S-matrix inverse scattering problem via the fixed-point theorem

James T. Cushing
Department of Physics, University of Notre Dame, Notre Dame, Indiana 46556
(Received 20 March 1978)

By use of the contraction mapping principle we show explicitly that the equations for the inverse scattering problem [i.e., constructing the full amplitude, $A(s,t)$, from just the s-wave amplitude $A_0(s)$ given at all real s] in nonrelativistic S-matrix theory have a locally unique solution when there are no pole terms and no subtractions and when the relevant norms are sufficiently small. The corresponding mapping problem for the relativistic, crossing-symmetric case is also formulated and the question of uniqueness is discussed.

In a previous paper[1] we discussed the inverse scattering problem of determining the full scattering amplitude $A(s,t)$, given just the s-wave projection $A_0(s)$. Since $A_0(s)$ is assumed to be analytic in s, values of $A_0(s)$ for positive values of s above threshold will in principle determine $A_0(s)$ for all values of s. We are interested here in establishing the existence of a unique determination of $A(s,t)$ from $A_0(s)$. In particular, we assume that $A(s,t)$ can be written as

$$A(s,t)=\frac{1}{\pi}\int_4^\infty \frac{dt'\sigma(t')}{t'-t} + \frac{1}{\pi^2}\int_4^\infty ds'\int_4^\infty \frac{dt'\rho(s',t')}{(s'-s)(t'-t)}, \qquad (1)$$

where the nonrelativistic double-spectral function $\rho(s,t)$ vanishes below the curve

$$t=\frac{16s}{s-4}. \qquad (2)$$

We established an iterative scheme [cf. Eqs. (27) and (12) of Ref. 1] to determine $\sigma(t)$ and $\rho(s,t)$ given the imaginary part of the s-wave amplitude on the left-hand cut. We did not prove that this iterative scheme converged, although we did show that if there was a solution, then it must be unique. We now formulate this as a mapping problem and establish the existence of a locally unique solution via the contraction mapping principle. In terms of the discontinuity of $A_0(s)$ on the left-hand cut $-\infty < s \leq 0$, we define

$$g(t)=-\mathrm{Im}[tA_0^L(4-t)], \quad t\geq 4. \qquad (3)$$

As shown in Ref. 1, the s-wave projection of Eq. (1) yields

$$g(t)=\int_4^t dt'\,\sigma(t')+\frac{1}{\pi}\int_4^\infty ds'\int_4^\infty \frac{dt'\rho(s',t')}{s'-4+t}, \qquad (4)$$

so that

$$\frac{dg(t)}{dt}=\sigma(t)+\frac{1}{\pi}\int_4^\infty \frac{ds'}{s'-4+t}\left[\rho(s',t)-\int_4^t \frac{dt'(s',t')}{s'-4+t}\right]. \qquad (5)$$

The discontinuity of $A(s,t)$ with respect to t is

$$D_t(s,t)=\sigma(t)+\frac{1}{\pi}\int_4^\infty \frac{ds'\rho(s',t)}{(s'-s)}. \qquad (6)$$

If we combine Eqs. (4) and (5) we obtain

$$D_t(s,t)=\frac{1}{\pi}\int_4^\infty ds'\left(\frac{1}{s'-s}-\frac{1}{s'-4+t}\right)\rho(s',t) + \frac{1}{\pi}\int_4^\infty ds'\int_4^t dt'\frac{\rho(s',t')}{(s'-4+t)^2}+\frac{dg(t)}{dt}. \qquad (7)$$

Elastic unitarity applied to Eq. (1) for $4\leq s<\infty$ yields the standard result

$$\rho(s,t)=\frac{2}{\pi[s(s-4)]^{1/2}} \times\int_4^\infty dt_1\int_4^\infty dt_2\,\frac{D_t^*(s,t_1)D_t(s,t_2)H(t-t_0)}{K^{1/2}(s,t;t_1,t_2)}, \qquad (8)$$

where $t_0(s;t_1,t_2)$ and $K(s,t;t_1,t_2)$ are given in Ref. 1 [Eqs. (13) and (14), respectively]. Equations (7) and (8) together define a nonlinear mapping which we denote by

$$\rho'=\Theta(\rho). \qquad (9)$$

In fact, the mapping of Eq. (9) is very similar to that considered by Atkinson and Warnock[2] for a relativistic crossing-symmetric problem with one subtraction. We shall simply indicate the modifications necessary to allow their results to be applied to our problem. As in Ref. 2 we take $\rho(s,t)$ to belong to that Banach space of real functions $F(s,t)$ defined over $s\in[4,\infty)$, $t\in[4,\infty)$ for which there exists the norm

