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Charge-monopole duality in spontaneously broken gauge theories

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The conjecture of Montonen and Olive concerning an electric-magnetic dual symmetry in the O(3) Georgi-Glashow model is generalized to theories with an arbitrary compact gauge group.

I. INTRODUCTION

In recent years our understanding of solitons in nonlinear field theories such as gauge theories has deepened. The solitons appear as classically bound configurations of the fundamental fields in the theory, and in most cases they carry charges which are conserved for topological reasons rather than because they generate a symmetry. An intriguing result is the formal equivalence on a quantum level, between the sine-Gordon model and the massive Thirring model as established by Coleman,[1] in which the soliton of the former corresponds to the fundamental field in the latter. The physics can be represented by either theory and the charge that is conserved because of symmetry in the Thirring model is so because of a topological conservation law in the sine-Gordon theory. This raises the question whether such a correspondence exists for any theory which exhibits topological solitons. In more than one dimension it is hard to establish a possible correspondence between different theories because little is known about the quantum properties of the solitons. This leaves in these cases the restricted knowledge about classical solutions as the only guide.

Recently Montonen and Olive[2] have made a conjecture concerning the SO(3) gauge theory broken down to U(1) by a single Higgs field in the adjoint representation, in the Bogomolny-Prasad-Sommerfield (BPS) limit. In this limit one lets the parameters of potential $V(\Phi)$ go to zero but keeps their ratio fixed, i.e., the symmetry breaking remains but the surviving scalar particle becomes massless. This model has two spherically symmetric nonsingular monopole solutions found by 't Hooft and Polyakov,[4] with charges $g=\pm 4\pi/e$. The conjecture is that the theory is equivalent to another SO(3) gauge theory with coupling constant $g=4\pi/e$, for which the gauge triplet consists of the two monopoles and the photon of the original theory. This equivalence is a non-Abelian generalization of the discrete dual invariance between electric and magnetic charges and fields in U(1) electrodynamics, where if one limits oneself to one vector potential the magnetic charge is topologically conserved.

In this paper we generalize this conjecture of "dual symmetry" to a similar limit of any theory with compact gauge group G, with only one Higgs field in the adjoint representation. The modest evidence is based on the same classical considerations as those of Ref. 2 for the SO(3) theory, and should perhaps be considered primarily as a remarkable hint.

In Sec. II we recall how the allowed magnetic charges of nonsingular monopoles in an arbitrary compact gauge theory correspond to the *inverse* root lattice of the gauge group G, which can be defined to be the root lattice of a dual group $*G$. The problem of finding all spherically symmetric nonsingular monopole solutions has been solved before[5] and in Sec. III we show that such solutions exist with charges corresponding to the nonzero weights of the adjoint representation of $*G$, and that these saturate the Bogomolny lower bound in the BPS limit. The charges will be topologically conserved if we choose the vacuum expectation value of the Higgs field such that it breaks the group G down to the maximal number of mutually commuting U(1) factors contained in G (the maximal torus in the group manifold).

In Sec. III we calculate the gauge-field masses due to this symmetry breaking and derive a mass relation between the monopole and gauge-field masses, which is very suggestive of the proposed symmetry. We observe that the vector fields in the G Lagrangian which remain massless after symmetry breaking carry over to the $*G$ Lagrangian under the duality transformation. In Sec. III we discuss problems related to the necessary symmetry breaking, which in general cannot be achieved by a single Higgs multiplet in the adjoint representation. We conclude the paper with a brief discussion.

II. MAGNETIC CHARGES AND INVERSE ROOTS

Consider a theory with compact simple gauge group G. We choose the following standard normalization of the Lie algebra of G

$$[\vec{H}, E_{\pm\alpha}] = \pm \vec{\alpha} E_{\pm\alpha}, \tag{2.1}$$

$$[E_{+\alpha}, E_{-\alpha}] = \vec{\alpha} \cdot \vec{H}, \tag{2.2}$$

$$g_{ij} = \sum_{\alpha} \alpha_i \alpha_j = \delta_{ij}. \tag{2.3}$$

Equation (2.1) defines the l-dimensional (l = rank of G) root vectors $\vec{\alpha}$ with components α_i ($i = 1, \ldots, l$). The generators are chosen such that the normalization constant which usually appears on the right-hand side of (2.2) is chosen to be equal to unity. This implies that

$$\mathrm{Tr}(H_r^{\,2}) = \mathrm{Tr}(E_{+\alpha}E_{-\alpha}) \equiv P. \tag{2.4}$$

[Proof: Multiply (2.2) from the right with H_r and take the trace.] The constant P (which is a property of the representation) in (2.4) is then fixed by condition (2.3).

