Instantons in (1 + 1)-dimensional Abelian gauge theories

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We study the Abelian Higgs model in $1+1$ dimensions with the addition of a fermion of arbitrary charge g to understand the effect of instantons on the gauge and chiral structure of the model. Semiclassical approximations are used assuming a small instanton density. Without fermions, the global gauge symmetry is found to be restored, resulting in a linearly rising potential between static charges ($\pm Q$) and a complete screening of integral charges (Q/e = integer with e the scalar charge). This gauge structure persists with the addition of a massive fermion. The spectrum of fermion bound states and the stability of the θ vacuums are discussed. For a massless fermion, the gauge symmetry is spontaneously broken. We explicitly show that the chiral symmetry is also spontaneously broken, but that the Nambu-Goldstone pole decouples from the gauge-invariant sector of the theory.

I. INTRODUCTION AND SUMMARY

The theory of quarks coupled to non-Abelian gauge bosons is the most promising field-theoretical model for understanding the reactions and spectroscopy of hadrons. Because of its asymptotically free behavior at short distances, it is capable of explaining the scaling behavior of high-energy hadronic reactions. The logarithmic violation of scaling which it predicts may even have been observed. Furthermore, the growing tendency of the effective coupling constant at large distances has led to the speculation that quarks and gluons may be permanently bound within hadrons and never appear as asymptotic particles.[1]

An attempt to implement these ideas is the formulation of gauge theories on a lattice.[2] It has been shown that in the strong-coupling limit quarks are indeed confined. This approach suffers from two drawbacks, however: the lack of Lorentz covariance at any intermediate stage of calculation, and the assumption, yet to be proven, that there is no phase transition at nonzero values of the coupling constant.[3] (The latter is important in understanding the approach to the continuum limit.)

Recently Polyakov pointed out[4] that certain classical gauge field configurations in the continuum Euclidean space-time, which have since been termed instantons, may be relevant to the problem of quark confinement. In fact, for "compact" quantum electrodynamics in $2+1$ dimensions, he has shown that instantons (which are Polyakov–'t Hooft monopoles[5]) boost the logarithmic Coulomb potential into a linearly rising one. For non-Abelian gauge theories in $3+1$ dimensions, Callan, Dashen, and Gross[6] have argued that the instanton discovered by Belavin *et al.*[7] does not confine quarks by itself, and proposed "half instantons" or "merons" as the relevant field configuration. Another important observation, by 't Hooft,[8] is that in the presence of massless quarks the instanton of Belavin *et al.* induces an effective interaction which breaks the chiral U(1) invariance down to a discrete symmetry, thus providing a possible way out of the U(1) problem.[8a]

With these ideas in mind, we have studied a simple model possessing instantons, namely, the U(1) gauge theory in $1+1$ dimensions specified by the Euclidean[9] Lagrangian density

$$\mathcal{L} = \tfrac{1}{4}F_{\mu\nu}F_{\mu\nu} + (\partial_\mu + ieA_\mu)\phi^*(\partial_\mu - ieA_\mu)\phi + \tfrac{1}{2}\lambda(\phi^*\phi - v^2)^2 + \bar\psi(\gamma_\mu\partial_\mu + m)\psi + ig\bar\psi\gamma_\mu\psi A_\mu \,. \tag{1.1}$$

Here $F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu$ is the curl of the U(1) gauge field A_μ, ϕ and ψ are complex scalar and Fermi fields of charges e and g, respectively, and we choose $\lambda, v > 0$, namely, the classical potential for ϕ has a degenerate set of minima at $|\phi| = v$. This Lagrangian density is invariant under the local gauge transformation $A_\mu(x) \to A_\mu(x) + \partial_\mu\lambda(x)$, $\phi(x) \to \phi(x)\exp[ie\lambda(x)]$, $\psi(x) \to \psi(x)\exp[ig\lambda(x)]$, and also under the global chiral transformation $A_\mu(x) \to A_\mu(x)$, $\phi(x) \to \phi(x)$, $\psi(x) \to \exp(\alpha\gamma_5)\psi(x)$ up to the mass term $m\bar\psi\psi$.

In the conventional perturbative approach, the system undergoes a spontaneous breakdown of the gauge symmetry signaled by a nonzero vacuum expectation value for ϕ,[10] which in turn generates a mass for the gauge field. In addition, the chiral symmetry which holds at $m = 0$ remains unbroken.[11] Thus the spectrum includes a massive vector boson, a massive neutral scalar boson, and a charged (massive or massless) fermion. As a result, electric charge is screened and the static potential between charges damps exponentially with their increasing separation.

