry considerations. In addition, since our gluon propagator represents an effective interaction, we sidestep problems associated with a field theory based on singular propagators.[8]

In Sec. II we treat the Dyson equation for the quark propagator. Heuristic arguments are offered to make our choice of interaction plausible. The discussion is sufficiently general to include the effects of infrared-finite corrections. The infrared behavior of the quark propagator is isolated exactly, and integral equations are developed for the finite functions that occur in the propagator.

Section III is concerned with the Bethe-Salpeter equation for bound states. Care is taken to justify use of the same interaction here as was used in the propagator equation. A convenient spin decomposition of the 4×4 Bethe-Salpeter wave function permits explicit demonstration of the cancellation process which leads to a set of eight, infrared-finite, three-dimensional integral equations. The discussion in this section and in Sec. II does not require an explicit functional form for the interaction.

In Sec. IV we discuss solutions to the bound-state equations. A definite choice is made for the function $D(\vec{k}^2)$. Nonrelativistic reduction of the integral equations leads to a Schrödinger equation which manifestly has confined solutions. A variational method is used to estimate energies in the fully relativistic problem.

Appendix A contains a very important result. We show that only the interaction we use leads to an infrared-finite bound-state equation. Other interactions are explored, and all lead to the condition that the Bethe-Salpeter wave functions vanish in the infrared limit. Appendix B is concerned with the infrared behavior of higher-order corrections to the Bethe-Salpeter kernel. Appendix C examines in detail one of the approximations made in solving the bound-state equations.

II. QUARK PROPAGATOR

The quark propagator function $S_F(p)$ satisfies the following Dyson equation:

$$S_F^{-1}(p)=\gamma\cdot p(1+a)-m(1+b)\frac{-ig^2}{(2\pi)^4}\times\int d^4k\,[D((\vec{p}-\vec{k})^2)\gamma_0 S_F(k)\gamma_0+D_1(p,k)\Gamma S_F(k)\Gamma']\,. \quad (1)$$

The infrared singularity is contained in the function $D((\vec{p}-\vec{k})^2)$, a function of three-momentum only. This singularity is assumed to be controlled by a cutoff parameter similar to a photon mass in QED. In the limit this parameter vanishes, $D(0)\to\infty$ sufficiently rapidly to make integrals diverge at small momentum. The precise functional form of $D((\vec{p}-\vec{k})^2)$ need not be specified until we look for actual bound-state solutions. Nonsingular terms in (1) are represented by $D_1(p,k)$. The spin structure of this regular interaction is left arbitrary. The constants a and b are for renormalization.

The infrared-singular interaction in (1) has both a noncovariant spin structure and a noncovariant momentum dependence. It is a generalized, instantaneous, Coulomb interaction. Our use of this interaction must be justified. A pragmatic reason is that it is the only interaction which leads to an infrared-finite Bethe-Salpeter equation. This point is explored in great detail in Appendix A where we show that if we change either the γ_0 spin structure or if we make $D(\vec{k}^2)$ a covariant function, the methodology of this paper breaks down. The choice in (1) is the only one which allows complete isolation of the infrared behavior of $S_F(p)$ and which produces a Bethe-Salpeter equation with mesonlike bound states.

A heuristic justification of (1) appeals to an unspecified gauge theory which is supposed to underly our phenomenological model. We make the analogy with two-dimensional field theories[2] in which a noncovariant choice of gauge makes the solution possible. In such a gauge the effective interaction has a noncovariant form. Presumably a covariant gauge could be used to obtain the same results but would entail consideration of a very complicated set of interactions. The vector nature of our interaction means that if it confines quark-antiquark pairs to make mesons, it will not bind two quarks.

The Dyson equation (1) has an unfamiliar form. Conventionally, the Dyson equation for a fermion propagator depends on the exact fermion-fermion-meson vertex function.[9] Since the function $D_1(p,k)$ is unspecified, Eq. (1) can be made consistent with the usual form. We assume that the infrared singularity originates in the effective gluon propagator. Vertex functions are infrared finite in our theory. The consistency of this approach is checked in Appendix B where we calculate a vertex function. The nature and meaning of $D((\vec{p}-\vec{k})^2)$ will be made more precise in the next section when we argue that it also represents the infrared-singular part of the Bethe-Salpeter kernel.

The first step in solving (1) is to write

$$S_F^{-1}(p)=A_0(p)\gamma_0 p_0-A(p)\vec{\gamma}\cdot\vec{p}-mB(p)\,. \quad (2)$$

Separate nonlinear integral equations for A_0, A, and B are projected out of (1):

$$A_0(p)=a_0(p)-\frac{ig^2}{(2\pi)^4}\frac{1}{p_0}\int d^3k\,D((\vec{p}-\vec{k})^2)\int_{-\infty}^{\infty}\frac{dk_0k_0A_0(k)}{A_0{}^2k_0{}^2-A^2k^2-m^2B^2}, \tag{3a}$$

$$A(p)=a(p)+\frac{ig^2}{(2\pi)^4}\int d^3k\,D((\vec{p}-\vec{k})^2)\frac{\vec{p}\cdot\vec{k}}{p^2}\int_{-\infty}^{\infty}\frac{dk_0\,A(k)}{A_0{}^2k_0{}^2-A^2k^2-m^2B^2}, \tag{3b}$$

$$B(p)=b(p)+\frac{ig^2}{(2\pi)^4}\int d^3k\,D((\vec{p}-\vec{k})^2)\int_{-\infty}^{\infty}\frac{dk_0B(k)}{A_0{}^2k_0{}^2-A^2k^2-m^2B^2}. \tag{3c}$$

The functions $a_0(p)$, $a(p)$, $b(p)$ contain the subtraction constants and the infrared-finite integrals involving $D_1(p,k)$. To solve (3), we make the ansatz that $A_0(p)$ is a constant and $A(p)$ and $B(p)$ are functions of three-momentum only. With this assumption the k_0 integration (with $m\to m-i\epsilon$) yields

$$A_0=a_0(p), \tag{4a}$$

$$A(\vec{p})=a(p)+f\int d^3k\frac{D((\vec{p}-\vec{k})^2)(\vec{p}\cdot\vec{k}/p^2)A(\vec{k})}{(A^2k^2+m^2B^2)^{1/2}}, \tag{4b}$$

$$B(\vec{p})=b(p)+f\int d^3k\frac{D((\vec{p}-\vec{k})^2)B(\vec{k})}{(A^2k^2+m^2B^2)^{1/2}}, \tag{4c}$$

where $f=g^2/[2(2\pi)^3A_0]$. The infrared divergence is controlled by a single subtraction

$$\int d^3k\,D((\vec{p}-\vec{k})^2)F(\vec{k})=\lambda F(\vec{p})+\int d^3k\tilde{D}(\vec{p}-\vec{k})F(\vec{k}). \tag{5}$$

The constant $\lambda=\int d^3k\,D(k^2)$ is infrared divergent. The new kernel $\tilde{D}(\vec{p}-\vec{k})=D((\vec{p}-\vec{k})^2)-\lambda\delta^3(\vec{p}-\vec{k})$ is infrared finite. Using this subtraction procedure, we find

$$A=\bar{A}+\frac{\lambda fA}{(A^2p^2+m^2B^2)^{1/2}}, \tag{6a}$$

$$B=\bar{B}+\frac{\lambda fB}{(A^2p^2+m^2B^2)^{1/2}}. \tag{6b}$$

$\bar{A}(\vec{p})$ and $\bar{B}(\vec{p})$ are finite functions:

$$\bar{A}(\vec{p})=a(p)+f\int d^3k\tilde{D}(\vec{p}-\vec{k})\times\frac{A(\vec{k})}{(A^2k^2+m^2B^2)^{1/2}}\frac{\vec{p}\cdot\vec{k}}{p^2}, \tag{7a}$$

$$\bar{B}(\vec{p})=b(p)+f\int d^3k\tilde{D}(\vec{p}-\vec{k})\frac{B(\vec{k})}{(A^2k^2+m^2B^2)^{1/2}}. \tag{7b}$$

Solution of (6) is a straightforward problem in algebra.

$$A=\bar{A}\left(1+\frac{\lambda f}{\bar{w}}\right), \tag{8a}$$

$$B=\bar{B}\left(1+\frac{\lambda f}{\bar{w}}\right), \tag{8b}$$

where $\bar{w}(\vec{p})^2=\bar{A}(\vec{p})^2p^2+m^2\bar{B}(\vec{p})^2$. Since A and B have the same functional dependence on the divergent constant λ, all dependence on λ cancels out of (7) to leave a pair of infrared-finite equations for $\bar{A}$ and $\bar{B}$:

$$\bar{A}(\vec{p})=a(p)+f\int d^3k\tilde{D}(\vec{p}-\vec{k})\frac{\bar{A}(\vec{k})}{\bar{w}(\vec{k})}\frac{\vec{p}\cdot\vec{k}}{p^2}, \tag{9a}$$

$$\bar{B}(\vec{p})=b(p)+f\int d^3k\tilde{D}(\vec{p}-\vec{k})\frac{\bar{B}(\vec{k})}{\bar{w}(\vec{k})}. \tag{9b}$$

The final step is to examine the functions $a_0(p)$, $a(p)$, and $b(p)$ in order to check on the consistency of our ansatz of constant A_0 and no p_0 dependence in $A(p)$ and $B(p)$. Given the solution (8), we see that $S_F(p)\to 0$ as the infrared cutoff is removed and $\lambda\to\infty$. Hence, the integral in (1) containing the function $D_1(p,k)$ will vanish unless the k_0 contour encircles a pole of $S_F(k)$. In that case the integral becomes

$$\frac{g^2}{2(2\pi)^3}\int d^3k\,[\lim_{|k_0|\to\infty}D_1(p,k)]\times\frac{\Gamma(-\vec{\gamma}\cdot\vec{k}\bar{A}(k)+m\bar{B}(k))\Gamma'}{A_0\bar{w}(k)}. \tag{10}$$

A reasonable assumption is that

$$\tilde{D}_1(\vec{p},\vec{k})=\lim_{|k_0|\to\infty}D_1(p,k)$$

either vanishes or is independent of p_0. The functions $a(p)$ and $b(p)$ are in turn independent of p_0. The absence of a term proportional to γ_0 in (10) implies that $a_0(p)$ and $A_0(p)$ are constants.

