PHYSICAL REVIEW D VOLUME 18, NUMBER 4 15 AUGUST 1978

Two-dimensional Ising field theory in a magnetic field: Breakup of the cut in the two-point function

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(Received 12 April 1978)

We demonstrate that the cut which is present as the leading singularity in the two-point function of the Ising field theory for $T < T_c$ and $H = 0$ breaks up into a sequence of poles for $H \neq 0$. Both the positions and the residues of the low-lying poles are calculated.

I. INTRODUCTION

The two-point function of the two-dimensional Ising field theory for $T<T_c$ when the external symmetry-breaking field $H=0$ has the striking property that its singularity in momentum space which is nearest the real axis is a square-root branch cut[1] proportional to

$$(1+\tfrac{1}{4}k^2)^{1/2}(2/k)^3\ln[k/2+(1+\tfrac{1}{4}k^2)^{1/2}]-(2/k)^2, \quad (1.1)$$

instead of the usual one-particle Ornstein-Zernike pole

$$\frac{A}{k^2+1}, \quad (1.2)$$

which[1] is the singularity closest to the real axis for $T>T_c$. In a previous publication[2] we announced the result that when an external symmetry-breaking (magnetic) field H is applied to this Ising field theory the cut breaks up into a sequence of poles. The purpose of this paper is to derive and discuss that result in detail.

It has long been conjectured[3] that the cut of (1.1) does break up into a sequence of poles when a magnetic field is applied. More generally, one may consider going from $T>T_c$, $H=0$ to $T<T_c$, $H=0$ by following a path in the H,T plane as indicated in Fig. 1. At the points a ($T>T_c$, $H=0$) and e ($T<T_c$, $H=0$), the two-point function is completely known[4] and its singularities in momentum space are schematically shown in Figs. 2(a) and 2(e). In particular, at point a, because of the up-down symmetry there are only odd-particle thresholds at $k=\pm i(2n-1)$. At point b, $T>T_c$, $H>0$ the up-down symmetry is broken and even-particle thresholds appear, Fig. 2(b). In addition the location of the singularities will move by an amount proportional to H^2. As one proceeds along the path of Fig. 1, bound-state poles start to emerge from the two-particle branch cuts so that for a general H and T (point c) the singularities of the two-point function are given by Fig. 2(c). It is an open question how many bound-state poles there are at $T=T_c$, $H>0$. Finally, as one moves to the point d many poles have emerged from the two-particle cut [Fig. 2(d)] until they coalesce to form the cut of Fig. 2(e). The principal result of this paper is to show that in the scaling limit when the scaled magnetic field $h\geq 0$ (see the Appendix for the relation of the scaled magnetic field h to the lattice parameters and the external magnetic field H), the poles near the tip of the branch cut at $\pm 2i$ are located at

$$\pm i(2+h^{2/3}\lambda_l^{2/3}), \quad (1.3a)$$

where λ_l are the positive solutions of

$$J_{1/3}(\tfrac{1}{3}\lambda_l)+J_{-1/3}(\tfrac{1}{3}\lambda_l)=0 \quad (1.3b)$$

and $J_\nu(z)$ is the Bessel function of order ν. The residues of these poles are

$$\tfrac{1}{2}\pi^{-2}h. \quad (1.4)$$

The Green's function in the presence of an external symmetry-breaking field h is formally expressed in terms of the $h=0$ connected Green's function as

$$G_2^c(0,\vec{r};h)=\sum_{n=0}^{\infty}\frac{1}{n!}h^n\int d\vec{r}_1\cdots d\vec{r}_n G_{n+2}^c(0,\vec{r},\vec{r}_1\cdots\vec{r}_n), \quad (1.5)$$

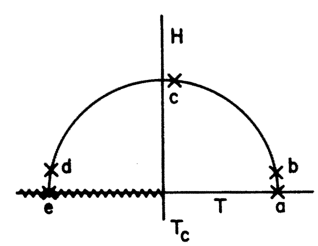

FIG. 1. A path in the (H,T) plane that goes from $T\gtrsim T_c$, $H=0$ (point a) to $T\lesssim T_c$, $H=0^+$ (point e). By the Lee-Yang circle theorem the only singularities of the correlation functions will occur at $H=0$ for $T<T_c$.

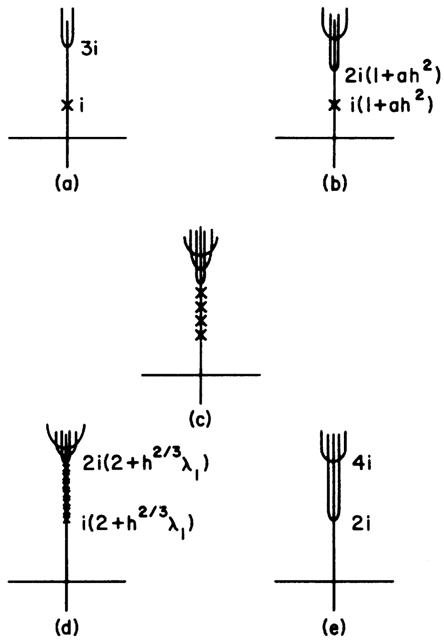

FIG. 2. The singularities in the complex k plane of the connected two-point function for the five points of Fig. 1. Only the upper half plane is shown because the singularities are symmetric about the real axis. In (a), (b), (d), and (e), the position of the lowest pole and lowest branch cut is indicated.