$$\|F\|\equiv \sup_{4\leq s_1,s_2,t_1,t_2<\infty}\frac{|F(s_1,t_1)-F(s_2,t_2)|\ln^2\bar{s}\ln^2\bar{t}}{\left|\frac{s_1-s_2}{s_1s_2\bar{t}}\right|^{\mu}+\left|\frac{t_1-t_2}{t_1t_2\bar{s}}\right|^{\mu}} \qquad (10)$$

and for which

$$\lim_{s\to\infty}F(s,t)=0=\lim_{t\to\infty}F(s,t). \qquad (11)$$

Here we have set $\bar{s}=\min(s_1,s_2)$, $\bar{t}=\min(t_1,t_2)$, and the Hölder index μ is restricted to the range $0<\mu<\frac{1}{2}$. We take $dg(t)/dt$ in Eq. (7) to belong to the Banach space of real continuous functions $f(t)$ on $[4,\infty)$ such that there exists the norm

$$\|f\| = \sup_{4\leq t<\infty} |t^{2\mu}f(t)|. \tag{12}$$

We now restrict $\rho(s,t)$ by

$$\|\rho(s,t)\| \leq b \tag{13}$$

and $dg(t)/dt$ by

$$\left\|\frac{dg(t)}{dt}\right\| \leq c. \tag{14}$$

Equations (13) and (14) will, for example, be satisfied for a superposition of Yukawa potentials provided the strength is sufficiently weak (i.e., bound states and resonances are certainly ruled out). Such an amplitude can be kept as small as we please. Our mapping problem of Eqs. (7)–(9) now corresponds to that of Atkinson and Warnock.[3] In light of the restrictions of Eqs. (13) and (14), we need only show that

$$|h(t)| \leq O(\psi)t^{-2\mu}, \tag{15}$$

where

$$h(t) \equiv \frac{1}{\pi}\int_4^t dt' \int_4^\infty ds' \frac{\rho(s',t')}{(s'-4+t)^2} \tag{16}$$

and

$$\psi = \max(b,c). \tag{17}$$

Here Eq. (15) means that $|h(t)| \leq M|\psi| t^{-2\mu}$ for all ψ. From Eqs. (10), (11), and (13) it follows that

$$|\rho(s,t)| \leq b(st)^{-\mu}\ln^{-2}(s)\ln^{-2}(t). \tag{18}$$

Since the integrals in Eq. (16) are not even principal-value ones, it is a simple matter to bound $h(t)$ as

$$\begin{aligned}|h(t)| &\leq \frac{b}{\pi}\int_4^t \frac{dt'}{t'^{\mu}\ln^2 t'} \int_4^\infty \frac{ds'}{s'^{\mu}\ln^2 s'(s'-4+t)^2}\\ &\leq \frac{b}{\pi}\int_4^t dt'\, t'^{-\mu}\left[\frac{1}{t^2}\int_4^t \frac{ds'}{s'^{\mu}\ln^2 s'} + t^{-1-\mu}\int_1^\infty \frac{dy}{y^{\mu}(y+1-4/t)^2}\right]\\ &\leq \frac{b}{\pi}\frac{(2-\mu)}{(1-\mu)^2}\, t^{-2\mu} = O(\psi)t^{-2\mu}.\end{aligned} \tag{19}$$

Therefore, the arguments in Refs. 2 and 4 show that the mapping of Eq. (9) is bounded and continuous and, in fact, a contraction mapping so that

$$\|\rho_1'(s,t)-\rho_2'(s,t)\| \leq O(\psi)\|\rho_1(s,t)-\rho_2(s,t)\|. \tag{20}$$

For b and c sufficiently small [cf. Eqs. (13) and (14)] we have a contraction mapping so that by the Banach-Cacciapoli theorem[5] there exists a unique fixed point $\rho_0(s,t)$ for the mapping of Eq. (9) such that

$$\rho_0 = \Theta(\rho_0). \tag{21}$$

It is important to note that this fixed point is nonzero since $\rho_0\equiv 0$ is not a solution to Eqs. (7) and (8) when $dg(t)/dt \neq 0$.