The exact nonlinear, coupled integral equations satisfied by the infrared-finite functions $\bar{A}(\vec{p})$ and $\bar{B}(\vec{p})$ are

$$\bar{A}(\vec{p})=1+a+f\int d^3k\frac{\bar{A}(\vec{k})}{\bar{w}(\vec{k})}\times\left[\tilde{D}(\vec{p}-\vec{k})\frac{\vec{p}\cdot\vec{k}}{p^2}-\tilde{D}_1(\vec{p},\vec{k})\frac{\mathrm{tr}(\vec{\gamma}\cdot\vec{p}\Gamma\vec{\gamma}\cdot\vec{k}\Gamma')}{4p^2}\right], \tag{11a}$$

$$\bar{B}(p)=1+b+f\int d^3k\frac{\bar{B}(\vec{k})}{\bar{w}(\vec{k})}\times\left[\tilde{D}(\vec{p}-\vec{k})+\tilde{D}_1(\vec{p},\vec{k})\mathrm{tr}\frac{(\Gamma\Gamma')}{4}\right]. \tag{11b}$$

Equations (8) and (11) with $\overline{w}=(\bar{A}^2p^2+m^2\bar{B}^2)^{1/2}$ constitute a complete solution of the infrared behavior of the quark propagator function. From (8) and the definition of $S_F^{-1}(p)$ in (2) we see that the quark propagator vanishes in the limit where infrared cutoff goes to zero. This result is a consequence of the fact that the quark energy becomes infinite in this limit and the quark pole moves to infinity. There are no free quarks in the infrared limit.

The next step is to use the exact quark propagator in the Bethe-Salpeter equation for quark-antiquark bound states. The infrared limit is to be taken after contour integrals in the k_0 plane have been evaluated. Only in this way is it possible to obtain sensible results with a quark propagator that vanishes.

III. BETHE-SALPETER EQUATION FOR BOUND STATES

The Bethe-Salpeter equation for quark-antiquark bound states is

$$\psi(p)=S_F(p+E)\left\{\frac{ig^2}{(2\pi)^4}\int d^4k\,[D((\vec{p}-\vec{k})^2)\gamma_0\psi(k)\gamma_0+D_2(p,k)\Gamma\psi(k)\Gamma]\right\}S_F(p-E)\,. \tag{12}$$

The wave function $\psi(p)$ is a 4×4 matrix in spinor space. The energy of a bound state is $2E$. The infrared singularity is contained in $D((\vec{p}-\vec{k})^2)$, the same function that appeared in the Dyson equation for the quark propagator. Nonsingular terms are represented by the unspecified kernel $D_2(p,k)$.

Equation (12) requires some discussion. It is not obvious that the same function controls the infrared behavior of both the quark propagator and the bound-state problem. If $D(\vec{k}^2)$ represents the infrared-singular portion of a single-vector-gluon propagator, then Eq. (12) implies that single-gluon exchange dominates the Bethe-Salpeter kernel, and Eq. (1) implies that the single-gluon term dominates the quark self-energy. This viewpoint is consistent in a weak-coupling theory if the higher-order corrections are no more singular than the one-gluon term. In Appendix B we calculate a vertex correction and show that it is infrared finite. We also show that the crossed-gluon correction to the Bethe-Salpeter kernel is no more singular than the single-gluon term. The divergences from gluon lines cancel against zeros from quark propagators (properly encircled by contour integrations) to produce these results. Under the assumption that these low-order results hold up to all values, we conclude that (1) and (12) are consistent.

There is an additional complication if the underlying field theory is non-Abelian. Trilinear and quadrilinear gluon couplings produce higher-order corrections which appear to be more singular than the single-gluon term. However, the momentum dependence of these couplings suppresses their infrared behavior. (Trilinear couplings vanish when two of the gluons have the same momentum.) If these corrections are not suppressed by a choice of a noncovariant gauge, we assume that their effects are incorporated into the effective gluon propagator and its pointlike coupling to quarks.

If vertex corrections are ignored, $D((k^2))$ can be regarded as the effective quark-antiquark interaction, and the restriction to weak coupling can be dropped. In the absence of vertex corrections, the exact Bethe-Salpeter equation is a sum of two-particle-irreducible diagrams [Fig. 1(a)].[10] Each diagram has the same order infrared singularity. There is a one-to-one correspondence between terms in this expansion of the Bethe-Sal-

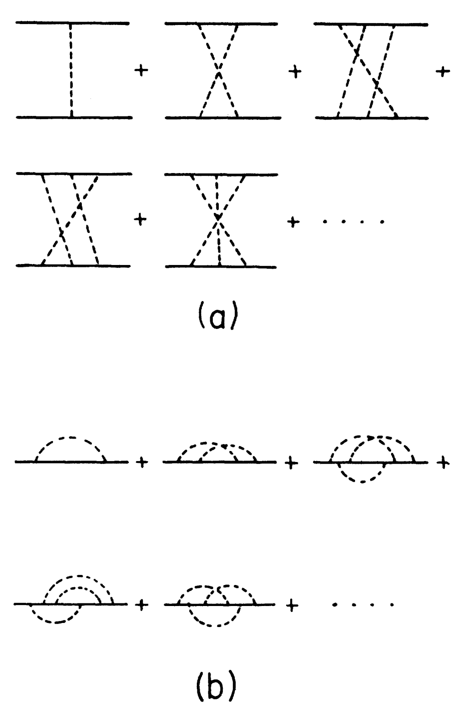

FIG. 1. (a) The diagrams contributing to the Bethe-Salpeter kernel. (b) The diagrams contributing to the quark self-energy. There is a one-to-one correspondence between the two sets. Solid lines are quarks and dotted lines are gluons.

peter kernel [Fig. 1(a)] and terms in the quark self-energy [Fig. 1(b)]. $D((\vec{p}-\vec{k})^2)$ represents the sum of all such diagrams. The nonsingular functions $D_1(p,k)$ and $D_2(p,k)$ correct for the fact that there are many more diagrams, order by order in the coupling constant, in the Bethe-Salpeter kernel than in the quark self-energy.

The singular kernel in (12) is independent of k_0. The integral equation can then be converted into an equation for the amplitude

$$\phi(\vec{p})=\int_{-\infty}^{\infty} dp_0\psi(p_0,\vec{p}). \tag{13}$$

The logic that was used to reduce $D_1(p,k)$ to $\tilde{D}_1(\vec{p},\vec{k})$ in (10) applies here to reduce $D_2(p,k)$ to $\tilde{D}_2(\vec{p},\vec{k})$, where $\lim_{p_0\to\infty}D_2(p,k)=\tilde{D}_2(\vec{p},\vec{k})$. The p_0 integration in (13) leads to evaluation of $D_2(p,k)$ with p_0 equal to the infinite quark energy. Using the "exact" solution for $S_F(p)$, we find that the ingral equation for $\phi(p)$ is

$$\begin{aligned}[(\overline{w}+\lambda f)^2-E^2]\phi(\vec{p})=\frac{f}{2\overline{w}^2}\Big\{&(\overline{w}+\lambda f)\Big[\overline{w}\gamma_0\Big(\int D\gamma_0\phi\gamma_0\Big)\gamma_0\overline{w}-\overline{\Lambda}(\vec{p})\Big(\int D\gamma_0\phi\gamma_0\Big)\overline{\Lambda}(\vec{p})\Big]\\&+E\Big[\overline{\Lambda}(\mathrm{p})\Big(\int D\gamma_0\phi\gamma_0\Big)\gamma_0\overline{w}-\overline{w}\gamma_0\Big(\int D\gamma_0\phi\gamma_0\Big)\overline{\Lambda}(\vec{p})\Big]\Big\}\,,\end{aligned}\tag{14}$$

where $\overline{\Lambda}(\vec{p})=-\overline{A}\vec{\gamma}\cdot\vec{p}+m\overline{B}$. The constant A_0 has been absorbed into the definition of E. The infrared nature of (14) is determined by the divergent constant λ and the divergence of the integral

$$\int D\phi=\int d^3k\,D((\vec{p}-\vec{k})^2)\phi(\vec{k})=\lambda\phi+\int\tilde{D}\phi\,.$$

Again $\tilde{D}(\vec{p},\vec{k})$ is infrared finite. The dependence on $\tilde{D}_2(p,k)$ has been suppressed in (14). We will add it back in at the end.