where $G_2^c(0,\vec{r};h)=G_2(0,\vec{r};h)-G_2(0,\infty;h)$ (see the Appendix). When $\vec{r}$ is large we may use the results of the two preceding papers[5,6] to demonstrate for $T<T_c$ the intimate connection between the string structure of the n-point functions and the destruction of the cut in the 2-point function. Without loss of generality we may take the point $\vec{r}$ to be on the y axis. Then, calling m_i the vertical coordinate and n_i the horizontal coordinate of $\vec{r}_i$, we have from paper II that if $0<m_i<r$,

$$\bar{m}_i=m_i/r=O(1)$$

and (1.6)

$$\bar{n}_i=n_i/r^{1/2}=O(1)\,,$$

then

$$G_{n+2}^c(0,\vec{r},\vec{r}_1,\ldots,\vec{r}_n)\sim r^{-2}e^{-2r}h_{n+2}\left(\frac{m_i}{r}\,,\;\frac{n_i}{r^{1/2}}\right)\,. \tag{1.7}$$

Inserting this into (1.5) we see that each integral over $\vec{r}_i$ gives a factor of $r^{3/2}$ and hence, for $r\gg 1$,

$$G_2^c(0,\vec{r};h)\sim r^{-2}e^{-2r}\sum_{n=0}^{\infty}\frac{1}{n!}\,(r^{3/2}h)^n c_n\,, \tag{1.8a}$$

where

$$c_n=\int_0^1\{d\bar{m}_i\}\int_{-\infty}^{\infty}\{d\bar{n}_i\}h_{n+2}(\bar{m}_i,\bar{n}_i)\,. \tag{1.8b}$$

Now if $G_2^c(0,\vec{r};h)$ is represented as a sum of poles in momentum space we must have the coordinate-space asymptotic expansion

$$G_2^c(0,\vec{r};h)\sim\pi^{3/2}r^{-1/2}e^{-2r}\sum_l a_l(h)e^{-r\kappa_l(h)}\,. \tag{1.9}$$

However, in (1.8) h always appears as $hr^{3/2}$. Therefore the only way for (1.8) to be of the form (1.9) is for

$$\kappa_l(h)=h^{2/3}\kappa_l \tag{1.10a}$$

and

$$a_l(h)=ha_l\,. \tag{1.10b}$$

Thus the functional dependence of $\kappa_l(h)$ and $a_l(h)$ on h as given by (1.3) and (1.4) follows from the string property on the n-point functions alone.

The result (1.9), while only an approximation, is already sufficient to demonstrate that the magnetization $M(h)$ is not analytic at $h=0$ even though $M(h)$ is infinitely differentiable[7] at $h=0$. To see this we note that $\partial M(h)/\partial h=\chi(h)$ may be expressed as the value of $G_2^c(k;h)$ at $k=0$. However, the factor $h^{2/3}$ in (1.10a) guarantees that $\chi(e^{\pi i}|h|)$ cannot equal $\chi(e^{-\pi i}|h|)$. Therefore we conclude that $M(h)$ has a discontinuity across the negative h axis and hence that hystersis in the sense of analytic continuation of $M(h)$ through $h=0$ does not exist for the two-dimensional Ising field theory. This lack of analyticity has been previously discussed in the context of the cluster or droplet models of condensation by Langer[8] and Fisher.[9]

Of course we must actually demonstrate that (1.8) does indeed have the form (1.9). To do this we need the explicit formulas for the functions h_n.

II. AN EXPRESSION FOR c_n

From the explicit expression (II3.5), we have for $n=0$

$$h_2=\tfrac{1}{8}(2\pi)^{-2}\int_{-\infty}^{\infty}dx_1dx_2(x_1+x_2)^2e^{-(x_1^2+x_2^2)/2}\,. \tag{2.1}$$

Therefore,

$$c_0=h_2=1/8\pi\,. \tag{2.2}$$

Moreover, when $n\geq 1$ and

$$0<\bar{m}_1<\bar{m}_2<\cdots<\bar{m}_n<\bar{m}_{n+1}=1\,, \tag{2.3}$$

$$h_{n+2}(\bar{m}_i,\bar{n}_i)=\tfrac{1}{2}(2\pi)^{-2(n+1)}2^{-(n-1)}\sum_{\{\epsilon_l\}}\int_{-\infty}^{\infty}dx_1\cdots dx_{2(n+1)}\,\frac{1}{x_1-x_2+i\epsilon}$$
$$\times\prod_{l=1}^{n-1}\left(\frac{2}{x_{l+1}-x_{l+2}+i\epsilon_l}\right)(x_{n+1}+x_{n+2})$$
$$\times\prod_{l=1}^{n-1}\left(\frac{2}{x_{n+l+2}-x_{n+l+1}+i\epsilon_l}\right)\frac{1}{x_{2n+2}-x_{2n+1}+i\epsilon}(x_{2n+2}+x_1)$$
$$\times\exp\left[-\tfrac{1}{2}\sum_{l=1}^{n}\tau_l(x_l^2+x_{2n-l+3}^2)\right.$$
$$\left.-i\sum_{l=1}^{n}\bar{n}_l(x_l-x_{l+1}-x_{2n-l+3}+x_{2n-l+2})\right],\qquad(2.4)$$

where

$$\tau_l=\bar{m}_{l+1}-\bar{m}_l\qquad(2.5)$$

and

$$\sum_{l=1}^{n+1}\tau_l=1\,.\qquad(2.6)$$

To obtain c_n we first integrate h_{n+2} over $-\infty<n_l<\infty$ to obtain

$$\int_{-\infty}^{\infty}d\bar{n}_1\cdots d\bar{n}_n h_{n+2}(\bar{m}_i,\bar{n}_i)=\tfrac{1}{2}(2\pi)^{-(n+2)}2^{(n-1)}$$
$$\times\sum_{\{\epsilon_l\}}\int_{-\infty}^{\infty}dx_1\cdots dx_{2(n+1)}\,\frac{1}{x_1-x_2+i\epsilon}$$
$$\times\prod_{l=1}^{n-1}\left(\frac{1}{x_{l+1}-x_{l+2}+i\epsilon_l}\right)(x_{n+1}+x_{n+2})$$
$$\times\prod_{l=1}^{n-1}\left(\frac{1}{x_{n+l+2}-x_{n+l+1}+i\epsilon_l}\right)\frac{1}{x_{2n+2}-x_{2n+1}+i\epsilon}(x_{2n+2}+x_1)$$
$$\times\exp\left[-\tfrac{1}{2}\sum_{l=1}^{n+1}\tau_l(x_l^2+x_{2n-l+3}^2)\right]$$
$$\times\prod_{l=1}^{n}\delta(x_l-x_{l+1}-x_{2n-l+3}+x_{2n-l+2})\,.\qquad(2.7)$$