Except for terms containing Fermi fields, (1.1) is identical to the Ginzburg-Landau free energy density for a two-dimensional superconductor. Such a system is known to possess vortex configurations, i.e., localized regions with $\phi(x)\approx 0$ through which a "magnetic" field ($F_{\mu\nu}$) penetrates perpendicular to the two-dimensional plane. They have the property that the phase of $\phi(x)$ changes by $2\pi N$ ($N\in Z$, the group of integers) along a closed path encircling a vortex. They are the instantons in our terminology, and we have studied how they modify the gauge and chiral structure of the system obtained in a perturbative approach.

The main tool of our analysis is the so-called dilute-instanton-gas approximation which takes into account field configurations consisting of an arbitrary number of instantons widely separated from each other as compared to their sizes. The range of validity of this approximation is limited to $S_{\rm cl}\gg 1$ where $S_{\rm cl}$ is the Euclidean action of one instanton. (This condition will be made more precise in Sec. II.) The conclusions we have reached are summarized as follows:

(1) *The system without fermions*: Instantons mediate quantum-mechanical tunneling from one classical vacuum configuration to another.[4,8,12] As a result, the true vacuum is a linear superposition of gauge vacuums centered around the classical zero-energy configurations which are connected to each other by an instanton. The new vacuum is parametrized by an angle θ and is called the θ vacuum.[12]

The θ vacuums respect the global U(1) gauge symmetry, i.e., in the θ vacuums, the expectation value of ϕ vanishes and the gauge field propagator acquires a massless pole[13] in addition to the massive pole present in a perturbative treatment.

As may be expected from this, the static potential between a pair of charges Q and $-Q$ separated by a distance R consists of two terms: a short-range piece damping exponentially with R, and a long-range piece rising linearly with R. The latter has two noteworthy features: (a) It becomes dominant over the former on a length scale characteristic of the mean instanton separation, i.e., the linear potential becomes appreciable only when the charges are far enough apart so that there are many instantons between them. (b) The coefficient of R is periodic in Q with period e, vanishing for $Q=ne$ ($n\in Z$). This is indicative of charge screening due to the scalar field ϕ (with charge $\pm e$) which is present under the guise of instantons. We verify this explicitly by calculating the expectation value of the total charge in the presence of an isolated external charge. Some of these results have previously been obtained by Callan *et al.*[16]

The θ vacuums have a background electric field proportional to $\sin\theta$. Thus, for $\theta\neq\theta$ and π (mod 2π), parity and time-reversal invariances are spontaneously broken.[8,12] In order to see whether these effects are physically meaningful, we examine the stability of the θ vacuums against charged pair production. The energy density of the θ vacuums is proportional to $-\cos\theta$.[14] The creation of a pair of charges Q and $-Q$ effectively changes the value of θ between the pair into $\theta \mp 2\pi Q/e$ [the sign depends on whether Q is on the right ($-$) or left ($+$) of $-Q$]. From these considerations, we show that the introduction of an additional field with charge g ($\neq ne$, $n\in Z$) induces the decay of the θ vacuums except for $\theta=0$ (mod 2π).[15]

(2) *The system with massive fermions*: All the results described above apply to this case. In particular, the existence of a linearly rising term in the static potential indicates that, unless $g=ne$ ($n\in Z$), a single fermion state has infinite energy and hence disappears from the spectrum. In order to confirm this and also to analyze the spectrum in general, we derive an effective Lagrangian for the Fermi field. This is done by integrating out the scalar and gauge fields in the Feynman path integral within the dilute-instanton-gas approximation. We then express the effective Lagrangian in terms of an equivalent boson representation and apply semiclassical techniques. The result: for $g/e=p/q$, a rational number ($p,q\in Z$), there are finite-energy bound states of l fermions and $\bar{l}$ antifermions with $l-\bar{l}=\pm q$ and with (fermionic) charge $Q_f=\pm pe$. (This is possible since the fermionic charge can then be screened by vacuum polarization of the scalar field.) There are also bound states containing an equal number of fermions and antifermions. If g/e is not a rational number, only the latter type of bound state exists.

We also show that unless θ and g/e satisfy $\theta + 2\pi mg/e = 2\pi n$ ($m,n\in Z$) the true vacuum breaks parity and time-reversal invariances spontaneously. The above criterion is interpreted as the condition for an exact cancellation of the background electric field by the creation of charged fermion and scalar pairs.