A finite integral equation for $\phi(\vec{p})$ emerges from (14) if the terms of order λ and λ^2 cancel. Demonstration of this cancellation is a lengthy algebraic exercise. However, since this step is crucial for the success of our approach, we give some of the details. Let $\phi(\vec{p})$ be written in the form[11]

$$\phi=\frac{1}{2}\begin{bmatrix} S+T+(\vec{V}+\vec{U})\cdot\vec{\sigma} & Q+R+(\vec{F}+\vec{G})\cdot\vec{\sigma}\\ Q-R+(\vec{F}-\vec{G})\cdot\vec{\sigma} & S-T+(\vec{V}-\vec{U})\cdot\vec{\sigma}\end{bmatrix}, \tag{15}$$

where $\vec{\sigma}$ is the vector of Pauli spin matrices and $S(\vec{p})$, $T(\vec{p})$, $V(\vec{p})$, ... are scalar and vector functions of $\vec{p}$. Equation (14) is equivalent to the following eight coupled integral equations.

$$\Omega S=\frac{f}{\overline{w}^2}\int D[(p^2\overline{A}^2S-m\overline{B}\overline{A}\vec{G}\cdot\vec{p})(\overline{w}+\lambda f)-\overline{w}E\overline{A}\vec{F}\cdot\vec{p}], \tag{16a}$$

$$\Omega T=0\,, \tag{16b}$$

$$\Omega Q=\frac{f}{\overline{w}^2}\int D[Q\overline{w}^2(\overline{w}+\lambda f)+\overline{w}Em\overline{B}R-\overline{w}E\overline{A}\vec{V}\cdot\vec{p}], \tag{16c}$$

$$\Omega R=\frac{f}{\overline{w}^2}\int D[(m^2\overline{B}^2R-m\overline{B}\overline{A}\vec{V}\cdot\vec{p})(\overline{w}+\lambda f)+\overline{w}Em\overline{B}Q], \tag{16d}$$

$$\Omega\vec{V}=\frac{f}{\overline{w}^2}\int D[(\overline{A}^2\vec{p}\vec{V}\cdot\vec{p}-m\overline{A}\overline{B}\vec{p}R)(\overline{w}+\lambda f)-\overline{w}E\overline{A}\vec{p}Q], \tag{16e}$$

$$\begin{aligned}\Omega\vec{U}=\frac{f}{\overline{w}^2}\int D[&(\overline{A}^2p^2\vec{U}-\overline{A}^2\vec{p}\vec{U}\cdot\vec{p}\\&-im\overline{A}\overline{B}(\vec{F}\times\vec{p}))(\overline{w}+\lambda f)\\&-i\overline{w}E\overline{A}\vec{G}\times\vec{p}],\end{aligned} \tag{16f}$$

$$\begin{aligned}\Omega\vec{F}=\frac{f}{\overline{w}^2}\int D[&(m^2\overline{B}^2\vec{F}+\overline{A}^2\vec{p}\vec{F}\cdot\vec{p}\\&-im\overline{A}\overline{B}(\vec{U}\times\vec{p}))(\overline{w}+\lambda f)\\&-\overline{w}E(\overline{A}\vec{p}S-m\overline{B}\vec{G})],\end{aligned} \tag{16g}$$

$$\begin{aligned}\Omega\vec{G}=\frac{f}{\overline{w}^2}\int D[&(\vec{G}\overline{w}^2-\overline{A}^2\vec{p}\vec{p}\cdot\vec{G}-m\overline{A}\overline{B}\vec{p}S)(\overline{w}+\lambda f)\\&-\overline{w}E(i\overline{A}(\vec{U}\times\vec{p})-m\overline{B}\vec{F})],\end{aligned} \tag{16h}$$

where $\Omega=(\overline{w}+\lambda f)^2-E^2$. The integrals in (16) are over three-momentum $\vec{k}$. The quantities $\overline{w}(\vec{p})$, $\overline{A}(\vec{p})$, and $\overline{B}(\vec{p})$ are all functions of external momentum $\vec{p}$. Since $\int D\phi=\lambda\phi+\int\tilde{D}\phi$, both sides of (16) have terms of order λ^2 and λ.

The quadratically divergent terms cancel if the Bethe-Salpeter amplitudes satisfy a set of constraint equations,

$$\begin{aligned}&S=-\frac{\overline{A}}{m\overline{B}}\vec{p}\cdot\vec{G}\,,\quad T=0\,,\quad \vec{V}=-\frac{\overline{A}}{m\overline{B}}\vec{p}R\,,\\&\vec{U}=-\frac{i\overline{A}}{m\overline{B}}(\vec{F}\times\vec{p}).\end{aligned} \tag{17}$$

The requirement that terms linear in λ cancel produces a set of integral equations for the remaining independent amplitudes:

$$Q=\frac{E}{m\overline{B}}R+\frac{1}{\overline{w}}f\int\tilde{D}Q\,, \tag{18a}$$

$$R=\frac{Em\bar{B}}{\bar{w}^2}Q+\frac{m^2\bar{B}^2}{\bar{w}^3}f\int\tilde{D}R+\frac{m\bar{B}\bar{A}}{\bar{w}^3}f\int\tilde{D}\vec{p}\cdot\vec{k}\left(\frac{\bar{A}}{m\bar{B}}\right), \quad (18b)$$

$$\vec{F}=\frac{Em\bar{B}}{\bar{w}^2}\left(\vec{G}+\frac{\bar{A}^2\vec{p}\,\vec{p}\cdot\vec{G}}{m^2\bar{B}^2}\right)+\frac{m^2\bar{B}^2}{\bar{w}^3}f\int\tilde{D}\left[\vec{F}+\frac{\bar{A}^2(p)\vec{p}\vec{F}\cdot\vec{p}}{m^2\bar{B}^2(p)}\right]-\frac{m\bar{A}\bar{B}f}{\bar{w}^3}\int\tilde{D}\vec{p}\times\left(\frac{\bar{A}}{mB}\vec{k}\times\vec{F}\right), \quad (19a)$$

$$\vec{G}=\frac{E}{m\bar{B}}\left(\vec{F}-\frac{\bar{A}^2\vec{p}\vec{F}\cdot\vec{p}}{\bar{w}^2}\right)+\frac{f}{\bar{w}}\int\tilde{D}\left[\vec{G}-\frac{\vec{p}\vec{G}\cdot\vec{p}\bar{A}^2}{\bar{w}(p)^2}\right]+mA\bar{B}\frac{\vec{p}f}{\bar{w}^3}\int D\left(\frac{\bar{A}\vec{G}\cdot\vec{k}}{m\bar{B}}\right). \quad (19b)$$

The final step is to examine the infrared-finite terms in (16). Here is where the real surprise occurs. From (16c) and (16d) we have

$$(\bar{w}^2-E^2)Q=\bar{w}f\int\tilde{D}Q+\frac{mE\bar{B}}{\bar{w}}\int\tilde{D}R+\frac{E}{\bar{w}}\bar{A}\int\tilde{D}\vec{p}\cdot\left(\frac{\bar{A}}{m\bar{B}}\vec{k}R\right), \quad (20a)$$

$$(\bar{w}^2-E^2)R=\frac{m^2\bar{B}^2}{\bar{w}}f\int\tilde{D}R+\frac{\bar{A}m\bar{B}}{\bar{w}}f\int\tilde{D}\vec{p}\cdot\left(\frac{\bar{A}}{m\bar{B}}\vec{k}R\right)+\frac{m\bar{B}E}{\bar{w}}f\int\tilde{D}Q. \quad (20b)$$

Although (20) appears to be very different from (18), it is, in fact, just a rearrangement of the same set of integral equations. The same surprise occurs for the λ^0 terms in (16g) and (16h).

Cancellation of divergent terms of each order in λ makes the eight equations in (16) equivalent to 24 equations for the eight Bethe-Salpeter amplitudes. (Actually there are 16 amplitudes and 48 equations if the vector nature of the amplitudes is taken into account.) The system appears to be overdetermined. However, for a γ_0-γ_0 interaction, all equations are satisfied if the amplitudes satisfy (17), (18), and (19). At this point we have no fundamental understanding as to why the equations work out so beautifully.

Still remaining is the question of the effect of the infrared-finite kernel $\tilde{D}_2(\vec{p},\vec{k})$. Define an amplitude ϕ' by

$$\gamma_0\phi'\gamma_0=\Gamma\phi\Gamma. \quad (21)$$

The matrices Γ describe the spin structure of the $\tilde{D}_2$ interaction. The amplitude ϕ has a representation similar to (15). Equation (16) is then modified by

$$\int DS\to\int(DS+\tilde{D}_2S'), \quad \int D\vec{V}\to\int(D\vec{V}+\tilde{D}_2\vec{V}''),$$

etc. The terms of order λ^2 in (16) are unaffected by this change. The constraints (17) are unchanged. Equation (18a) is modified by

$$\int\tilde{D}Q\to\int\tilde{D}Q+\int\tilde{D}_2Q',$$

and for (18b) we have

$$\int\tilde{D}R\to\int\tilde{D}R+\int\tilde{D}_2R'$$

and

$$\int\tilde{D}\left(\frac{\vec{k}\bar{A}}{m\bar{B}}R\right)\to\int\tilde{D}\left(\frac{\vec{k}\bar{A}}{m\bar{B}}R\right)-\int\tilde{D}_2\vec{V}'.$$

The complete program goes through in the presence of $\tilde{D}_2(p,k)$. The final, finite Bethe-Salpeter equations are modified by an additional finite interaction which could be used to control the large-momentum short-distance behavior. However, $\tilde{D}_2(p,k)$ does not affect the fact of confinement. Parity and charge-conjugation invariance guarantees that the Q,R equations do not mix with the $\vec{F},\vec{G}$ equations even with $\tilde{D}_2(p,k)$ present.