We have shown that *if* $A_0(s)$ comes from a Mandelstam amplitude $A(s,t)$ of the form of Eq. (1), then that Mandelstam amplitude is uniquely determined by $A_0(s)$. However, we are not guaranteed that this pair $\sigma(t)$, $\rho(s,t)$ will necessarily reproduce $A_0(s)$ everywhere, but only that $dg(t)/dt$ will be correctly reproduced for $t\geq 4$, that is, that

$$\frac{d}{ds}[(4-s)\,\mathrm{Im}A_0^L(s)], \quad -\infty<s\leq 0$$

will be reproduced. If we denote by $\bar{A}_0(s)$ that s-wave amplitude generated by $\sigma(t),\rho(s,t)$, then we are certain only that

$$\frac{d}{ds}[(4-s)\,\mathrm{Im}\bar{A}_0^L(s)] = \frac{d}{ds}[(4-s)\,\mathrm{Im}A_0^L(s)], \quad -\infty<s\leq 0 \tag{22}$$

so that

$$\mathrm{Im}\bar{A}_0^L(s) = \mathrm{Im}A_0^L(s) + \frac{a}{s-4}, \quad -\infty<s\leq 0 \tag{23}$$

where a is an arbitrary constant. This is consistent with Eq. (4) which tells us that

$$\mathrm{Im}\bar{A}_0^L(s) \underset{s\to-\infty}{\sim} \frac{1}{s}\int_4^\infty dt'\,\sigma(t'). \tag{24}$$

Therefore, we cannot conclude that $a=0$ no matter what we are willing to assume about the rate of falloff of $\mathrm{Im}A_0(s)$ unless it should happen that

$$\int_4^\infty dt'\,\sigma(t') = 0, \tag{25}$$

which need not be the case. This is not surprising since we would not expect an arbitrary s-wave amplitude $A_0(s)$ to have come from a Mandelstam representation. Since for a Mandelstam amplitude we expect

$$A_0(s) \underset{s\to\infty}{\sim} \frac{\ln s}{s} \int_4^\infty dt'\,\sigma(t'), \tag{26}$$

then if $A_0(s)$ falls off too rapidly we cannot in general have it reproduced by a Mandelstam amplitude of the form of Eq. (1) [or in terms of the language of potential scattering by a local potential $V(r)$, which is a superposition of Yukawa potentials].

We can now ask whether or not the relativistic inverse-scattering problem has a unique solution for the completely crossing-symmetric generalization of Eq. (1),

$$A(s,t,u) = \frac{1}{\pi}\int_4^\infty dt'\,\sigma(t')\left(\frac{1}{t'-s} + \frac{1}{t'-t} + \frac{1}{t'-u}\right) + \frac{1}{\pi^2}\int_4^\infty ds' \int_4^\infty dt'\,\rho(s',t')\left[\frac{1}{(t'-t)(s'-s)} + \frac{1}{(t'-u)(s'-t)} + \frac{1}{(t'-s)(s'-u)}\right], \tag{27}$$

where

$$\rho(s,t) = \rho(t,s) \tag{28}$$

and where $\rho(s,t)$ must vanish on or below the curve

$$t = \min\left(\frac{4s}{s-16}, \frac{16s}{s-4}\right). \tag{29}$$

In this case elastic unitarity holds only in the region $4 \le s \le 16$, so that unitarity for the s wave becomes

$$\mathrm{Im}A_0(s) = \left(\frac{s-4}{s}\right)^{1/2} |A_0(s)|^2 + \tfrac{1}{4}\left(\frac{s}{s-4}\right)^{1/2} [1-\eta^2(s)], \tag{30}$$

where $\eta(s)$ is the inelasticity function and must equal unity for $s \le 16$. Notice that since we are taking $A_0(s)$ as *given*, then we may consider $\eta(s)$ as known. The standard mapping problem for a $\rho(s,t)$ can now be stated as

$$D_t(s,t) = \frac{1}{\pi}\int_4^\infty ds'\,\rho(s',t)\left(\frac{1}{s'-s} + \frac{1}{s'+s+t-4}\right) - \frac{2}{\pi(s-4)}\int_4^\infty dt'\,\rho(s,t')\ln\left(1+\frac{s-4}{t'}\right) + \mathrm{Im}A_0^R(s), \tag{31}$$

$$\rho^{\mathrm{el}}(s,t) = \frac{4}{\pi[s(s-4)]^{1/2}} \int_4^\infty dt_1 \int_4^\infty dt_2\, \frac{D_t^*(s,t_1)D_t(s,t_2)H(t-t_0)}{K^{1/2}(s,t;t_1,t_2)}, \tag{32}$$

$$\rho(s,t) = \rho^{\mathrm{el}}(s,t) + \rho^{\mathrm{el}}(t,s) + v(s,t), \tag{33}$$

where $v(s,t) = v(t,s)$ is an arbitrary function satisfying Eqs. (10) and (11) and vanishing on and below the curve

$$t = 16\,\frac{t-4}{t-16}. \tag{34}$$

We summarize the mapping of Eqs. (31)–(33) as

$$\rho' = \mathcal{O}(\rho; v). \tag{35}$$

Atkinson and Warnock[2] have shown that Eq. (35) defines a contraction mapping provided ψ is small enough where ψ is now defined as