It has been shown that the charges g_a of nonsingular monopoles have to satisfy the quantization condition

$$\exp(ieg^a \Lambda_a) = 1, \tag{2.5}$$

where e is the gauge coupling and the Λ_a generate a faithful representation of $\overline{G}$ the universal covering group of G.[6] In the Abelian gauge for large R, one can without loss of generality choose the nonvanishing components of the magnetic charge g^a to be in the Cartan subalgebra of G; i.e.,

$$eg^a \Lambda_a = e\vec{g} \cdot \vec{H} = eg. \tag{2.6}$$

Now to each basis vector ψ^k (where $\{\psi^k\}$ forms the basis for a representation of $\overline{G}$) there corresponds a set of eigenvalues or weight vector $\vec{m}_k$:

$$\vec{H}\psi^k = \vec{m}_k \psi^k. \tag{2.7}$$

By a standard result of Lie algebra theory the following relation holds for any $\vec{\alpha}$,

$$\sum_{r=1}^{l} m_k^r \alpha^r = \vec{m}_k \cdot \vec{\alpha} = \frac{n}{2}|\vec{\alpha}|^2 \quad (n = \text{integer}). \tag{2.8}$$

Combining equations (2.6)–(2.8) one finds that (2.5) will be satisfied only if the charge matrix $eg/4\pi$ satisfies

$$\frac{eg}{4\pi} = \sum_{\alpha} n_\alpha \frac{\vec{\alpha}}{|\alpha|^2} \cdot \vec{H} \equiv \sum_{\alpha} n_\alpha \vec{\alpha}^{-1} \cdot \vec{H}, \tag{2.9}$$

where we have introduced the inverse root vectors $\vec{\alpha}^{-1} = \vec{\alpha}/|\alpha|^2$.

To define what we mean by the dual group $*G$, we recall the result of Goddard *et al.*,[7] who showed that the set of inverse roots $\{\vec{\alpha}^{-1}\}$ defines a root system

$$*\Sigma = \{\vec{\beta} = \vec{\alpha}^{-1}/N\}, \tag{2.10}$$

where N is some known normalization constant that depends on the group. $*\Sigma$ defines uniquely the (semi) simple algebra of $*G$. Because the roots of a Lie algebra have at most two different lengths, the duality operation of inverting the roots corresponds to interchanging the long and short roots. As one can most easily see from the Dynkin diagrams (by interchanging black and white dots) this implies that most algebras are self-dual except the algebras of SO($2N+1$) and Sp($2N$) which are dual to each other. To fully define $*G$ one has to specify its connectivity properties as well, which can be done uniquely once the global structure of G is given.[7] This definition of $*G$ establishes an explicit correspondence between the nonzero weights of the adjoint representation of $*G$ (which are per definition equal to the nonzero roots $\vec{\alpha}'^{-1}$) and the charges $eg/4\pi = \vec{\alpha}^{-1} \cdot \vec{H}$ of a subset of nonsingular monopoles in a gauge theory based on G.

III. MONOPOLE MASSES

Since we are interested in the identification of monopoles in G with gauge fields in $*G$, we can restrict our attention to the monopoles with the lowest allowed nonvanishing magnetic charges, i.e., $eg/4\pi = \vec{\alpha}^{-1} \cdot \vec{H}$. Monopoles with higher charges $eg/4\pi = \sum_\alpha n_\alpha \vec{\alpha}^{-1} \cdot \vec{H}$ are possible and will in some cases be spherically symmetric as well.[5] Their masses will, however, satisfy the inequality

$$E_g \geq \sum_{\alpha} n_\alpha E_\alpha,$$

where the equal sign applies if the monopole mass saturates the Bogomolny lower bound in the BPS limit. Therefore it appears that the multiply charges monopoles are unstable against decay into monopoles with the lowest charges.