We now may carry out n of the x_l integrations by use of the δ functions. To do this, define the variables

$$2w=x_1-x_{2(n+1)}=x_2-x_{2n+1}=\cdots=x_{n+1}-x_{n+2}\,,\qquad(2.8a)$$

which incorporates the δ-function constraint, and

$$2z_l=x_l+x_{2(n+1)+1-l}\,,\quad l=1,\ldots,n+1\,.\qquad(2.8b)$$

Thus,

$$x_l=z_l+w\,,\qquad(2.9a)$$

$$x_{2(n+1)+1-l}=z_l-w\,,\qquad(2.9b)$$

and

$$[dx_l]=2dw\,[dz_l]\qquad(2.10)$$

and, hence,

$$\int_{-\infty}^{\infty} d\bar{n}_1 \cdots d\bar{n}_n h_{n+2}(\bar{m}_i, \bar{n}_i) = (2\pi)^{-(n+2)} \sum_{\{\epsilon_l\}} \int_{-\infty}^{\infty} dw \int_{-\infty}^{\infty} dz_1 \cdots dz_{n+1} \left(\frac{1}{z_1 - z_2 + i\epsilon}\right)^2$$
$$\times \prod_{l=1}^{n-1} \left(\frac{1}{z_{l+1} - z_{l+2} + i\epsilon_l}\right)^2 z_1 z_{n+1} \exp\left[-\sum_{l=1}^{n+1} \tau_l (z_l^2 + w^2)\right]. \quad (2.11)$$

The w integral is trivially evaluated using (2.6), and hence

$$\int_{-\infty}^{\infty} d\bar{n}_1 \cdots d\bar{n}_n h_{n+2}(\bar{m}_i, \bar{n}_i) = \tfrac{1}{2}\pi^{-(n+3/2)} \sum_{\{\epsilon_l\}} \int_{-\infty}^{\infty} dz_1 \cdots dz_{n+1} z_1 z_{n+1} \left(\frac{1}{z_1 - z_2 + i\epsilon}\right)^2$$
$$\times \prod_{l=1}^{n-1} \left(\frac{1}{z_{l+1} - z_{l+2} + i\epsilon_l}\right)^2 \exp\left(-\sum_{l=1}^{n+1} \tau_l z_l^2\right). \quad (2.12)$$

We now must carry out the integrations over all n of the $\bar{m}_l$ satisfying $0 < \bar{m}_l < 1$. The form (2.4) is valid only in the region (2.3); however, the other $n! - 1$ regions are equal by symmetry. The symmetry factor cancels the $n!$ in (1.8) and hence we find for $n \geq 1$

$$\frac{1}{n!} c_n = \tfrac{1}{2}\pi^{-(n+3/2)} \sum_{\{\epsilon_l\}} \int_{-\infty}^{\infty} dz_1 \cdots dz_{n+1} \int_0^{\infty} d\tau_1 \cdots d\tau_{n+1} \delta\left(\sum_{l=1}^{n+1} \tau_l - 1\right) z_1 z_{n+1} \left(\frac{1}{z_1 - z_2 + i\epsilon}\right)^2$$
$$\times \prod_{l=1}^{n-1} \left(\frac{1}{z_{l+1} - z_{l+2} + i\epsilon_l}\right)^2 \exp\left(-\sum_{l=1}^{n+1} \tau_l z_l^2\right). \quad (2.13)$$

Then, using the integral representation

$$\delta\left(\sum_{l=1}^{n+1} \tau_l - 1\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\rho \exp\left[i\rho\left(1 - \sum_{l=1}^{n+1} \tau_l\right)\right], \quad (2.14)$$

we have

$$\int_0^{\infty} d\tau_1 \cdots d\tau_{n+1} \delta\left(\sum_{l=1}^{n+1} \tau_l - 1\right) \exp\left(-\sum_{l=1}^{n+1} \tau_l z_l^2\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\rho\, e^{i\rho} \prod_{l=1}^{n+1} \frac{1}{z_l^2 + i\rho}$$
$$= \sum_{j=1}^{n+1} \exp(-z_j^2) \prod_{l=1}^{n+1}{}' (z_l^2 - z_j^2)^{-1}, \quad (2.15)$$

where in $\prod_{l=1}^{\prime n+1}$ the term $l = j$ is omitted. Therefore

$$\frac{1}{n!} c_n = \tfrac{1}{2}\pi^{-(n+3/2)} \sum_{\{\epsilon_l\}} \int_{-\infty}^{\infty} dz_1 \cdots dz_{n+1} z_1 z_{n+1} \left(\frac{1}{z_1 - z_2 + i\epsilon}\right)^2$$
$$\prod_{l=1}^{n-1} \left(\frac{1}{z_{l+1} - z_{l+2} + i\epsilon_l}\right)^2 \sum_{j=1}^{n+1} \exp(-z_j^2) \prod_{l=1}^{n+1}{}' (z_l^2 - z_j^2)^{-1}. \quad (2.16)$$

