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Non-Abelian structure of Yang-Mills theory and infrared-finite asymptotic states

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The idea of Kulish and Faddeev for treating asymptotic dynamics is extended to derive a set of states with which the matrix element of quark scattering from a color-singlet external potential is shown to be free of the leading-order non-Abelian class of infrared divergences to fifth order in g.

An open and intriguing question in the study of the infrared region of non-Abelian gauge theory[1] is the existence of a set of asymptotic states, analogous to the coherent states of QED and gravitation, with an "S-matrix operator" with finite matrix elements. In QED, the initial and major contribution toward an affirmative answer was provided by Chung's ansatz[2] for the asymptotic states. We show here that the perhaps less well remembered constructive approach[3,4] to the asymptotic dynamics by Kulish and Faddeev, when extended, also solves the problem in non-Abelian gauge theory, at least to the extent of the lowest nontrivial order we have examined so far. It should be noted that an affirmative answer is complementary and not *ab initio* orthogonal to the confinement property for quantum chromodynamics (QCD) because the present analysis does not incorporate the important nonperturbative, long-range effects which are conjectured to forbid colored quarks, color-octet gluons, and all systems with color-nonsinglet quantum numbers from existing as isolated objects.[5]

We consider spin-$\frac{1}{2}$ fermion scattering in an external potential which transforms as a singlet representation of the non-Abelian gauge group. In the Schrödinger representation the interaction operator coupling the fermion fields and a set of gauge fields is

$$V=-g\int :\ \bar{\psi}(x)\gamma_\mu t_a\psi(x):\ A_a^\mu(x)d^3x\,, \tag{1}$$

where $[t_a,t_b]=if_{abc}t_c$, with t_a the representation matrices for the fermions and f_{abc} the real and totally antisymmetric structure constants of the gauge group. We work in the Feynman–'t Hooft gauge. Following Kulish and Faddeev, by studying the $|t|\to\infty$ behavior of this expression in the interaction representation, one finds that the asymptotic dynamics are described in the Schrödinger representation by means of $H_{as}(t)=H_0+V_{as}(t)$, where

$$V_{as}(t)=\frac{1}{(2\pi)^{3/2}}\int J^\mu_{a,as}(\vec{k},t)\times[a_{a\mu}(k_0,\vec{k})+a^\dagger_{a\mu}(k_0,-\vec{k})]\,\frac{d^3k}{\sqrt{2k_0}}\,,$$

$$J^\mu_{a,as}(k,t)=-g\int p^\mu \exp\left(i\,\frac{\vec{p}\cdot\vec{k}}{p_0}\,t\right)\rho_a(p)\,\frac{d^3p}{p_0}\,,$$

$$\rho_a(p)=\sum_{\pm s}[b^\dagger(p,s)t_ab(p,s)-d^\dagger(p,s)t_a^Td(p,s)]\,. \tag{2}$$

The asymptotic S-matrix operator is defined as

$$S^g(t_1,t_2)=\lim_{\substack{t_1\to\infty\\ t_2\to-\infty}}U_{as}(t_1)^\dagger\exp[-iH(t_1-t_2)]U_{as}(t_2)\,, \tag{3}$$

where the operator $U_{as}(t)$ satisfies $idU_{as}(t)/dt=H_{as}(t)U_{as}(t)$. The solution is $U_{as}(t)=\exp(-iH_0t)Z(t)$, $Z(t)=\exp[\Omega(t)]$, where, in terms of the asymptotic potential in the interaction representation $V^I_{as}(t)$,

$$\frac{d\Omega}{dt}=\{-iV^I_{as}(t),\Omega/(1-e^{-\Omega})\}=\sum_{n=0}\beta_n\{-iV^I_{as}(t),\Omega^n\}\,, \tag{4}$$

in a notation employing Lie elements $\{y,x^l\}=[\cdots[[y,x]x]\cdots x]$, $\{y,x^0\}=y$. The β_n vanish for n odd, except $\beta_1=\frac{1}{2}$, $\beta_0=1$, and $\beta_n=B_n/n!$ for n even with B_n the Bernoulli numbers. Thus, by this theorem[7] of Magnus, in the non-Abelian theory