$$\psi = \max(b, c, \|v\|, \|1-\eta\|). \tag{36}$$

That is, there is a locally unique fixed point $\rho(s,t)$ for any *given* $v(s,t)$. Any one of these $\rho(s,t)$ will correctly reproduce $\mathrm{Im}A_0^R(s)$, the discontinuity of $A_0(s)$ on the right-hand cut. However, we do not know which, if any, of these will also yield the correct value for $\mathrm{Im}A_0^L(s)$, the discontinuity of $A_0(s)$ on the left-hand cut. For $-\infty < s \le 0$, this is given as

$$\mathrm{Im}A_0^L(s) = \frac{2}{(s-4)}\int_4^{4-s} dt'\,\sigma(t') + \frac{2}{\pi(s-4)}\int_4^\infty ds' \int_4^{4-s} dt'\,\rho(s',t')\left(\frac{1}{s'-s} + \frac{1}{t'+s+s-4}\right) + \frac{1}{\pi(s-4)}\int_4^{-s} dt'\,\rho(4-s-t',t')\ln\left(\frac{4-s}{t'}-1\right). \tag{37}$$

Whether or not any of the fixed points of Eq. (35) [i.e., some choice or choices for $v(s,t)$] will satisfy the constraint of Eq. (37) is an open and difficult question.

Even the answer to the uniqueness question for the simultaneous solution of Eqs. (35) and (37) is not immediate. Let us consider the case of no subtractions [i.e., $\sigma(t) \equiv 0$] and assume that there were two different functions $\rho_1(s,t)$ and $\rho_2(s,t)$ which would both reproduce a given s-wave amplitude $A_0(s)$. If we define

$$f(s,t) \equiv \rho_1(s,t) - \rho_2(s,t) = f(t,s)\,, \tag{38}$$

then $f(s,t)$ must simultaneously satisfy the conditions

$$\int_4^\infty dt' f(s,t') \ln\left(1 + \frac{s-4}{t'}\right) \equiv 0, \quad s \geq 4 \tag{39}$$

$$\int_4^s ds' \int_4^\infty dt' f(s',t')\left(\frac{1}{t'+s-4} + \frac{1}{t'+s'-s}\right) + \tfrac{1}{2}\int_4^{s-4} ds'\, f(s-s',s') \ln\left(\frac{s}{s'} - 1\right) \equiv 0\,, \quad s \geq 4\,. \tag{40}$$

In fact, because of Eq. (29), the second integral in Eq. (40) does not begin to contribute until $s > 36$. Although it is fairly evident that either Eq. (39) or Eq. (40) above has nontrivial solutions for $f(s,t)$, it may not be so apparent that together they can be satisfied except when $f(s,t) \equiv 0$.

However, we can make plausible the existence of a nontrivial solution to Eqs. (39)–(40) as follows. We begin by choosing $f(s,t) \equiv 0$ except inside the triangular region bounded by

$$t = R, \quad s = R, \quad t = R + 4 - s\,, \tag{41}$$

where R is some large, finite quantity. If we then replace the integral equations of Eqs. (39) and (40) by a set of linear algebraic ones for the unknown coefficients $\{\beta_{ij}\}$, $i,j = 4,\ldots,n$, we find that as n increases [i.e., a finer and finer grid covering the triangle of Eq. (41)], the number of unknowns soon far exceeds the number of algebraic equations. Therefore, there exists a set of $\{\beta_{ij}\}$ not all of which are zero, which indicates a nonzero solution to Eqs. (39) and (40). Of course, unitarity has played no role in this argument so that we have not established that Eqs. (35) and (37) necessarily can have more than one solution in common.

In summary, then, we have shown that a Mandelstam amplitude of the form of Eq. (1) can be uniquely reconstructed from its s-wave projection, provided certain smallness constraints on norms are satisfied [cf., Eqs. (13) and (14)]. The corresponding question for the relativistic case remains open, having been reduced to a mapping problem [Eqs. (31)–(33)] subject to a constraint [Eq. (37)].

The author wishes to thank Professor R. L. Warnock for comments on and discussion of the relativistic case which produces the mapping problem subject to a constraint.

[1]J. T. Cushing, Phys. Rev. D 15, 1790 (1977).

[2]D. Atkinson and R. L. Warnock, Phys. Rev. 188, 2098 (1969).

[3]In particular, the first integral in our Eq. (7) corresponds to $d_1(s,t)$ of Eq. (4.3) of Ref. 2, the second integral in our Eq. (7) to their $d_2(t)$ of Eq. (4.3), and our $dg(t)/dt$ to their $\sigma(t)$.

[4]D. Atkinson, Nucl. Phys. B13, 415 (1969).

[5]W. Pogorzelski, *Integral Equations and Their Applications* (Pergamon, London, 1966), p. 197.