Equations (18) and (19) represent the culmination of this work. If they have finite bound-state solutions, we will have demonstrated a relationship between quark confinement and infrared singularities.

IV. BOUND-STATE SOLUTIONS

In the preceding two sections we have shown how an infrared-singular interaction produces an infrared-finite equation for quark-antiquark bound states. To complete the program, we need to show that (18) and (19) have bound-state solutions. The ideal program would have the following steps: (i) Assume a specific but general form for the infrared-singular function $D(\vec{k}^2)$. (ii) Solve (9) for $\bar{A}(p)$ and $\bar{B}(p)$. (iii) Use the solutions in (18) and (19) and solve for the bound-state spectrum. (iv) Test various possibilities for the infrared-finite interactions $D_1(p,k)$ and $D_2(p,k)$. (v) Compare with the experimental meson spectrum to fix the parameters of the model. This ideal program is beyond the scope of the present paper. We demonstrate the existence of confined solutions first by looking at the nonrelativistic limits of (18) and (19) and second by a variational calculation with the full equations. We ignore effects of possible infrared-finite corrections. The explicit form we choose for $D(k^2)$ is

$$D(k^2)=\frac{(m^2)^{n-1}}{(\vec{k}^2+\mu^2)^n}, \quad (22)$$

where $\mu^2\to 0$ produces the infrared limit. The quark mass dependence in the numerator is for dimensional consistency. The divergent constant λ is

$$\lambda=\int d^3k\,D(k^2)$$
$$=\frac{2\pi}{\mu^{2n-3}}\frac{\Gamma(\frac{3}{2})\Gamma(n-\frac{3}{2})}{\Gamma(n)}(m^2)^{n-1}, \qquad (23)$$

and n must lie in the range $\frac{3}{2}\le n<\frac{5}{2}$ in order that $\lambda\to\infty$ as $\mu\to 0$, yet the divergence can be controlled by a single subtraction. The Fourier representation of the finite kernel $\tilde{D}(\vec{p}-\vec{k})=D((\vec{p}-\vec{k})^2)-\lambda\delta(\vec{p}-\vec{k})$ is

$$\tilde{D}(k)=\frac{\pi^{3/2}}{2^{2n-3}}\frac{(n-\frac{1}{2})\Gamma(\frac{5}{2}-n)}{\Gamma(n)}\times\int\frac{d^3r}{(2\pi)^3}e^{i\vec{k}\cdot\vec{r}}(r)^{2n-3}(m^2)^{n-1}. \qquad (24)$$

This form for $\tilde{D}(k)$ is obtained by first calculating the Fourier transform of (22) and then letting $\mu^2\to 0$.

The amplitudes Q and R describe the quark-antiquark pair in a singlet-spin configuration.[11] Solutions to (18) are states with parity $P=(-1)^{J+1}$ and charge-conjugation parity $C=(-1)^J$. Examples of such states are the π, η, and B mesons. The $\vec{F}$ and $\vec{G}$ amplitudes describe the triplet configuration.[11] Parity invariance separates the triplet states with $J=L$ from those states with $J=L+1$. The uncoupled triplet states with $J=L$ state with $J=1$ and have $P=(-1)^{J+1}$ and $C=(-1)^{J+1}$. The A_1 meson is an example of such a state. Examples of coupled triplet states with $J=L\pm 1$, $P=(-1)^J$, and $C=(-1)^J$ are the ϵ, ρ, w, ϕ, f, A_2, g.

The representation (24) for $\tilde{D}(\vec{p}-\vec{k})$ should be used in (9) to solve for $\bar{A}(\vec{p})$ and $\bar{B}(\vec{p})$ with $a(p)$ and $b(p)$ constant. An exact solution is difficult. However, we note that the variation in these functions, as $|\vec{p}|$ ranges from 0 to ∞, is of order f, the effective coupling constant. In the weak-coupling limit the functions are nearly constant. We choose them to be constants and absorb them into the bare quark mass m and the coupling constant f, through a rescaling of all momenta. In Appendix C we explore more carefully the validity of this approximation. For the rest of this section we use $\bar{A}=\bar{B}=1$.

The nonrelativistic limits of (18) and (19) are obtained by multiplying through by $\bar{w}$ and setting $E=m+\epsilon$ and $\bar{w}=(p^2+m^2)^{1/2}=m+p^2/2m$. For (18) we have

$$\left(m+\frac{p^2}{2m}\right)Q=\left(m+\epsilon+\frac{p^2}{2m}\right)R+f\int\tilde{D}Q,$$
$$\left(m+\frac{p^2}{2m}\right)R=\left(m+\epsilon-\frac{p^2}{2m}\right)Q+f\int\tilde{D}R. \qquad (25)$$

Adding and subtracting these equations leads to

$$\left(m+\frac{p^2}{2m}\right)(Q+R)=(m+\epsilon)(Q+R)-\frac{p^2}{2m}(Q-R)+f\int\tilde{D}(Q+R), \qquad (26a)$$
$$\left(m+\frac{p^2}{2m}\right)(Q-R)=-(m+\epsilon)(Q-R)+\frac{p^2}{2m}(Q+R)+f\int\tilde{D}(Q-R). \qquad (26b)$$

The $m\to\infty$ limit of (26b) requires $Q=R$. Equation (26a) becomes

$$\frac{p^2}{2m}Q(\vec{p})=\epsilon Q(\vec{p})+f\int\tilde{D}(\vec{p}-\vec{k})Q(\vec{k}). \qquad (27)$$

In configuration space with $\tilde{D}(k)$ given by (24), Eq. (27) becomes a Schrödinger differential equation for $Q(\vec{r})$,

$$-\frac{\hbar^2}{2m}\nabla^2Q(\vec{r})+\gamma r^{2n-3}Q(\vec{r})=\epsilon Q(\vec{r}), \qquad (28)$$

where

$$\gamma=\frac{f\pi^{3/2}}{2^{n-3/2}}\frac{(n-\frac{1}{2})\Gamma(\frac{5}{2}-n)}{\Gamma(n)}(m^2)^{n-1}.$$

When $n>\frac{3}{2}$, there are confined solutions to (28). The linear potential favored by phenomenologists[12] is obtained when $n=2$.

The reduction of (19) proceeds along similar lines and the resulting equations are identical to (28) with Q replaced by $\vec{G}=\vec{F}$. In the nonrelativistic limit all spin states have the same energy spectrum. Relativistic corrections to (28) are readily calculated and produce spin-dependent interactions.

An exact treatment of (18) or (19) begins with a partial-wave decomposition to produce a set of coupled one-dimensional integral equations. Equation (19) produces a pair of coupled equations for the $J=L$ triplet states and four coupled integral equations for the $J=L\pm 1$ states. These equations do not lend themselves to solution by straightforward numerical techniques. The subtracted kernel $\tilde{D}(\vec{p}-\vec{k})$ is very difficult to handle numerically. We use a variational method for estimating bound-state energies. For example, we set $R=m\bar{B}R'/\bar{w}$, $Q=Q'$ and write (18) in the form

$$E\begin{bmatrix}Q'\\R'\end{bmatrix}=\begin{bmatrix}0 & \bar{w}-H_1\\ \bar{w}-H_2 & 0\end{bmatrix}\begin{bmatrix}Q'\\R'\end{bmatrix}. \qquad (29)$$

H_1 and H_2 are symmetric integral operators:

$$H_1=f\int d^3k\,\tilde{D}(\vec{p}-\vec{k}),$$
$$H_2=f\int d^3k\,D(\vec{p}-\vec{k})\left[\frac{m^2+\vec{p}\cdot\vec{k}}{\bar{w}(\vec{p})\bar{w}(\vec{k})}\right].$$

The invariant inner product is[13]

$$\int (Q'R'^{\dagger}+Q'^{\dagger}R')d^3k=\text{const}. \tag{30}$$

Bound-state energies are estimated with the trial wave functions of definite angular momentum

$$\begin{aligned} Q'_J(\vec{p}) &= \frac{p^J}{(p^2+m^2)^{q+J/2}}Y_{Jm}(\Omega_p),\\ R'_J(\vec{p}) &= \lambda Q_J(\vec{p}). \end{aligned} \tag{31}$$

The parameters q and λ are varied to minimize the expectation value of the energy calculated with the inner product (30). Results for the bound-state energies in units of the quark mass are shown in Fig. 2. No attempt is made to fit the spectrum of low-lying states.

Results for variational calculations of the triplet-state energies appear in Figs. 3 and 4. Trial wave functions for the uncoupled triplet states are

$$\begin{aligned} \vec{G}_0 &= \frac{p^J}{(p^2+m^2)^{q+J/2}}\vec{Y}_{JJm},\\ \vec{F}_0 &= \lambda\frac{m}{w}\vec{G}_0, \end{aligned} \tag{32}$$

where $\vec{Y}_{JLm}$ is a vector spherical harmonic.[14] Since $\vec{p}\cdot\vec{Y}_{JJm}(\Omega_p)=0$, it is straightforward to write (19) in the form of (29) with new operators H_1 and H_2. The expectation value of E is calculated with the inner product[13]

$$\int d^3k\,\frac{\overline{w}}{m\overline{B}}(\vec{F}^{\dagger}\cdot\vec{G}+\vec{F}\cdot\vec{G}^{\dagger})=\text{const}. \tag{33}$$

Again E is minimized as a function of λ and q.