$$\Omega(t)=-i\int^t V^I_{as}(\tau)d\tau-\tfrac{1}{2}\int^t d\tau\int^\tau d\sigma\,[V^I_{as}(\tau),V^I_{as}(\sigma)]+(-i)^3\tfrac{1}{4}\int^t d\tau\int^\tau d\sigma\int^\sigma d\rho[V^I_{as}(\tau),[V^I_{as}(\sigma),V^I_{as}(\rho)]]$$
$$+(-i)^3\tfrac{1}{12}\int^t d\tau\int^\tau d\sigma\int^\tau ds[[V^I_{as}(\tau),V^I_{as}(\sigma)],V^I_{as}(s)]+\cdots\,, \tag{5}$$

by iterative integration. We assume that the contributions to the integrals from the lower limits of integration must vanish although we have not yet carefully shown the desirability or necessity of this. In the Abelian case[3] this assumption is necessary or else $\Omega(t)$ does not commute asymptotically with the total-momentum operator.

We use the usual Dyson S operator and consider the asymptotic states in the non-Abelian case in

$$\mathcal{H}_{as} = \lim_{t\to-\infty} \exp[-R(t) - i\Phi(t)]\mathcal{H}_F\,. \qquad (6)$$

Proceeding formally in the complete infinite tensor product of the spaces of the individual oscillators, one could try to separate the anti-Hermitian $\Omega(t) = R(t) + i\Phi(t)$, with $R(t)$ identified as symmetrically separated series of terms linear in $\rho_c(p)$, and then define R_f to be the time-independent quantity obtained from $R(t)$ by replacing each $(p_\mu/p\cdot k)\exp(itp\cdot k/p_0)$ by the transverse form factor $f_\mu(\vec{k},p) = (p_\mu/p\cdot k - c_\mu)\phi(k,p)$, $\phi(k,p)=1$ in the neighborhood of $k=0$, where c_μ is a light-like vector such that $k_\mu c^\mu = 1$.[8] Then, following Kulish and Faddeev, one would define the asymptotic space of states by $\exp(-R_f)\mathcal{H}_F$, $\mathcal{H}_F$ = usual Fock space, and the asymptotic S-matrix operator corresponding to (3), but acting in $\exp(-R_f)\mathcal{H}_F, \mathcal{H}_F$ would be

$$S = \lim_{\substack{t_1\to\infty\\ t_2\to-\infty}} \exp[-i\Phi(t_1)]S_D(t_1,t_2)\exp[i\Phi(t_2)]\,, \qquad (7)$$

where S_D is the Dyson S operator. However, this construction would assume $[R,\Phi]_- = 0$.

To the extent that we have considered the asymptotic dynamics for the non-Abelian case, it is necessary to verify for perturbation theory that the matrix elements $\langle\psi|S_D|\psi'\rangle$ for arbitrary ψ and ψ' in $\mathcal{H}_{as}$ are indeed infrared finite. For spin-$\frac{1}{2}$ fermion scattering in a color-singlet external potential to order g^5, we make a gauge transformation[3] and find the initial asymptotic state

$$\psi_{as}(p) = \lim_{t\to-\infty} \exp[-R(t) - i\Phi(t)]b^\dagger(p)|0\rangle$$
$$= (c + c_i a_i^\dagger + c_{ij}a_i^\dagger a_j^\dagger + \cdots)|0\rangle\,, \qquad (8)$$
$$a_i^\dagger = e_\mu^{(m)} a_{a\mu}(k)\,,$$

where $e_\mu^{(m)}(k)$, $m = 1,2$, are transverse polarization vectors with i denoting a for the gauge-group index, k for the gauge quantum four-momentum, and m for the polarization, and where the coefficients