Now the sum in (2.15) is continuous when $z_l = z_j$, but term by term there are poles. However, since in (2.16) z_1 appears as $(z_1 - z_2 + i\epsilon)^{-2}$, we may write in (2.15)

$$z_l^2 - z_1^2 = (z_l + z_1 + i\epsilon)(z_l - z_1 - i\epsilon). \quad (2.17)$$

Then in all terms in the sum over j in (2.16) with $j \neq 1$, all singularities in the z_1 integration lie in the lower half plane, and since there is no factor of $e^{-z_1^2}$, the contour may be closed in the upper half plane to give zero. Thus only the term with $j = 1$ survives in (2.16) and we obtain.

$$\frac{1}{n!} c_n = (-1)^n \tfrac{1}{2}\pi^{-(n+3/2)} \sum_{\{\epsilon_l\}} \int_{-\infty}^{\infty} dz_1 \cdots dz_{n+1} z_1 z_{n+1} \left(\frac{1}{z_1 - z_2 + i\epsilon}\right)^2$$
$$\times \prod_{l=1}^{n-1} \left(\frac{1}{z_{l+1} - z_{l+2} + i\epsilon_l}\right)^2 e^{-z_1^2} \prod_{l=2}^{n+1} \left[\frac{1}{(z_1 + z_l + i\epsilon)(z_1 - z_l + i\epsilon)}\right]. \quad (2.18)$$

From this we may evaluate c_n for arbitrary n by elementary means. For example, if $n = 1$

$$c_1 = -\tfrac{1}{2}\pi^{-5/2} \int_{-\infty}^{\infty} dz_1 dz_2 z_1 z_2 \left(\frac{1}{z_1 - z_2 + i\epsilon}\right)^2 \frac{1}{(z_1+z_2+i\epsilon)} \frac{1}{(z_1 - z_2 + i\epsilon)} e^{-z_1^2}. \tag{2.19}$$

The z_2 integral may be evaluated by closing in the lower half plane at $z_2 = -z_1 - i\epsilon$ to find

$$c_1 = -i\tfrac{1}{8}\pi^{-3/2} \int_{-\infty}^{\infty} dz \frac{1}{z_1 + i\epsilon} e^{-z_1^2} = -\tfrac{1}{8}\pi^{-1/2}. \tag{2.20}$$

To proceed, in general, we let

$$z_l = z_1 y_l, \quad l = 2, \ldots, n+1. \tag{2.21}$$

Then

$$\frac{1}{n!} c_n = (-1)^n \tfrac{1}{2}\pi^{-(n+3/2)} \sum_{\{\epsilon_l\}} \int_{-\infty}^{\infty} dz_1 (z_1 + i\epsilon)^{-3n+2} e^{-z_1^2} \int_{-\infty}^{\infty} dy_2 \cdots dy_{n+1} y_{n+1} \frac{1}{(1 - y_2 + i\epsilon)^2}$$
$$\times \prod_{l=1}^{n-1} \left(\frac{1}{y_{l+1} - y_{l+2} + i\epsilon_l}\right)^2 \prod_{l=2}^{n+1} \left[\frac{1}{(1 + y_l + i\epsilon)(1 - y_l + i\epsilon)}\right]. \tag{2.22}$$

We now define an operator

$$K(x,y) = -\pi^{-1} \frac{1}{(1 + x + i\epsilon)(1 - x + i\epsilon)} \left[\frac{1}{(x - y + i\epsilon)^2} + \frac{1}{(x - y - i\epsilon)^2}\right] \tag{2.23}$$

and hence obtain the desired expression for $n \geq 1$

$$\frac{1}{n!} c_n = -\tfrac{1}{2}\pi^{-5/2} \int_{-\infty}^{\infty} dz_1 (z_1 + i\epsilon)^{-3n+2} e^{-z_1^2} \int_{-\infty}^{\infty} dy_2 dy_{n+1} \frac{1}{(1 - y_2 + i\epsilon)^2} K^{n-1}(y_2, y_{n+1}) \frac{y_{n+1}}{(1 + y_{n+1} + i\epsilon)(1 - y_{n+1} + i\epsilon)}, \tag{2.24}$$

where $K^{n-1}(y_2, y_{n+1})$ is the $n-1$ iteration of the operator K.

III. SUMMATION OF THE SERIES

We now make use of (2.24) in the series (1.8) to formally obtain

$$G_2^c(0, \vec{r}; h) \sim r^{-2} e^{-2r} \left[\frac{1}{8\pi} - \tfrac{1}{2}\pi^{-5/2} h r^{3/2} \int_{-\infty}^{\infty} dz_1 e^{-z_1^2} \int_{-\infty}^{\infty} dy_2 dy_{n+2} \frac{1}{(1 - y_2 + i\epsilon)^2} \right.$$
$$\left. \times \frac{1}{(z_1 + i\epsilon)^3 - h r^{3/2} K(y_2, y_{n+1})} \frac{y_{n+1}}{(1 + y_{n+1} + i\epsilon)(1 - y_{n+1} + i\epsilon)}\right]. \tag{3.1}$$

To proceed further we need the eigenvalues and eigenfunctions of the operator $K(x,y)$. Accordingly, we define the eigenfunction $f_l(x)$ by

$$\int_{-\infty}^{\infty} dy\, K(x,y) f_l(y) = \lambda_l f_l(x) \tag{3.2a}$$

and the adjoint eigenfunction $f^a(x)$ by

$$\int_{-\infty}^{\infty} dx\, f_l^a(x) K(x,y) = \lambda_l f_l^a(y). \tag{3.2b}$$

Using the explicit form of K (2.23) we find

$$f_l^a(x) = (1 + x + i\epsilon)(1 - x + i\epsilon) f_l(x). \tag{3.3}$$

In terms of $f_l(x)$ and λ_l we may formally write the resolvent operator as

$$\frac{1}{(z + i\epsilon)^3 - r h^{3/2} K(x,y)} = \sum_l \frac{1}{(z + i\epsilon)^3 - r h^{3/2} \lambda_l} f_l(x) f_l^a(y), \tag{3.4}$$

where we have the normalization condition

$$1=\int_{-\infty}^{\infty}dx\,f_l^a(x)f_l(x)=\int_{-\infty}^{\infty}dx(1+x+i\epsilon)(1-x+i\epsilon)f_l^2(x)\,. \tag{3.5}$$