The variational calculation for the coupled triplet amplitudes is more complicated since there are four amplitudes and four equations. The curves in Fig. 4 are produced with trial wave functions of the form

$$\begin{aligned} \vec{F}_\pm(p) &= f_\pm\left(\frac{m}{\overline{w}}\right)^{1/2}\frac{p^{J\pm1}}{(p^2+m^2)^{q+(J\pm1)/2}}\vec{Y}_{JJ\pm1m}(\Omega_p),\\ \vec{G}_\pm(p) &= g_\pm\vec{F}_\pm(p), \end{aligned} \tag{34}$$

with $f_+=1$ and f_-, g_+, g_-, q as variational parameters. Many pages of algebra are necessary to rearrange (19), minimize the expectation value of the energy, and calculate the variational inte-

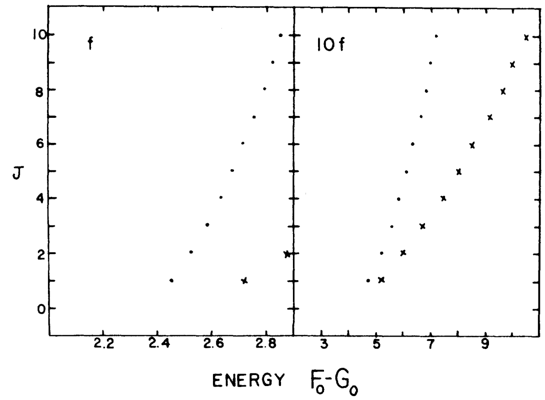

FIG. 3. The energy spectrum vs angular momentum J for the uncoupled triplet amplitudes $\vec{F}_0$-$\vec{G}_0$. The dots are for $n=1.75$ and the crosses are for $n=2.00$. The left figure shows results for $f=0.01$ ($n=1.75$) and $f=0.007$ ($n=2.00$). The right figure shows the effect of increasing the couplings by a factor of 10. Energy is in units of the quark mass. There are no $J=0$ states in this case.

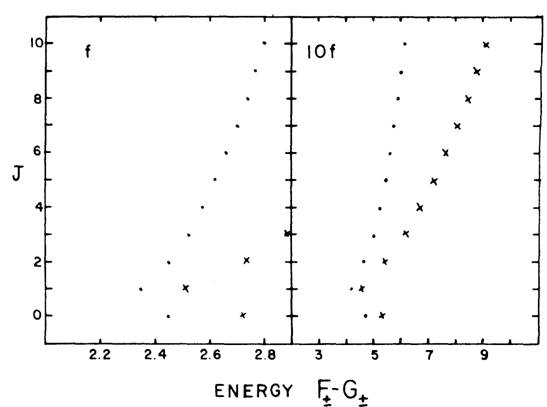

FIG. 4. The energy spectrum vs angular momentum J for the coupled triplet amplitudes $\vec{F}_\pm$-$\vec{G}_\pm$. The dots are for $n=1.75$ and the crosses are for $n=2.00$. The left figure shows results for $f=0.01$ ($n=1.75$) and $f=0.007$ ($n=2.00$). The right figure shows the effect of increasing the couplings by a factor of 10. Energy is in units of the quark mass. The $J=0$ solution is not a smooth extrapolation of higher-J states, since the equations partially decouple for $J=0$.

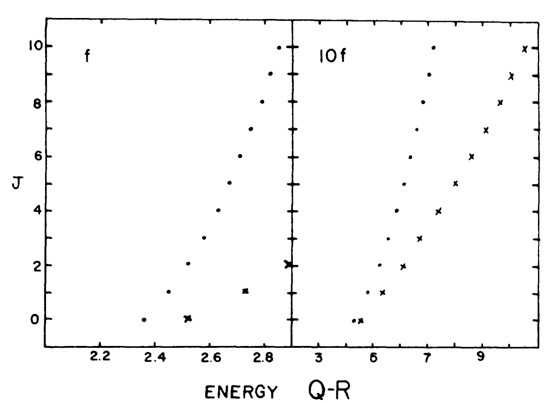

FIG. 2. The energy spectrum vs angular momentum J for the singlet states, Q-R amplitudes. The dots are for $n=1.75$ and the crosses are for $n=2.00$. The left figure shows results for $f=0.01$ ($n=1.75$) and $f=0.007$ ($n=2.00$). The right figure shows the effect of increasing the couplings by a factor of 10. Energy is in units of the quark mass.

grals.

In every case our estimates of the bound-state energy agree with what would be expected from arguments based on the Schrödinger equation (28). Energies lie above the quark "threshold" at $2m$. As the coupling constant f is decreased, the ground-state energy decreases and the problem becomes nonrelativistic. As the degree of the infrared singularity, the parameter n in (22) is increased, the bound-state energies increase. For fixed n and coupling f, a comparison of the singlet-triplet energies shows that singlet and uncoupled triplet states are nearly degenerate in energy for a given angular momentum J. The energy of the coupled triplet states, except for $J=0$, is less than that of the singlet states in qualitative agreeement with the experimental fact that the ϵ is more massive than the π meson but the f and ω are less massive than the B meson.

V. DISCUSSION AND CONCLUSIONS

We have developed a "theory" which relates the infrared behavior of the quark interaction to the effective potential which confines quarks inside mesons. There are no free quarks, yet there is a well-defined equation for bound states. The question of whether infrared slavery can work in a space with three space dimensions has an affirmative answer. We find that any infrared-singular interaction generates a quark propagator whose pole is at infinite energy in the infrared limit. However, both the vector nature of our interaction and its noncovariant dependence on momentum transfer are essential for the complete separation of infrared divergences in the Dyson equation and for the delicate cancellations which produce a finite Bethe-Salpeter equation. The cancellations depend on the fact that contour integrations in the k_0 plane give zero unless they encircle a quark pole. Why is the Coulomb-type, instantaneous, interaction singled out? All we offer is the explicit calculation in Appendix A.

The important, and very difficult, question to be answered now is whether an infrared singularity of the type assumed here is produced by a realistic fundamental theory of quarks and gluons.

Quark confinement, as developed in this work, appears to be a dynamical effect. In lattice gauge theories,[15] confinement is a direct consequence of local gauge invariance. Our discussion of the problem has ignored the effects of internal symmetry. Explanation of the absence of bound states with color, when SU(3) of color is the gauge group, requires analysis of symmetry effects. Extension of our model to describe baryons also requires an understanding of symmetry effects. The baryon problem is very difficult not only because of the horrors of the three-body problem but also because the existence of three-quark, but not two-quark, bound states implies an important role for three-body forces. The fact that it is possible to construct a successful infrared model of quark confinement for mesons suggests that the baryon problem is worth pursuing.

APPENDIX A: ALTERNATIVE INTERACTIONS

The quark interaction used in this paper is a function of three-momentum transfer. The spin structure is that of the fourth component of a four-vector. In this appendix we discuss alternate forms of the infrared-singular interaction and show that our choice is unique in that it permits exact isolation of the infrared behavior of the quark propagator and it leads to an infrared-finite Bethe-Salpeter equation.

Possible modifications of the interaction are to change its spin structure and/or to make it a covariant function of four-momentum transfer. Consider first changes in the spin structure and the effect of these changes on the quark propagator.

$$S_F^{-1}(p)=\gamma\cdot p(1+a)-m(1+b) \times\frac{-ig^2}{(2\pi)^4}\int d^4k\,D((\vec{p}-\vec{k})^2)\Gamma S_F(k)\Gamma\,. \quad \text{(A1)}$$

Again $S_F^{-1}(p)$ is given by (2). As in Sec. II, the k_0 contour integration yields equations similar to (3). The infrared behavior is controlled by a single subtraction to ultimately yield a set of equations corresponding to (6):

$$A=\bar{A}-\frac{\lambda f\eta_A A}{(A^2p^2+m^2B^2)^{1/2}}\,, \quad \text{(A2a)}$$

$$B=\bar{B}+\frac{\lambda f\eta_B B}{(A^2p^2+m^2B^2)^{1/2}}\,. \quad \text{(A2b)}$$

The constants η_A and η_B are defined by

$$\Gamma\vec{\gamma}\cdot\vec{k}\Gamma=\eta_A\vec{\gamma}\cdot\vec{k}\,,\quad \Gamma\Gamma=\eta_B\,.$$

For the γ_0-γ_0 interaction used in this paper we have $\eta_B=-\eta_A=1$. In Table I we present η_A, η_B

TABLE I. Values of η_A and η_B for various Γ where $\Gamma\vec{\gamma}\cdot\vec{p}\,\Gamma=\eta_A\vec{\gamma}\cdot\vec{p}$, $\Gamma\Gamma=\eta_B$.

Γ	η_A	η_B
1	1	1
$i\gamma_5$	−1	1
γ_μ	−2	4
γ_0	−1	1
$\gamma_\mu\gamma_5$	2	4
$\gamma_0\gamma_5$	1	1
$\sigma_{\mu\nu}$	0	12

for several different spin structures.