$$\begin{aligned}
c &= 1 - \tfrac{1}{2}C_F\sum_m \int d^3k|\tilde{S}^{(m)}(k)|^2 + \tfrac{1}{8}C_F^2\Big(\sum_m\int d^3k|\tilde{S}^{(m)}(k)|^2\Big)^2 \\
&\quad + \tfrac{1}{4}C_F C_{\rm YM}\sum_{m,n}\int d^3k\,d^3l|\tilde{S}^{(m)}(k)|^2|\tilde{S}^{(n)}(l)|^2\left[1 + \frac{p\cdot k\,p\cdot l}{p\cdot(k+l)p\cdot(k+l)}\right] + \cdots\,, \\
c_i &= \left[1 - \tfrac{1}{2}C_F\sum_n\int d^3l|\tilde{S}^{(n)}(l)|^2 - \tfrac{1}{8}C_{\rm YM}\sum_n\int d^3l|\tilde{S}^{(n)}(l)|^2\frac{p\cdot l}{p\cdot(k+l)} + \cdots\right]t_a\tilde{S}^{(m)}(k)\,, \\
c_{ij} &= \tfrac{1}{2}t_a t_b\tilde{S}^{(m)}(k)\tilde{S}^{(n)}(l) + i\,\tfrac{1}{2}t_c f_{cab}\tilde{S}^{(m)}(k)\tilde{S}^{(n)}(l)\frac{p\cdot l}{p\cdot(k+l)} + \cdots\,,
\end{aligned} \qquad (9)$$

with

$$\tilde{S}^{(m)}(k) \equiv g\,p\cdot e^{(m)}/\{[2(2\pi)^3k_0]^{1/2}p\cdot k\}\,, \qquad (10)$$

where the Casimir operators are defined by $t_a t_b = C_F 1$ and $f_{abc}f_{abd} = C_{\rm YM}\delta_{cd}$. To order g^5, the unlisted terms do not contribute to the set of leading-order divergences which appear in the analysis of the matrix element; the listed coefficients do include significant contributions from the third-degree Lie elements of (5) but those possible from Lie elements of fourth degree vanish. It is easy to check that this state, with the listed coefficients, is normalized to unity to order g^4.

To study the infrared divergences we use the technique of dimensional regularization with space-time dimension $4+\epsilon$. Conscious of the similar calculation[10] based on the Bloch-Nordsieck idea, we group the contributions according to the topological sets which appear in this asymptotic approach in Feynman-type graphs, but with on-shell particles bridging the initial state, graph, and final-state parts. The nontrivial weighting factors are provided by (4). The C_F^2 divergences appear correctly and cancel, as they should. Factoring out $C_F C_{\rm YM}[g^4 M_0/\epsilon^2(2\pi)^4]$, with M_0 the order-g basic interaction, we find the non-Abelian class of graphs yield, as coefficients for successive terms of higher order in a quadratic polynomial in

$$\{(1+r^2)\ln[(1+r)/(1-r)]/2r-1\},$$

where $r=1/[(1-4m^2/q^2)]^{1/2}$ with fermion mass m and four-momentum transfer q, coefficients $(0,-\frac{1}{2},0)$ from the g^4 virtuals and again from the sum of g^4 states plus two gluons disconnected plus g^2 states with one gluon disconnected, coefficients $(0,1,\frac{1}{2})$ from g^3 graphs with one real gluon and again from the sum of one real gluon with one gluon disconnected plus one real gluon with g^2 states, and coefficients $(0,-1,-1)$ from two-real-gluon contributions. If this cancellation occurs for all orders, the simple mathematical structure to all orders revealed by (4) for the asymptotic states *must be present* in the graph part and, hence, be exploitable in proofs and practical calculations based on the Bloch-Nordsieck approach.

We wish to caution that to order g^5 there are also contributions of first and second degree in the three-point gluon vertex. The above type of graphical inspection, for example, indicates this and also shows that such contributions occur naturally in the asymptotic dynamics. We are presently evaluating the associated integrals so as to demonstrate the cancellation of infrared singularities for this subset.

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