Note that because $K(x,y)$ is an unbounded operator our operations at this stage are purely formal and questions of convergence will be discussed later. Using (3.4) we then may write

$$G_2^c(0,\vec{r};h)\sim r^{-2}e^{-2r}\left\{\frac{1}{8\pi}-\tfrac{1}{2}\pi^{-5/2}\int_{-\infty}^{\infty}dz_1e^{-z_1^2}z_1^2\sum_l\left[\frac{1}{(z_1+i\epsilon)^3-hr^{3/2}\lambda_l}\right.\right.$$
$$\left.\left.\times\int_{-\infty}^{\infty}dy_2\frac{f_l(y_2)}{(1-y_2+i\epsilon)^2}\int_{-\infty}^{\infty}dy_{n+1}y_{n+1}f_l(y_{n+1})\right]\right\}, \tag{3.6}$$

We must now explicitly solve the integral equation (3.2a) which, using (2.23), is explicitly written as

$$-\pi^{-1}\int_{-\infty}^{\infty}dy\left[\frac{1}{(x-y+i\epsilon)^2}+\frac{1}{(x-y-i\epsilon)^2}\right]f(y)=\lambda(1+x+i\epsilon)(1-x+i\epsilon)f(x)\,. \tag{3.7}$$

The left-hand side of this equation is a convolution. Therefore, we introduce the Fourier transform

$$h(k)=(2\pi)^{-1/2}\int_{-\infty}^{\infty}dx\,e^{ikx}f(x) \tag{3.8}$$

and, using the fact that if $\mathrm{Re}k>0$,

$$\int_{-\infty}^{\infty}dx\,e^{ikx}\left[\frac{1}{(x+i\epsilon)^2}+\frac{1}{(x-i\epsilon)^2}\right]=-2\pi k\,, \tag{3.9a}$$

while if $\mathrm{Re}k<0$,

$$\int_{-\infty}^{\infty}dx\,e^{ikx}\left[\frac{1}{(x+i\epsilon)^2}+\frac{1}{(x-i\epsilon)^2}\right]=2\pi k\,, \tag{3.9b}$$

we find the second-order equation for $\mathrm{Re}k>0$

$$2kh(k)=\lambda\left(1+i\epsilon+\frac{\partial^2}{\partial k^2}\right)h(k)\,, \tag{3.10a}$$

and for $\mathrm{Re}k<0$

$$-2kh(k)=\lambda\left(1+i\epsilon+\frac{\partial^2}{\partial k^2}\right)h(k)\,. \tag{3.10b}$$

To complete the definition of $h(k)$ and to determine λ we need boundary conditions for (3.10). First of all, we see that in (3.7), if we send $x\to -x$ and $y\to -y$, $f(x)$ and $f(-x)$ satisfy the same equation. Therefore, the symmetric and the antisymmetric parts of f separately satisfy (3.7). However, in the integral over y_{n+1} in (3.6), only the antisymmetric eigenfunctions give a nonzero contribution. Therefore we may restrict our attention to antisymmetric eigenfunctions satisfying

$$h(-k)=-h(k)\,. \tag{3.11}$$

Moreover, in order for the second derivative to exist at $k=0$, $h(k)$ must be continuous and differentiable at $k=0$. Therefore, from (3.11) the quantization condition for λ is

$$h(0)=0\,. \tag{3.12}$$

Because of (3.11), it suffices to consider the differential equation (3.10a) derived for $\mathrm{Re}k>0$ considered as an equation valid for all k. To solve the equation let

$$\xi=(\tfrac{1}{2}\lambda)^{-1/3}[k-\tfrac{1}{2}\lambda(1+i\epsilon)]\,, \tag{3.13}$$

where the cube root is defined to be real and positive for λ real and positive and by analytic continuation elsewhere. Then, writing

$$h(k)=N(\lambda)\tilde{h}(\xi)\,, \tag{3.14}$$

we have

$$\left(\frac{d^2}{d\xi^2}-\xi\right)\tilde{h}(\xi)=0\,. \tag{3.15}$$

This is Airy's equation.

To determine which solution to (3.15) we need first consider λ real and positive. Then for the normalization integrals to exist we need

$$h(k)\to 0 \ \text{ as } k\to\infty\,. \tag{3.16}$$

The solution which satisfies this boundary condition is[10] (for $\xi>0$)

$$\tilde{h}(\xi)=\pi^{-1}(\xi/3)^{1/2}K_{1/3}(\tfrac{2}{3}(\xi)^{3/2})\,. \tag{3.17}$$

Now from the quantization condition (3.12) we need

$$(\tilde{h}-(\tfrac{1}{2}\lambda)^{2/3})=0\,. \tag{3.18}$$

When $\xi<0$, the solution (3.17) may be written as[10]

$$\tilde{h}(\xi)=\tfrac{1}{3}(-\xi)^{1/2}[J_{1/3}(\tfrac{2}{3}(-\xi)^{3/2})+J_{-1/3}(\tfrac{2}{3}(-\xi)^{3/2})]\,. \tag{3.19}$$

Therefore λ_l are determined from the equation

$$J_{1/3}(\tfrac{1}{3}\lambda_l)+J_{-1/3}(\tfrac{1}{3}\lambda_l)=0\,. \tag{3.20}$$