According to (A2)

$$A=\frac{\eta_A \bar{A}}{(\eta_B+\eta_A)-\eta_A\bar{B}/B}. \tag{A3}$$

The nature of the solutions to (A2) depends on whether or not $\eta_A+\eta_B=0$. If $\eta_A+\eta_B=0$, as it does for a γ_0-γ_0 interaction, the solution is given in Sec. II by (8). When $\eta_A+\eta_B\neq 0$, exact solutions to (A2) are complicated algebraically. However, we are interested in the $\lambda\to\infty$ limit; there are two possible solutions in this limit:

$$B_{\mathrm{I}}=\eta_B\frac{\lambda f}{m}+\bar{B}+O\left(\frac{1}{\lambda}\right), \tag{A4a}$$

$$A_{\mathrm{I}}=\frac{\eta_B}{\eta_B+\eta_A}\bar{A}+O\left(\frac{1}{\lambda}\right), \tag{A4b}$$

or

$$B_{\mathrm{II}}=\frac{\eta_A}{\eta_B+\eta_A}\bar{B}+O\left(\frac{1}{\lambda}\right), \tag{A5a}$$

$$A_{\mathrm{II}}=-\eta_A\frac{\lambda f}{p}+\bar{A}+O\left(\frac{1}{\lambda}\right). \tag{A5b}$$

The solution in (A4) has the interpretation of the quark mass becoming infinite in the infrared limit. This is the solution we use in the Bethe-Salpeter equation.

If the interaction $D((p-k)^2)$ is an infrared-singular function of four-momentum transfer, the infrared behavior of the Dyson equation can be isolated only after a rotation of the k_0 contour to a Euclidean four-space. The validity of the contour rotation is difficult to verify. However, having done it, we proceed as before. The equations equivalent to (6) and (A2) are

$$A=\bar{A}+\frac{\lambda f\eta_A A}{A^2p^2+m^2B^2}, \tag{A6a}$$

$$B=\bar{B}-\frac{\lambda f\eta_B B}{A^2p^2+m^2B^2}, \tag{A6b}$$

where η_A and η_B are given in Table I. In this case $f=g^2/(2\pi)^4$ and p^2 is a four-momentum squared. If $\eta_B=-\eta_A=1$, the solution of (A6) is

$$B=A\bar{B}/\bar{A}, \quad \bar{w}=(A^2p^2+m^2B^2)^{1/2},$$
$$A=\frac{\bar{A}}{2}\left[1\pm i\left(\frac{4\lambda f}{\bar{w}^2}-1\right)^{1/2}\right]. \tag{A7}$$

When $\eta_A+\eta_B\neq 0$, there are two possible solutions. The one corresponding to infinite quark mass is

$$B=\pm\frac{i(\lambda f\eta_B)^{1/2}}{m}+\bar{B}, \tag{A8a}$$

$$A=\frac{\eta_B}{\eta_A+\eta_B}\bar{A}. \tag{A8b}$$

TABLE II. Values of η_i where i refers to the set of amplitudes $\phi_i=S,T,\vec{V},\ldots,\Gamma\phi_i\Gamma=\eta_i\phi_i$.

Γ	S	T	$\vec{V}$	$\vec{U}$	Q	R	$\vec{F}$	$\vec{G}$
1	1	1	1	1	1	1	1	1
$i\gamma_5$	1	−1	1	−1	1	−1	1	−1
γ_0	1	1	1	1	−1	−1	−1	−1
γ_μ	4	−2	0	2	−4	2	0	−2
$\gamma_0\gamma_5$	1	−1	1	−1	−1	1	−1	1
$\gamma_\mu\gamma_5$	4	2	0	−2	−4	−2	0	2
$\sigma_{\mu\nu}$	12	0	0	4	12	0	0	4

The quark mass is imaginary in the infrared limit unless the coupling g^2 is negative or $\eta_B<0$. The validity of the contour rotation is questionable if the quark mass is imaginary.

Comparing (A4), (A6), and (A7) with (8), we see that except for an $i\gamma_5$ noncovariant interaction, none of the other interactions match our choice of γ_0 in terms of elegance and simplicity of solutions to the Dyson equation.

Next we compare Bethe-Salpeter equations for the various alternative interactions. When the kernel is a function of three-momentum transfer, an integration over the fourth momentum component produces a generalized version of (14).

$$\phi(\vec{p})=\frac{1}{w^2-E_2}\frac{f}{2w}\int d^3k\,D((\vec{p}-\vec{k})^2)$$
$$\times\{w(\vec{p})^2\gamma_0\phi'(\vec{k})\gamma_0-\Lambda(\vec{p})\phi'(\vec{k})\Lambda(\vec{p})$$
$$+E[\Lambda(\vec{p})\phi'(\vec{k})\gamma_0-\gamma_0\phi'(\vec{k})\Lambda(\vec{p})]\}, \tag{A9}$$

where $\Lambda(\vec{p})=-\vec{\gamma}\cdot\vec{p}A(\vec{p})+mB(\vec{p})$, $w(\vec{p})^2=A(\vec{p})^2p^2+m^2B(\vec{p})^2$, and $\phi'=\Gamma\phi\Gamma$. If ϕ has the representation (15) and the spin structures of Table I are considered, we find $S'=\eta_S S$, $T'=\eta_T T$, $\vec{V}'=\eta_V\vec{V},\ldots$. The set of η's is given in Table II. Equation (A9) is equivalent to eight coupled equations for the amplitudes $S,T,\ldots,$

$$KS=f\int D(p^2A^2S'+2EA\vec{p}\cdot\vec{F}'+2mAB\vec{G}'\cdot\vec{p}), \tag{A10a}$$

$$KT=0, \tag{A10b}$$

$$K\vec{V}=f\int D(A^2\vec{p}\vec{p}\cdot\vec{V}'+mAB\vec{p}R'+EA\vec{p}Q'), \tag{A10c}$$

$$K\vec{U}=f\int D[p^2A^2\vec{U}'-A^2\vec{p}\vec{p}\cdot\vec{U}'+iEA(\vec{G}'\times\vec{p})], \tag{A10d}$$

$$KQ=f\int D(-w^2Q'-mEBR'-EA\vec{p}\cdot\vec{V}'), \tag{A10e}$$

$$KR=f\int D(-m^2B^2R'-mBEQ'-mAB\vec{p}\cdot\vec{V}')\,, \tag{A10f}$$

$$K\vec{F}=f\int D[-m^2\vec{F}'-A^2\vec{p}\vec{F}'\cdot\vec{p}-imAB(\vec{U}'\times\vec{p}) -EAS'\vec{p}-EmB\vec{G}]\,, \tag{A10g}$$

$$K\vec{G}=f\int D[-w^2\vec{G}'+A^2\vec{p}\vec{G}'\cdot\vec{p}-m\vec{p}S'-EmB\vec{F}' -iEA(\vec{U}'\times\vec{p})]\,, \tag{A10h}$$

where $K=w(w^2-E^2)$. The infrared limit of (A10) is obtained by using (8) for γ_0 and $i\gamma_5$ and (A4) for the others to calculate the infrared limit of $A(\vec{p})$ and $B(\vec{p})$.

If the Bethe-Salpeter kernel has an $i\gamma_5$ spin structure, the requirement that the λ^2 divergence cancel on the two sides of (A10) leads to a set of constraint equations

$$S=-\frac{\bar{A}}{m\bar{B}}\vec{p}\cdot\vec{G}\,,\quad T=0\,,\quad \vec{V}=-\frac{\bar{A}}{m\bar{B}}\vec{p}R\,, \qquad Q=0\,,\quad \vec{U}=\vec{F}=0\,. \tag{A11}$$

The difference between (A11) and (17) is due to the different values of the η parameters for γ_0 and $i\gamma_5$ in Table II. Given (A11), we find that the terms in (A10) linear in λ require that all amplitudes vanish. There is no consistent set of infrared-finite Bethe-Salpeter equations for an $i\gamma_5$ interaction.

For the other five interactions in Table I, the solution to the Dyson equation is given by (A4) ($B\to\infty$, A finite, $w\to mB\to\infty$). The quadratic constraint equations in this case are

$$S=T=\vec{V}=\vec{u}=0\,,$$
$$Q=-\frac{\eta_Q}{|\eta_B|}Q\,,\quad R=-\frac{\eta_R}{|\eta_B|}R\,, \tag{A12}$$
$$\vec{F}=-\frac{\eta_F}{|\eta_B|}\vec{F}\,,\quad \vec{G}=-\frac{\eta_G}{|\eta_B|}\vec{G}\,,$$

where η_B comes from Table I and the other η's are in Table II. When $\Gamma=1$ or $\Gamma=\sigma_{\mu\nu}$, there are no nonzero solutions to (A12). For $\Gamma=\gamma_\mu$, $\gamma_0\gamma_5$, $\gamma_\mu\gamma_5$, only the Q amplitude can be nonzero. A short calculation of the order-λ constraints shows that Q must also vanish. The conclusion is that only the γ_0 spin structure produces a nontrivial set of equations in the infrared limit.