To ensure that the charges $eg/4\pi = \vec{\alpha}^{-1} \cdot \vec{H}$ for all $\vec{\alpha}^{-1}$ will be topologically conserved we have to break the symmetry such that only the fields corresponding to the generators of the Cartan subalgebra remain massless, i.e., the vacuum expectation value Φ_0 of the Higgs multiplet should be such that $[\Phi_0, \vec{H}] = 0$ but $[\Phi_0, E_{\pm\alpha}] \neq 0$ ($\forall\alpha$). It then follows that Φ_0 can be written as

$$\Phi_0 = \vec{f} \cdot \vec{H}, \tag{3.1}$$

where $\vec{f}$ is some appropriate constant vector.

Next we turn to the calculation of the monopole masses. For a purely magnetic time-independent solution we write the Hamiltonian density:

$$\begin{aligned} \mathcal{H} = -\mathcal{L} &= \frac{1}{4P}\mathrm{Tr}(F_{ij}F_{ij}) + \frac{1}{P}\mathrm{Tr}(D_i\Phi)^2 + V(\Phi) \\ &= \frac{1}{2P}\mathrm{Tr}(B_k \mp D_k\Phi)^2 \pm \frac{1}{P}\mathrm{Tr}(B_k D_k \Phi) + V(\Phi), \end{aligned} \tag{3.2}$$

where we have defined the magnetic field as usual by

$$B_k = \tfrac{1}{2}\epsilon_{ijk}F_{ij} \,. \tag{3.3}$$

If we now consider the BPS limit in which we neglect $V(\Phi)$ but maintain the nonvanishing vacuum expectation value for Φ, then any solution to the first-order differential equations

$$B_k = \pm D_k\Phi \tag{3.4}$$

will saturate the lower bound on the energy.[8] This energy is directly related to the topological charge and therefore independent of the details of the solution of (3.4), for we can write

$$E = \int d^3x\,\mathcal{H} = \pm \int d^3x\, \frac{1}{P}\,\mathrm{Tr}(B_k D_k \Phi)\,, \tag{3.5}$$

which can be partially integrated to yield

$$E = \pm \lim_{r^2\to\infty} r^2 \int d\Omega\, \frac{1}{P}\,\mathrm{Tr}(B_r\Phi_0)\,, \tag{3.6}$$

where we have used $D_kB_k \equiv 0$ so that only the boundary term (3.6) survives. Remembering that for a monopole $B_r = g/4\pi r^2 = \vec{\alpha}^{-1}\cdot\vec{\mathrm{H}}/er^2$ we find for the corresponding monopole mass

$$E_\alpha = \frac{4\pi}{e}|\vec{\mathrm{f}}\cdot\vec{\alpha}|\,. \tag{3.7}$$

Let us now verify that for the monopoles we are interested in, the equations (3.4) can indeed be satisfied. First observe that

$$eg = \vec{\alpha}^{-1}\cdot\vec{\mathrm{H}} = T_{3\alpha}\,, \tag{3.8}$$

i.e., that for each $\vec{\alpha}$ the charge matrix $eg/4\pi$ defines a regular SU(2) embedding in G:

$$\vec{\mathrm{T}}_\alpha = \left\{\frac{1}{(2\alpha^2)^{1/2}}(E_\alpha + E_{-\alpha}), \frac{1}{i(2\alpha^2)^{1/2}}(E_\alpha - E_{-\alpha}), \vec{\alpha}^{-1}\cdot\vec{\mathrm{H}}\right\}. \tag{3.9}$$