It is most important to realize that (3.20) has two sets of solutions. First, there are an infinite number of solutions which are real and positive. Indeed, when l is large[10]

$$\lambda_l\sim 3\pi(l-\tfrac{1}{4})\,. \tag{3.21}$$

However, if we analytically continue λ and use

$$J_{\pm 1/3}(e^{3\pi i}\tfrac{1}{3}|\lambda_l|)=e^{\pm\pi i}J_{\pm 1/3}(\tfrac{1}{3}|\lambda_l|)\,, \tag{3.22}$$

we see that, if λ_l is real and positive and satisfies (3.20), then

$$\lambda_l' = e^{3\pi i}\lambda_l$$

also is a set of eigenvalues, provided the boundary condition at $|k| \to \infty$ is analytically continued simultaneously with λ. This continuation may be made without upsetting either the normalization integrals or the integrals over the eigenfunctions in (3.6), and indeed must be done because in the integration over z_1, the phase of $(z_1+i\epsilon)^3$ goes from real and positive to $e^{3\pi i}$. With this continuation of boundary conditions, the eigenfunctions for λ_l' on the continued contours are the same as the eigenfunctions for λ_l on the original contours. Therefore we may write (3.6) as

$$G_2^c(0,\vec{r};h) \sim r^{-2}e^{-2r}\left\{\frac{1}{8\pi} - \tfrac{1}{2}\pi^{5/2}hr^{3/2}\int_{-\infty}^{\infty} dz_1 e^{-z_1^2} z_1^2 \sum_l \left[\left(\frac{1}{(z_1+i\epsilon)^3 - hr^{3/2}\lambda_l} + \frac{1}{(z_1+i\epsilon)^3 + hr^{3/2}\lambda_l}\right) \times \int_{-\infty}^{\infty} dy_2 \frac{f_l(y_2)}{(1-y_2+i\epsilon)^2}\int_{-\infty}^{\infty} dy_{n+1} y_{n+1} f_l(y_{n+1})\right]\right\}, \tag{3.23}$$

where we now may restrict our attention to the eigenfunctions with λ_l real and positive satisfying (3.16).

We next need to evaluate the y_2 and y_{n+1} integrals in (3.23). First we have, using (3.8),

$$\int_{-\infty}^{\infty} dy\, y f_l(y) = (2\pi)^{-1/2}\int_{-\infty}^{\infty} dk\, h_l(k)\int_{-\infty}^{\infty} dy\, y e^{iky}$$
$$= -i(2\pi)^{1/2}h_l'(0)\,, \tag{3.24}$$

where the prime indicates differentiation with respect to k.

Second we have

$$\int_{-\infty}^{\infty} dy \frac{f_l(y)}{(1-y+i\epsilon)^2}$$
$$= (2\pi)^{-1/2}\int_{-\infty}^{\infty} dk\, h_l(k)\int_{-\infty}^{\infty} dy \frac{e^{-iky}}{(1-y+i\epsilon)^2}$$
$$= (2\pi)^{1/2}\int_{-\infty}^{0} dk\, h_l(k) k e^{-ik}. \tag{3.25}$$

Now we use the differential equation (3.10b) with $k<0$ to write

$$(2\pi)^{1/2}\int_{-\infty}^{0} dk\, h_l(k) k\, e^{-ik}$$
$$= -(\pi/2)^{1/2}\lambda_l \int_{-\infty}^{0} dk\, e^{-ik}\left(1+\frac{\partial^2}{\partial k^2}\right)h_l(k)\,. \tag{3.26}$$

Then we may integrate by parts to transfer $\partial^2/\partial k^2$ to e^{-ik} using the boundary condition (3.16) and (3.12) to obtain the desired result

$$\int_{-\infty}^{\infty} dy \frac{f_l(y)}{(1-y+i\epsilon)^2} = -(\pi/2)^{1/2}\lambda_l h_l'(0)\,. \tag{3.27}$$

Therefore, (3.23) becomes

$$G_2^c(0,\vec{r};h) \sim r^{-2}e^{-2r}\left\{\frac{1}{8\pi} - \tfrac{1}{2}\pi^{-3/2}hr^{3/2} i\int_{-\infty}^{\infty} dz_1 e^{-z_1^2} z_1^2 \sum_l \lambda_l [h'(0)]^2\left[\frac{1}{(z_1+i\epsilon)^3 - hr^{3/2}\lambda_l} + \frac{1}{(z_1+i\epsilon)^3 + hr^{3/2}\lambda_l}\right]\right\}. \tag{3.28}$$

It remains to use the normalization condition (3.5) to evaluate $h'(0)$. From (3.5) we first obtain

$$1 = \int_{-\infty}^{\infty} dk\, h_l(-k)\left(1+\frac{\partial^2}{\partial k^2}\right)h_l(k)$$
$$= -4\lambda_l^{-1}\int_0^{\infty} dk\, k h^2(k)\,, \tag{3.29}$$

where, to obtain the last line, both (3.10) and (3.11) were used. To evaluate this integral consider for the moment h_1 and h_2, which for $k>0$ satisfy

$$2kh_1(k) = \lambda_1\left(1 + i\epsilon + \frac{\partial^2}{\partial k^2}\right)h_1(k) \tag{3.30a}$$

and

$$2kh_2(k) = \lambda_2\left(1 + i\epsilon\frac{\partial^2}{\partial k^2}\right)h_2(k) \tag{3.30b}$$

with the boundary condition (3.16). Then

$$\frac{d}{dk}\left(h_2\frac{d}{dk}h_1 - h_1\frac{d}{dk}h_2\right) = 2(\lambda_1^{-1} - \lambda_2^{-1})kh_1h_2 \tag{3.31}$$

and thus

$$-4\int_0^{\infty} dk\, kh_1(k)h_2(k)$$
$$= \frac{2}{\lambda_1^{-1}-\lambda_2^{-1}}\left(h_2\frac{d}{dk}h_1 - h_1\frac{d}{dk}h_2\right)\Big|_{k=0}. \tag{3.32}$$