If the kernel is a function of four-momentum transfer, the Bethe-Salpeter equation (after contour rotation) is

$$\psi=-\frac{f}{A^2}\frac{[\gamma_0(ip_4+E)-\vec{\gamma}\cdot\vec{p}+\bar{m}]}{[(p_4-iE)^2+\vec{p}^2+\bar{m}^2]}\left(\int D\Gamma\psi\Gamma\right) \times\frac{[\gamma_0(ip_4-E)-\vec{\gamma}\cdot\vec{p}+\bar{m}^2]}{[(p_4+iE)^2+\vec{p}^2+\bar{m}^2]}\,, \tag{A13}$$

where $\bar{m}=mB/A$. The functions A and B are given by (A7) or (A8) in the infrared limit. If (A7) is appropriate, $\bar{m}$ is independent of the infrared cutoff. When (A13) is multiplied through by A^2, we have an equation of the form

$$\frac{\bar{A}^2}{4}\left[-\frac{4\lambda f}{\bar{w}^2}+2\pm2i\left(\frac{4\lambda f}{\bar{w}^2}-1\right)^{1/2}\right]\psi =O_1\left(\lambda\psi+\int\tilde{D}\psi\right)O_2\,, \tag{A14}$$

where O_1 and O_2 are combinations of γ matrices. Although the order-λ terms can cancel, there is a $\lambda^{1/2}$ term on the left of (A14) but not on the right. The wave function ψ must vanish in the infrared limit.

If the appropriate solution to the Dyson equation is given by (A8), then the infrared limit of (A13) is obtained by letting $\bar{m}=mB/A\to\infty$. The leading equation produces the constraint

$$\psi=-\frac{f}{A^2}\frac{\lambda}{m^2}\Gamma\psi\Gamma=\frac{1}{\eta_B}\Gamma\psi\Gamma\,. \tag{A15}$$

If $\Gamma=1$, this equation is automatically satisfied. If Γ is anything else, there are only a few amplitudes which satisfy (A15). However, the next-order constraint equation (order $\lambda^{3/2}$) is

$$\psi=\frac{A}{m\bar{B}}[\gamma\cdot(p+E)\psi+\psi\gamma\cdot(p-E)]\,. \tag{A16}$$

This equation has no solutions unless the momentum dependence of $A/m\bar{B}$ is in a way which contradicts the equations satisfied by A and $\bar{B}$. Moreover, it is necessary to cancel terms of order λ and $\lambda^{1/2}$ in order to produce a finite equation. The conclusion is that there is no covariant interaction which generates an infrared-finite Bethe-Salpeter equation.

In this appendix we have shown that, given our methodology, the only interaction which produces an infrared-finite Bethe-Salpeter equation is the one actually used in this paper.

APPENDIX B: HIGHER-ORDER CORRECTIONS

In Sec. III it was argued that $D((\vec{p}-\vec{k})^2)$ is the infrared-singular interaction both in the Dyson equation for the quark propagator and in the Bethe-Salpeter equation for bound states. Part of the justification relied on the fact that, for our choice of an interaction, vertex corrections are infrared finite and higher-order corrections to the Bethe-Salpeter kernel are no more singular than the lowest-order terms.

The one-gluon modification of the quark-gluon vertex in Fig. 5(a) is calculated from

$$\Gamma(p,p') = -\frac{c}{2\pi i}\int d^4k\,\gamma_0 S_F(p-k)\gamma_0 S_F(p'-k)\gamma_0 D(\vec{k}^2)\,, \tag{B1}$$

where c is a constant. For the quark propagator we use (2) with $A_0=1$. The k_0 integration has contributions from the two quark poles

$$\Gamma(p,p') = c\int d^3k\,D(k^2)\gamma_0\left\{\frac{[\gamma_0 w - A\vec{\gamma}\cdot(\vec{p}-\vec{k})+mB]\gamma_0[\gamma_0(p_0'-p_0+w)-A'\vec{\gamma}\cdot(\vec{p}'-\vec{k})+mB']}{2w[(p_0'-p_0+w)^2-w'^2]} + \frac{[\gamma_0(p_0-p_0'+w')-A\vec{\gamma}\cdot(\vec{p}-\vec{k})+mB]\gamma_0[\gamma_0 w'-A'\vec{\gamma}\cdot(\vec{p}'-\vec{k}')+mB']}{[(p_0-p_0'+w')^2-w^2]2w'}\right\}\gamma_0\,, \tag{B2}$$

where $w^2=A^2(\vec{p}-\vec{k})^2+m^2B^2$, $w'^2=A'^2(\vec{p}'-\vec{k})^2+m^2B'^2$, and $A=A(\vec{p}-\vec{k})$, $A'=A(\vec{p}'-\vec{k})$, etc. In the infrared limit, $A(\vec{q})$ and $B(\vec{q})$ are given by (8). The integrand in (B2) vanishes as λ^{-1}, $\lambda\to\infty$. Hence, $\Gamma=0$ unless this zero is compensated by a divergence at $k^2=0$ from the integral over $D(\vec{k}^2)$. In the infrared limit we find

$$\Gamma(p,p') = -\frac{c}{f}\frac{\gamma_0}{2\bar{w}\bar{w}'}\left[\Lambda(\vec{p})+\Lambda(\vec{p}') + \frac{(p_0-p_0')\Lambda(p)\gamma_0\Lambda(p')}{p_0+\bar{w}'-p_0'-\bar{w}}\right]\gamma_0\,, \tag{B3}$$

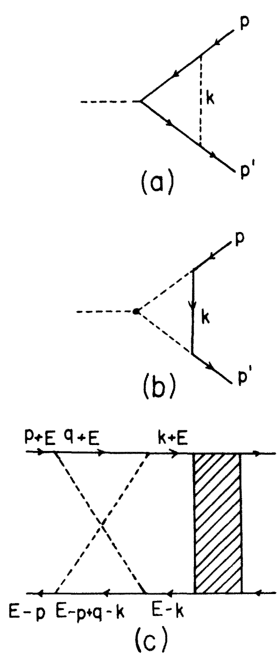

FIG. 5. (a) One-gluon vertex correction which is infrared finite. (b) A vertex correction introduced by trilinear gluon couplings. (c) First-order correction to the one-gluon Bethe-Salpeter kernel. In the infrared limit this amplitude is no more singular than the one-gluon term.

with $\Lambda(\vec{p})=\gamma_0\bar{w}(\vec{p})-\bar{A}(p)\vec{\gamma}\cdot\vec{p}+m\bar{B}(\vec{p})$. [Each of the two terms inside the curly brackets in (B2) is constant as $\lambda\to\infty$, but the constants cancel out to leave terms of order λ^{-1}.] Note only is $\Gamma(p,p')$ finite in the infrared limit, but it does not enhance the infrared behavior at $\vec{p}=\vec{p}'$ when inserted in another diagram.

A second class of vertex corrections are typified by Fig. 5(b). The amplitude for this vertex part is determined by the integral

$$\Gamma'(p,p') = -\frac{c'}{2\pi i}\int d^4k\,\gamma_0 S_F(k)\gamma_0 D((\vec{p}-\vec{k})^2) \times D((\vec{p}'-\vec{k})^2)F(p-k,p'-k)\,. \tag{B4}$$

The k_0 contour integral cancels out the λ^{-1} behavior of the propagator. $\Gamma'(p,p')$ is proportional to

$$\Gamma'\sim\lambda D((p'-p)^2)\,[F(0,p'-p)+F((p-p'),0)]\,. \tag{B5}$$

The combination $\lambda D((\vec{p}-\vec{p}')^2)$ is very singular. However, the function $F(p,p')$ is introduced into (B4) to describe the three-gluon vertex. The behavior of this function is unspecified in our model. In gauge theories the three-gluon vertex vanishes when all gluons are purely timelike or when two of the gluons have the same momentum. The function $F(p,p')$ will suppress the (B4) contributions and could even render it nonsingular. The other possibility is that, since (B4) is generated by pure gluon interactions, it is already contained in the effective propagator $D(k^2)$. In that case, Fig. 5(b) does not represent a permissible vertex correction.

Next we compare the infrared behavior of the crossed-gluon Bethe-Salpeter kernel [Fig. 5(c)] with the single-gluon kernel. Equation (12) is to be compared with

$$\int_{-\infty}^{\infty} dp_0 \int_{-\infty}^{\infty} dq_0 \int d^3q \int d^3k\, D((\vec{p}-\vec{q})^2) D((\vec{q}-\vec{k})^2) \times S_F(p+E)\gamma_0 S_F(q+E)\gamma_0 \phi(\vec{k},E)\gamma_0 S_F(k+p-q-E)\gamma_0 S_F(p-E)\,. \tag{B6}$$

A complete calculation of the infrared behavior of (B5) is complicated but possible. A qualitative argument produces the same result. Each propagator in (B6) vanishes like λ^{-1} in the infrared limit. The p_0 and q_0 integrations both enclose quark poles and reduce the net effect of the four propagators to order λ^{-2}. Each singular kernel contributes a factor of λ to make the overall integral infrared finite. The crossed-gluon kernel is no more singular than the single-gluon terms. The divergence of (B6) is the same order as produced by single-gluon exchange in (12). It is straightforward to generalize the result to amplitudes with any number of gluon exchanges.

APPENDIX C SOLUTION TO THE DYSON EQUATIONS

In the discussion of solutions to the bound-state equations we made the approximation $\bar{A}=\bar{B}=1$. Here we study more carefully the problem of solving (9) for the functions $\bar{A}(\vec{p})$ and $\bar{B}(\vec{p})$. First (9) is written in the form

$$\bar{A}(\vec{p}) = 1 + f\int d^3k\, \tilde{D}(\vec{p}-\vec{k})\frac{\vec{p}\cdot\vec{k}}{p^2}\frac{1}{[k^2+m^2\Gamma(k)^2]^{1/2}}\,, \tag{C1}$$

$$\bar{B}(\vec{p}) = 1 + f\int d^3k\, \tilde{D}(\vec{p}-\vec{k})\frac{\Gamma(k)}{[k^2+m^2\Gamma(k)^2]^{1/2}}\,, \tag{C2}$$

where $\Gamma(\vec{k})=\bar{B}(\vec{k})/\bar{A}(\vec{k})$. The constant terms in (C1) and (C2) have been adjusted so that $\bar{A}(\infty)=\bar{B}(\infty)=1$. Since the integrals depend only on the ratio $\Gamma(\vec{k})$, a convenient ansatz for a solution is that $\Gamma(\vec{k})=\Gamma(\infty)=1$. In other words, $\bar{A}(\vec{k})$ and $\bar{B}(\vec{k})$ are not separately constant, but their ratio is constant. Corrections to this guess are determined by an iteration procedure.