In fact from condition (3.8) together with

$$[\Phi_0, eg] = 0\,, \tag{3.10}$$

it follows that the corresponding nonsingular solution will be spherically symmetric with respect to the generalized angular momentum operator $\vec{\mathrm{J}} = \vec{\mathrm{L}} + \vec{\mathrm{T}}$ with $\vec{\mathrm{L}} = \vec{\mathrm{r}}\times(-i\vec{\nabla})$.[5] It is therefore not hard to generate an ansatz for the fields. First we decompose the Higgs field as

$$\Phi_0 = \varphi_0 + a\,T_{3\alpha}\,, \tag{3.11}$$

where $\varphi_0 = [(\vec{\mathrm{f}} - (\vec{\mathrm{f}}\cdot\vec{\alpha})\vec{\alpha}^{-1}]\cdot\vec{\mathrm{H}}$ which satisfies

$$[\varphi_0, \vec{\mathrm{T}}] = 0\,. \tag{3.12}$$

Because of (3.12) we can make the usual 't Hooft-Polyakov ansatz in the SU(2) subalgebra spanned by $\vec{\mathrm{T}}_\alpha$,

$$e\vec{\mathrm{W}} = [1 - K(r)]\vec{\mathrm{T}}_\alpha \times \hat{r}/r\,, \tag{3.13a}$$

$$\Phi(r) = \varphi_0 + H(r)\vec{\mathrm{T}}_\alpha\cdot\hat{r}\,, \tag{3.13b}$$

with boundary conditions

$$K(\infty) = 0, \quad H(\infty) = \vec{\mathrm{f}}\cdot\vec{\alpha}\,. \tag{3.14}$$

Substituting (3.13) in the Bogomolny equations (3.4) one obtains the following set of first-order radial equations:

$$\begin{aligned} eK' &= \pm HK\,, \\ r^2H' &= \pm e(K^2 - 1)\,, \end{aligned} \tag{3.15}$$

which have the familiar scale-invariant Prasad-Sommerfield[3] solution

$$\begin{aligned} K &= \frac{\lambda r}{\sinh\lambda r}\,, \\ H &= \pm e\left(\frac{1}{r} - \lambda\coth\lambda r\right)\,. \end{aligned} \tag{3.16}$$

Recalling that we must use the + (−) sign for $\vec{\mathrm{f}}\cdot\vec{\alpha}$, the boundary conditions (3.14) are satisfied if we choose $\lambda = -(\vec{\mathrm{f}}\cdot\vec{\alpha}^{-1})/e$.

IV. VECTOR MASSES AND MASS RELATIONS

First we calculate the masses of the gauge fields generated by the symmetry breaking $\Phi = \Phi_0 = \vec{\mathrm{f}}\cdot\vec{\mathrm{H}}$, and then compare the results with the monopole masses (3.7).

It is convenient to expand W as

$$W_\mu = \sum_\alpha (W^{+\alpha}T_{-\alpha} + W^{-\alpha}T_{+\alpha}) + \sum_{r=1}^{l} W_r H_r\,, \tag{4.1}$$

where the sum over α runs over the positive nonzero roots of G [i.e., there are $\frac{1}{2}(d-l)$ terms in $\sum_\alpha$, where $d = \dim G$ and $l = \mathrm{rank}\, G$]. The mass term in the Lagrangian is generated by

$$\frac{1}{2P}\,\mathrm{Tr}(D_\mu\Phi_0)^2 = -\frac{e^2}{2P}\,\mathrm{Tr}[W,\Phi_0]^2\,, \tag{4.2}$$

with

$$[W,\Phi_0] = \sum_\alpha (\vec{\mathrm{f}}\cdot\vec{\alpha})(W^{+\alpha}T_{-\alpha} - W^{-\alpha}T_{+\alpha})\,. \tag{4.3}$$

We obtain

$$\begin{aligned} [W,\Phi_0]^2 = \sum_{\alpha,\gamma} (\vec{\mathrm{f}}\cdot\vec{\alpha})(\vec{\mathrm{f}}\cdot\vec{\gamma})(&W^{+\alpha}W^{+\gamma}T_{-\alpha}T_{-\gamma} + W^{-\alpha}W^{-\gamma}T_{+\alpha}T_{+\gamma} \\ &- W^{+\alpha}W^{-\gamma}T_{-\alpha}T_{+\gamma} \\ &- W^{-\alpha}W^{+\gamma}T_{+\alpha}T_{-\gamma})\,. \end{aligned} \tag{4.4}$$

Taking the trace of (4.4) as required by (4.2) the only contributions come from terms where $\vec{\gamma} = \vec{\alpha}$, it follows that

$$-\frac{e^2}{2P}\,\mathrm{Tr}[W,\Phi_0]^2=e^2\sum_\alpha(\vec{f}\cdot\vec{\alpha})^2W^{+\alpha}W^{-\alpha}. \qquad (4.5)$$

So the masses of the gauge fields become

$$M_{W\pm\alpha}=e\,|\vec{f}\cdot\vec{\alpha}\,|. \qquad (4.6)$$

This amusing result tells us that the masses of a gauge field, after spontaneously breaking the symmetry with one Higgs multiplet in the adjoint representation, is just given by the inner product in root space of the vector $\vec{f}$ which determines the vacuum expectation value of Φ and the root vector $\vec{\alpha}$ corresponding to the gauge field.