We now let $\lambda_1 \to \lambda_2$. Then, since

$$h_2\frac{d}{dk}h_1-h_1\frac{d}{dk}h_2$$
$$=N(\lambda_1)N(\lambda_2)\left[\tilde{h}_2(\xi)\frac{d}{dk}\tilde{h}_1(\xi)-\tilde{h}_1(\xi)\frac{d}{dk}\tilde{h}_2(\xi)\right], \tag{3.33}$$

and since

$$\frac{d}{dk}\tilde{h}(\xi)\Big|_{k=0}=-3\frac{d}{d\lambda}\tilde{h}(\xi)\Big|_{k=0}, \tag{3.34}$$

we have

$$\left[\tilde{h}_2(\xi)\frac{d}{dk}\tilde{h}_1(\xi)-\tilde{h}_1(\xi)\frac{d}{dk}\tilde{h}_2(\xi)\right]\Big|_{k=0}$$
$$=-3\left[\tilde{h}_2(\xi)\frac{d}{d\lambda_1}\tilde{h}_1(\xi)-\tilde{h}_1(\xi)\frac{d}{d\lambda_2}\tilde{h}_2(\xi)\right]\Big|_{k=0}. \tag{3.35}$$

Then expanding

$$\tilde{h}_2=\tilde{h}_1(\xi)+(\lambda_2-\lambda_1)\frac{d}{d\lambda_1}\tilde{h}_1(\xi) \tag{3.36a}$$

and

$$\frac{d}{d\lambda_2}\tilde{h}_2(\xi)=\frac{d}{d\lambda_1}\tilde{h}_1(\xi)+(\lambda_2-\lambda_1)\frac{d^2}{d\lambda_1^2}\tilde{h}_1(\xi), \tag{3.36b}$$

we find

$$\lim_{\lambda_2\to\lambda_1}\frac{2}{\lambda_1^{-1}-\lambda_2^{-2}}\left[\tilde{h}_2(\xi)\frac{d}{dk}\tilde{h}_1(\xi)-\tilde{h}_1(\xi)\frac{d}{dk}\tilde{h}_2(\xi)\right]\Big|_{k=0}=-6\lambda_1^2\left[\left(\frac{d}{d\lambda_1}\tilde{h}_1(\xi)\right)^2-\tilde{h}_1(\xi)\frac{d^2}{d\lambda_1^2}\tilde{h}_1(\xi)\right]\Big|_{k=0}. \tag{3.37}$$

Therefore, using (3.34) and the boundary condition (3.12), we obtain

$$1=-N^2(\lambda_l)\tfrac{2}{3}\lambda_l[\tilde{h}_l'(0)]^2=-\tfrac{2}{3}\lambda_l[h'(0)]^2. \tag{3.38}$$

This may now be substituted in (3.28) to find

$$G_2^c(0,\vec{r};h)\sim r^{-2}e^{-2r}\left\{\frac{1}{8\pi}+\tfrac{3}{4}\pi^{-3/2}hr^{3/2}i\int_{-\infty}^{\infty}dz_1e^{-z_1^2}z_1^2\sum_l\left[\frac{1}{(z_1+i\epsilon)^3-hr^{3/2}\lambda_l}+\frac{1}{(z_1+i\epsilon)^3+hr^{3/2}\lambda_l}\right]\right\}. \tag{3.39}$$

We may now return to the questions of convergence which we postponed earlier. First of all, because of the behavior of λ_l for large l given by (3.21), the terms with $+\lambda_l$ and $-\lambda_l$ must be grouped as in (3.39) to obtain a convergent expression. Moreover, by construction, when $h\to 0$

$$G_2^c(0,\vec{r};h)\to r^{-2}e^{-2r}\frac{1}{8\pi}. \tag{3.40}$$

However, when $h\to 0$ the z_1 integral diverges. Therefore in our formal manipulations we have lost a constant term (independent of h) which must be determined by the requirement (3.40).

The integrand in the z_1 integral is antisymmetric in z_1. Therefore only the contribution comes from the poles at

$$z_1+i\epsilon=\pm r^{1/2}(h\lambda_l)^{1/3}. \tag{3.41}$$

Thus the integral is carried out and we find

$$\tfrac{3}{4}\pi^{-3/2}hr^{3/2}i\int_{-\infty}^{\infty}dz_1e^{-z_1^2}z_1^2\sum_l\left[\frac{1}{(z_1+i\epsilon)^3-hr^{3/2}\lambda_l}+\frac{1}{(z_1+i\epsilon)^3+hr^{3/2}\lambda_l}\right]=-\tfrac{1}{2}\pi^{-1/2}hr^{3/2}\sum_l e^{-r(h\lambda_l)^{2/3}}. \tag{3.42}$$

As $h\to 0$, we use (3.21) to find

$$\tfrac{1}{2}\pi^{-1/2}hr^{3/2}\sum_l e^{-r(h\lambda_l)^{2/3}}$$
$$\to\tfrac{1}{2}\pi^{-1/2}hr^{3/2}\int_0^{\infty}dl\,e^{-r(3\pi hl)^{2/3}}. \tag{3.43}$$

Let

$$l=\frac{1}{3\pi hr^{3/2}}t \tag{3.44}$$

to find

$$\tfrac{1}{2}\pi^{-1/2}hr^{3/2}\int_0^{\infty}dl\,e^{-r(3\pi hl)^{2/3}}=\tfrac{1}{6}\pi^{-3/2}\int_0^{\infty}dt\,e^{-t^{2/3}}$$
$$=\tfrac{1}{4}\pi^{-3/2}\int_0^{\infty}dx\,x^{1/2}e^{-x}$$
$$=\frac{1}{8\pi}. \tag{3.45}$$

Therefore we must subtract $r^{-2}e^{-2r}/8\pi$ from the formal expression (3.39) to obtain the correct answer which satisfies (3.40). This just cancels the first term of (3.39) and we obtain the final result

$$G_2^c(0,\vec{r};h)\sim r^{-1/2}e^{-2r}\tfrac{1}{2}\pi^{-1/2}h\sum_l e^{-rh^{2/3}\lambda_l 2/3}, \tag{3.46}$$

where λ_l satisfies (3.20). Comparing this with the definition $\kappa_l(h)$ and $a_l(h)$ in (1.9), we find the results (1.3) and (1.4).