If $\Gamma(\vec{k})=1$ and $D(k^2)$ is given by (22), the integrals in (C1) and (C2) produce hypergeometric functions

$$\bar{A}_n(p) = 1 - f\alpha(n)\tfrac{2}{3}nF(n+1, n-1, \tfrac{5}{2}; -p^2/m^2)\,, \tag{C3}$$

$$\bar{B}_n(p) = 1 - f\alpha(n)F(n, n-1, \tfrac{3}{2}; -p^2/m^2)\,, \tag{C4}$$

where

$$\alpha(n) = 2(n-\tfrac{1}{2})\sqrt{\pi}\,\Gamma(\tfrac{5}{2}-n)\Gamma(n-1)\,,$$

and $\Gamma(n)$ is a gamma function. The hypergeometric function $F(a,b,c;-z^2)$ has the properties that $F(0)=1$, $\lim_{z\to\infty}F(-z^2)\to 0$. When $n=\frac{3}{2}$, we have an exact solution,

$$\bar{A}_{3/2}(p) = \bar{B}_{3/2}(p) = 1 - \frac{2\pi f}{(1+p^2/m^2)^{3/2}}\,. \tag{C5}$$

Since $\bar{A}_n(p)$ and $\bar{B}_n(p)$ approach $\bar{A}=\bar{B}=1$ monotonically from below as p increases, a measure of the extent to which (C3) and (C4) disagree with our starting ansatz is given by

$$\Gamma_n(0) - 1 = \frac{f\alpha(n)(\frac{2}{3}n-1)}{1-f\alpha(n)\frac{2}{3}n}\,. \tag{C6}$$

Thus, for weak coupling and/or $n\approx\frac{3}{2}$, $\bar{A}_n(p)$ and $\bar{B}_n(p)$ as given in (C3) and (C4) are good approximate solutions to (C1) and (C2). Moreover, (C6) provides a measure of what is meant by small coupling. Small coupling means the denominator in (C6) is approximately unity or

$$f\alpha(n)\tfrac{2}{3}n \ll 1\,.$$

For $n=\frac{3}{2}$ this means $2\pi f\ll 1$, and for $n=2$ it means $4\pi f\ll 1$. As $n\to\frac{5}{2}$, the coupling constant goes to zero.

It is possible to calculate analytically a correction to (C3) and (C4) with the assumption that $\Gamma(k)=1+\epsilon h(k)$, where $\epsilon\ll 1$ and $h(k)$ has a form that leads to tractable integrals. The requirement that $\Gamma(0)$ and $\Gamma'(0)=d\Gamma/dk^2|_{k^2=0}$ match for the input and output function fixes ϵ and parameters in $h(k)$. The result is that $|\Gamma(k)-1|$ is decreased over what it is when calculated from (C3) and (C4). In effect, iteration of the solution improves agreement with the ansatz.

The conclusion of this appendix is that for small couplings and n not too much larger than $\frac{3}{2}$, the approximation made in the bound-state equations is a good approximation.

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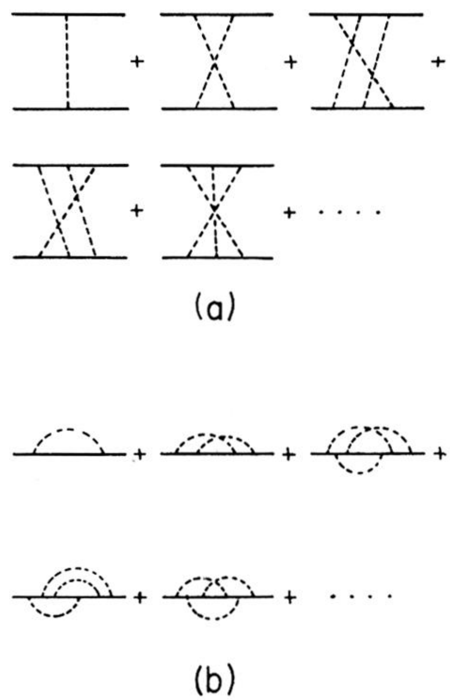

FIG. 1. (a) The diagrams contributing to the Bethe-Salpeter kernel. (b) The diagrams contributing to the quark self-energy. There is a one-to-one correspondence between the two sets. Solid lines are quarks and dotted lines are gluons.

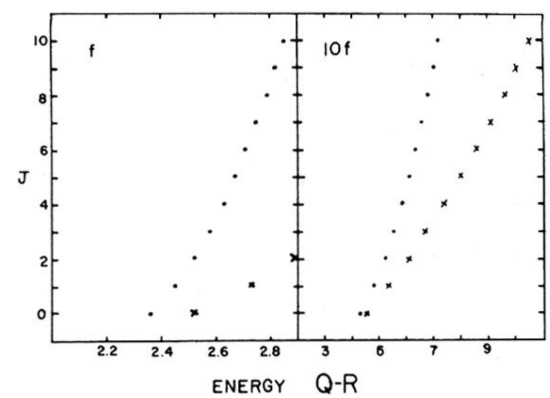

FIG. 2. The energy spectrum vs angular momentum J for the singlet states, $\boldsymbol{Q}$–$\boldsymbol{R}$ amplitudes. The dots are for $n = 1.75$ and the crosses are for $n = 2.00$. The left figure shows results for $f = 0.01$ ($n = 1.75$) and $f = 0.007$ ($n = 2.00$). The right figure shows the effect of increasing the couplings by a factor of 10. Energy is in units of the quark mass.

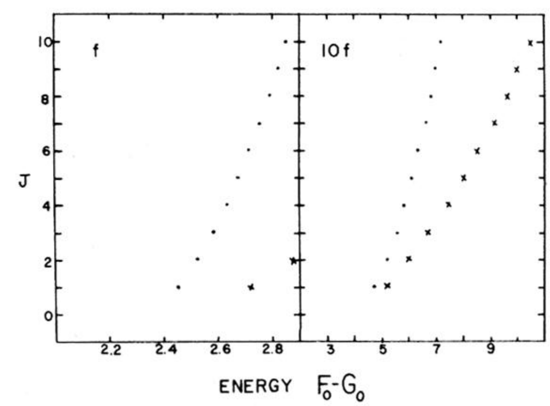

FIG. 3. The energy spectrum vs angular momentum J for the uncoupled triplet amplitudes $\vec{F}_0-\vec{G}_0$. The dots are for $n=1.75$ and the crosses are for $n=2.00$. The left figure shows results for $f=0.01$ ($n=1.75$) and $f=0.007$ ($n=2.00$). The right figure shows the effect of increasing the couplings by a factor of 10. Energy is in units of the quark mass. There are no $J=0$ states in this case.

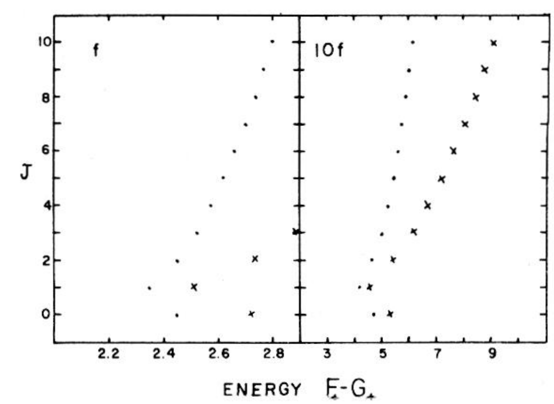

FIG. 4. The energy spectrum vs angular momentum J for the coupled triplet amplitudes $\vec{F}_{\pm} - \vec{G}_{\pm}$. The dots are for $n = 1.75$ and the crosses are for $n = 2.00$. The left figure shows results for $f = 0.01$ ($n = 1.75$) and $f = 0.007$ ($n = 2.00$). The right figure shows the effect of increasing the couplings by a factor of 10. Energy is in units of the quark mass. The $J = 0$ solution is not a smooth extrapolation of higher-J states, since the equations partially decouple for $J = 0$.

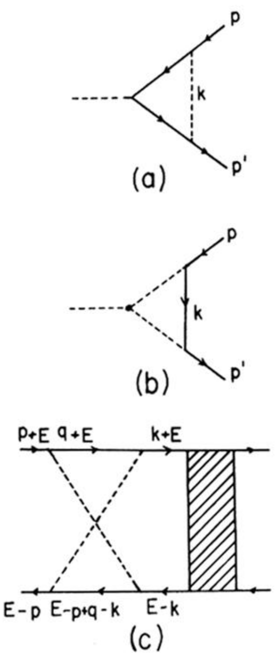

FIG. 5. (a) One-gluon vertex correction which is infrared finite. (b) A vertex correction introduced by trilinear gluon couplings. (c) First-order correction to the one-gluon Bethe-Salpeter kernel. In the infrared limit this amplitude is no more singular than the one-gluon term.