If we perform a similar calculation in the dual theory based on $*G$, where we have roots $\vec{\beta}=\vec{\alpha}^{-1}/N$ and a coupling constant g_0 we would find the monopole and gauge field masses to be

$$*E_{\pm\beta}=\frac{4\pi}{g_0}|\vec{f}\cdot\vec{\beta}^{-1}|, \qquad (4.7a)$$

$$*M_{\pm\beta}=g_0\,|\vec{f}\cdot\vec{\beta}\,|, \qquad (4.7b)$$

and we obtain the remarkable relation between masses of monopoles (3.7), (4.7b) and masses of gauge fields (4.7a) and (4.6) of the dual theory:

$$*M_{\pm\beta}=E_{\pm\alpha}\,,\qquad *E_{\pm\beta}=M_{\pm\alpha}\,, \qquad (4.8)$$

provided that the coupling constants satisfy

$$g_0=\frac{4\pi N}{e}. \qquad (4.9)$$

Note that the constant N is the same whether we go from $G\to *G$ or vice versa [because $(\vec{\alpha}^{-1})^{-1}=\vec{\alpha}$]. The question arises what happens to the massless vector fields under the duality transformation. In general these are l massless fields associated with the l-fold-degenerate zero root, but there is only one monopole state with zero charge (the vacuum). Therefore one is led to think that the massless fields carry over to the corresponding massless gauge fields in the dual theory, in analogy with the SO(3) case.

V. SYMMETRY BREAKING

In the SO(3) model one can break the theory such that only the symmetry generated by the Cartan subalgebra [U(1)] survives with a single Higgs field Φ in the adjoint representation. In this section we will consider this question for a general group. If one imposes a discrete symmetry $\Phi\to-\Phi$ on the Lagrangian of the SU(3) theory, the desired breaking is possible in the tree approximation, but not if one includes loop corrections as we will show next. The potential takes the form

$$V(\Phi)=\frac{\mu^2}{2}\,\mathrm{Tr}\,\Phi^2+\frac{\lambda}{4}(\mathrm{Tr}\Phi^2)^2, \qquad (5.1)$$

where we have used the fact that $\mathrm{Tr}\Phi^4\propto(\mathrm{Tr}\Phi^2)^2$. The value of $V(\Phi)$ depends therefore on $\mathrm{Tr}\Phi^2$ only and is consequently independent of the direction of $\vec{\Phi}$ in group space. It is then possible to choose for example $f\lambda_3$, which breaks the symmetry down to two U(1) factors (one generated by λ_3, the other by λ_8). The proposed dual symmetry however can be a property of the full quantum theory only. On a classical level the monopole has zero angular momentum, for example, and one has to check whether the choice for Φ_0 is quantum mechanically stable for a finite range of the parameters μ and λ. We therefore calculate the difference $\Delta V(\Phi)$ due to the one-loop corrections for the case at hand, which is in general given by[9]

$$\Delta V=\frac{1}{64\pi^2}\,\mathrm{Tr}[M_W^{\,4}\ln M_W^{\,2}]+\frac{3}{64\pi^2}\,\mathrm{Tr}[m^4\ln m^2]. \qquad (5.2)$$