ACKNOWLEDGMENTS

We are most grateful to the members of the Institute for Theoretical Physics at Utrecht, especially Professor Martinus J. G. Veltman, for their hospitality. It is a pleasure to acknowledge many useful conversations with Dr. Craig A. Tracy. This work was supported in part by the National Science Foundation under Grant No. DMR-7707863A01 and by the U.S. Department of Energy under Contract No. EY-76-S-02-3227.

APPENDIX

We here derive (1.5) from a lattice theory and express h in terms of lattice parameters. The interaction energy of the two-dimensional Ising model in the presence of an external magnetic field is

$$\mathcal{E}=\mathcal{E}_0-H\sum\sigma_{jk}, \tag{A1}$$

with

$$\mathcal{E}_0=-E_1\sum\sigma_{jk}\sigma_{j\,k+1}-E_2\sum\sigma_{jk}\sigma_{j+1\,k}. \tag{A2}$$

The correlation functions at finite magnetic field are, by definition,

$$\langle\sigma_{00}\sigma_{MN}\rangle_H=\frac{1}{Z(H)}\sum_{\{\sigma\}}\sigma_{00}\sigma_{MN}\exp\left(-\beta\mathcal{E}_0-\beta H\sum\sigma_{jk}\right), \tag{A3}$$

where

$$Z(H)=\sum_{\{\sigma\}}e^{-\beta\mathcal{E}} \tag{A4}$$

is the partition function at finite H. This expression for the correlation function may be expanded in a power series in H as

$$\langle\sigma_{00}\sigma_{M,N}\rangle_H-\mathfrak{M}^2(H)=\sum_{n=0}^{\infty}\frac{1}{n!}(\beta H)^n\times\sum_{M_nN_n}\langle\sigma_{00}\sigma_{MN}\sigma_{M_1N_1}\cdots\sigma_{M_nN_n}\rangle^c, \tag{A5}$$

where $\mathfrak{M}(H)$ is the magnetization at finite H. On the lattice we have the representation[11]

$$\langle\sigma_{00}\sigma_{MN}\cdots\sigma_{M_nN_n}\rangle=\mathfrak{M}^{2+n}e^{F_{n+2}}, \tag{A6}$$

where F_{n+2} is a function of the $2(n-1)$ variables M,N and M_i,N_i, and

$$\mathfrak{M}=[1-(\sinh 2\beta E_1\sinh 2\beta E_2)^{-2}]^{1/8} \tag{A7}$$

is the spontaneous magnetization.

We take the continuum limit by defining

$$m=\lim M[z_2(1-z_1^2)]^{-1/2}|z_1z_2+z_1+z_2-1|, \tag{A8a}$$

$$n=\lim N[z_1(1-z_2^2)]^{-1/2}|z_1z_2+z_1+z_2-1|, \tag{A8b}$$

$$h=\lim\beta H\mathfrak{M}[z_1z_2(1-z_1^2)(1-z_2^2)]^{1/2}|z_1z_2+z_1+z_2-1|^{-2}, \tag{A9}$$

and

$$G(0,\vec{r};h)=\lim\mathfrak{M}^{-2}\langle\sigma_{00}\sigma_{MN}\rangle_H, \tag{A10}$$

with

$$z_i=\tanh\beta E_i \tag{A11}$$

and the limit is

$$z_1z_2+z_1+z_2-1\to 0, \tag{A12}$$

with m, n, and h fixed. In this limit, (A4) reduces to (1.5).

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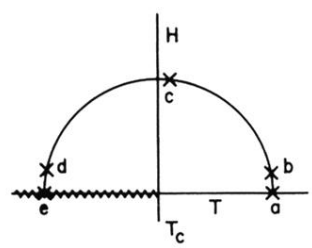

FIG. 1. A path in the (H, T) plane that goes from $T \gtrsim T_c$, $H = 0$ (point a) to $T \lesssim T_c$, $H = 0^+$ (point e). By the Lee-Yang circle theorem the only singularities of the correlation functions will occur at $H = 0$ for $T < T_c$.

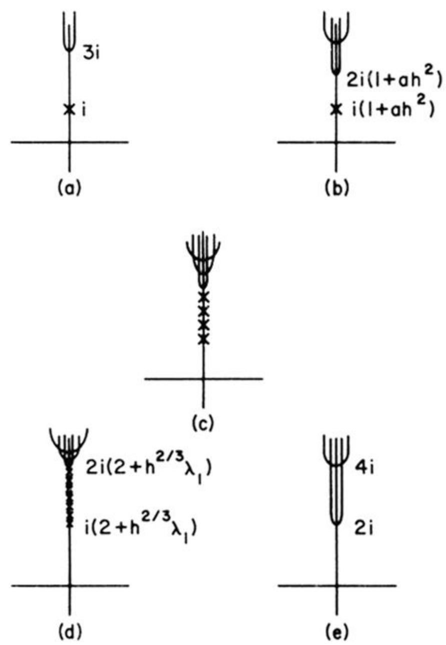

FIG. 2. The singularities in the complex k plane of the connected two-point function for the five points of Fig. 1. Only the upper half plane is shown because the singularities are symmetric about the real axis. In (a), (b), (d), and (e), the position of the lowest pole and lowest branch cut is indicated.