(3) *The system with massless fermions*: As has been shown by Callan *et al.*,[16] massless fermions induce a logarithmic interaction between instantons. This long-range interaction prohibits tunneling in the global sense although it is still allowed locally.[17] The question then is to understand the effect of the long-range interaction on the gauge and chiral structure of the theory.

In order to analyze this problem, we continue to use the boson representation of the effective Lagrangian (with $m=0$) obtained in (2), and invoke two statistical-mechanical analogies. One is the two-dimensional Coulomb gas analogy discovered by Callan *et al.*[16] in which the instantons are identified with the Coulomb charges. The other is an analogy to the continuum limit of the classical X-Y model in two dimensions. The equivalent Bose field (with a certain smearing due to the finite instanton size) is regarded as the angle of the two-dimensional classical spin relative to a fixed direction, and the instantons induce an "external magnetic field." The temperature for the former is given by $T'=e^2/4\pi g^2$, while for the latter, $T=4\pi g^2/e^2$.

For the Coulomb gas, Kosterlitz[18] has carried out a detailed renormalization group analysis. His results relevant to our study will be summarized in Sec. IV. To further facilitate the analysis, we have constructed a simple mean-field approximation for the X-Y model which is valid for low temperature. (For a more precise statement, see Sec. IV.) Roughly speaking, the behavior of the two systems that emerged is as follows: They both have a phase transition at $g^2=g_c^2 \simeq 2e^2$.[19] For $g^2>g_c^2$, the Coulomb gas is in a dipole phase (i.e., two oppositely signed charges bound together) and the X-Y model is in a "disordered" phase (i.e., the free energy is analytic at vanishing external magnetic field). For $g^2<g_c^2$, on the other hand, the Coulomb gas behaves as a uniform plasma, and the free energy of the X-Y model develops a singularity at vanishing external magnetic field ("ordered" phase).

In terms of these analogies, the gauge and chiral structure of the theory that we have found is described as follows:

(i) the gauge symmetry is spontaneously broken for arbitrary values of g^2. The mechanism responsible for its breakdown is quite different below and above g_c^2, however. For $g^2>g_c^2$, a pair of instantons with the phase change of $\phi(x)$ equal to 2π and -2π are closely bound together. (They correspond to Coulomb charges +1 and −1.) Thus, as far as the gauge symmetry is concerned, the θ vacuums have essentially the same structure as that of the perturbation-theory vacuums, and hence the gauge symmetry is spontaneously broken. For $g^2<g_c^2$, on the other hand, the instantons form a uniform plasma which exhibits a Debye-Hückel-type screening. This means that the gauge field propagator has a massive pole in addition to that present in a perturbative approach. The former becomes massless as $g^2 \to g_c^2 - 0$, but the residue also vanishes in this limit, ensuring a smooth transition to the region $g^2>g_c^2$. The gauge symmetry is therefore still spontaneously broken, and there is no linearly rising potential between a pair of charges.

(ii) The chiral structure is as follows: In terms of the X-Y model analogy, the angle θ determines the direction of the external magnetic field induced by the instantons. Obviously, the behavior of the system does not depend on the value of θ. Thus the θ vacuums form a degenerate set of vacuums which, we note, are mutually connected by chiral transformations. In addition, the X-Y model spin, which rotates under a chiral transformation, has a nonvanishing expectation value for all g^2. Thus the chiral symmetry is spontaneously broken[10] and moreover the θ vacuums are the correct vacuums satisfying the cluster decomposition property[17] (see Sec. IV C for details).

The corresponding Nambu-Goldstone pole appears in the Green's function $\langle\theta|T^*(\hat{J}^5_\mu(x)\sin\phi(y))|\theta\rangle$ where $\hat{J}^5_\mu$ is the conserved but gauge-noninvariant axial-vector current and $\sin\phi(y)$ is one of the components of the X-Y model spin.[13] This Nambu-Goldstone pole decouples from the gauge-invariant sector of the theory: The "seizing" mechanism proposed by Kogut and Susskind[20] is at work.

Actually the effective Lagrangian for the fermion is still invariant under a discrete chiral transformation $\psi \to \exp[\pi(e/g)n\gamma_5]\psi$ $(n\in Z)$ which, if not spontaneously broken, is enough to prohibit a nonzero fermion mass.[21] We show that this symmetry is spontaneously broken only for $g^2<g_c^2$. This is made explicit through the observation that for $g^2<g_c^2$ our effective Lagrangian is a cut-off version of the sine-Gordon Lagrangian.