Let us choose

$$\Phi_0=\vec{f}\cdot\vec{H}=f\left(\frac{\lambda_3}{2}\cos\theta+\frac{\lambda_8}{2}\sin\theta\right) \qquad (5.3)$$

and determine which value of θ minimizes ΔV in (5.2). First observe that it follows from the simple form of (5.1) that the symmetry of $V(\Phi)$ is in fact O(8), which after fixing f reduces to O(7). This means that there will be seven Goldstone bosons so that only the one component of $\varphi=\Phi-\Phi_0$ corresponding to $\vec{f}$ becomes massive. Consequently, the second term of (5.2) becomes simple $f^4\log f^2$ and is independent of θ. In the previous section we demonstrated that $M_{W\pm\alpha}=e\,|\vec{f}\cdot\vec{\alpha}\,|$, so that (5.1) becomes

$$\Delta V=\frac{1}{64\pi^2}\sum_\alpha e^4(\vec{f}\cdot\vec{\alpha})^4\ln[e^2(\vec{f}\cdot\vec{\alpha})^2]. \qquad (5.4)$$

This can be calculated directly from the projections $(\vec{f}\cdot\vec{\alpha})$ in the root diagram; defining $c=ef|\alpha|$ we get

$$\Delta V=\frac{c^2}{64\pi^2}\Big[\tfrac{1}{8}(\cos\theta+\sqrt{3}\,\sin\theta)^4\ln\frac{c}{4}(\cos\theta+\sqrt{3}\,\sin\theta)^2$$
$$+\tfrac{1}{8}(\cos\theta-\sqrt{3}\,\sin\theta)^4$$
$$\times\ln\frac{c}{4}(\cos\theta-\sqrt{3}\,\sin\theta)^2$$
$$+2\cos^4\theta\ln c\cos^2\theta\Big]. \qquad (5.5)$$

The expression above is minimized for $\theta=\pi/2$ (independent of c), i.e., breaking occurs along the λ_8 direction and the residual symmetry is not U(1)× U(1) but U(2). For larger groups one does not have the relation $\mathrm{Tr}\Phi^4\alpha(\mathrm{Tr}\Phi^2)^2$, and it has been shown at least for the unitary and orthogonal groups of arbitrary rank that the desired symmetry breaking cannot be achieved by only one Higgs multiplet in the adjoint representation, not even in the tree approximation.[10]

The previous considerations raise the important question of whether the conjectured dual symmetry will persist beyond the BPS limit. The classical argument given so far seems to convey a negative answer. Taking the potential $V(\Phi)$ into account (without taking the BPS limit) will leave the gauge field masses unchanged, but will increase the monopole mass,[11] so that the remarkable mass relation breaks down on this superficial level.

At this point we note another peculiar aspect of the dual symmetry, which we mentioned briefly in Sec. II, namely the fact that Sp($2N$) and SO($2N+1$) are each others' dual. Except for the cases $N=1,2$, the algebras are not isomorphic so that if we use formally the same potential $V(\Phi)$ in the theory and its dual, we expect to obtain a different symmetry-breaking pattern. In those cases it is not clear what happens to the potential under the duality transformation, or how to generate the correct BPS limit.

VI. DISCUSSION

In the preceding sections we have established the striking relation between the masses of monopoles and of gauge particles in the dual theory. Subsequently we pointed out some problems related to the necessary type of symmetry breaking. We conclude with some additional comments.

(i) In the SO(3) case the dual symmetry conjecture is supported by another observation, namely that in the BPS limit the force between two well-separated equally charged monopoles vanishes.[12] The same is true for the force between two equally charged gauge particles[2]; the repulsive contribution to the force from the one "photon" exchange is canceled by the attractive contribution of the corresponding massless scalar field. For oppositely charged monopoles (gauge fields) the force becomes twice the one expected, so that $F=2g^2/|\vec{r}_1-\vec{r}_2|^2$. This result naturally carries over to the case of an arbitrary gauge group G, where in the BPS limit one has to take the effects of l massless gauge fields and l massless scalar fields into account.

(ii) The dual transformation maps a weak-coupling theory onto a strong-coupling theory, which means that perturbation theory only applies in one of the two images. This raises the question of what happens to the charge-quantization condition $eg=n$ under renormalization. It is hard to determine what this means, since renormalization itself is only defined perturbatively and, furthermore, e and g are not independent. Conceivably the quantization condition is invariant, i.e., $\bar{e}\bar{g}=n$, which suggests that if $\bar{e}$ becomes large, $\bar{g}$ would become small and calculations in the dual theory would become meaningful.

After this work had been completed two papers by P.G.O. Freund appeared in which similar topics are discussed.[13]

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