The rest of the paper is a detailed account of the results described above. Sections II, III, and IV, respectively, deal with the three cases (1), (2), and (3). Section V is devoted to discussion. In particular, we compare our results with the recent work of Callan *et al.*[16] in which, instead of one fermion with an arbitrary charge g, they introduced N species of fermions all with charge e and studied the chiral structure as a function of N.

II. SCALAR ELECTRODYNAMICS IN 1+1 DIMENSIONS

We consider the system having the Euclidean Lagrangian density

$$\mathcal{L}=\tfrac{1}{4}F_{\mu\nu}F_{\mu\nu}+(\partial_\mu+ieA_\mu)\phi^*(\partial_\mu-ieA_\mu)\phi + \tfrac{1}{2}\lambda(\phi^*\phi-v^2)^2\,, \tag{2.1}$$

which is invariant under the U(1) gauge transformation $A_\mu(x)\to A_\mu(x)+\partial_\mu\lambda(x)$, $\phi(x)\to\phi(x)\exp[ie\lambda(x)]$, and we choose $\lambda, v>0$.

A. Instantons and the θ vacuums

In the usual perturbative treatment, the system described by (2.1) undergoes a spontaneous breakdown of the gauge symmetry due to the degenerate minima $[|\phi(x)|\sim v]$ of the scalar field potential[10]; which in turn generates a mass for the gauge field.

As a result, the particle spectrum consists of a vector boson with mass $m_W = \sqrt{2}\, ev$ and a neutral scalar boson with mass $m_s = \sqrt{2\lambda}\, v$. Consequently, charges are completely screened and the static potential between a pair of charges separated by a distance R decreases as $e^{-m_W R}$.

However, as discussed by Callan *et al.*,[14] the existence of instantons in this system completely alters the perturbative picture described above. In order to see how this comes about, we describe below the properties of one-instanton solutions and briefly outline the effect of instantons on the vacuum structure.

The Euclidean Lagrangian density (2.1) is identical to the Ginzburg-Landau free energy density for a superconductor in two space dimensions. Such a system has space-dependent field configurations with finite action known as vortices, which are the instantons in our terminology.

Recall that a field configuration has finite action only if it satisfies

$$A_\mu(x) \to \partial_\mu \lambda(x),$$
$$\phi(x) \to v \exp[ie\lambda(x)], \qquad (2.2)$$
$$\text{as } |x| \equiv (x_4^2 + x_1^2)^{1/2} \to +\infty\,.$$

The function $z \equiv \exp[ie\lambda(x)]$ then defines a mapping from the one-dimensional sphere at Euclidean infinity $|x| = +\infty$ to the unit sphere in the complex z plane. Since $\phi(x)$ must be a single-valued function, such mappings fall into homotopy classes characterized by an integer winding number

$$q = \frac{e}{4\pi} \int d^2x\, \epsilon_{\mu\nu} F_{\mu\nu} = \frac{e}{2\pi} \oint_{|x|=\infty} dx_\mu A_\mu \in Z. \qquad (2.3)$$

In the case that the inequality $e \ll \sqrt{\lambda}$ is satisfied,[22] the explicit form for an instanton with winding number $q = N (\neq 0)$ in the Landau gauge ($\partial_\mu A_\mu = 0$) is given by

$$A_\mu^{(N)}(x-R) = \frac{N}{e}\, \partial_\mu \Theta(x-R) A_N(|x-R|)\,, \qquad (2.4a)$$

$$\phi^{(N)}(x-R) = \exp[iN\Theta(x-R)]\rho_N(|x-R|), \qquad (2.4b)$$

where $\Theta(x) = \tan^{-1}(x_4/x_1)$ is the angle of the vector (x_4, x_1) measured from the direction of the x_1 axis, R is an arbitrary two-dimensional vector representing the position of the instanton, and[23]

$$A_N(|x-R|) \simeq 1 - m_W |x-R| K_1(m_W|x-R|) \quad \text{for } |x-R| \gtrsim 1/m_s$$

$$\sim 1 + O(e^{-m_W|x-R|}), \quad \text{as } |x-R| \to \infty \qquad (2.4c)$$

$$\rho_N(|x-R|) \sim v + O(e^{-m_s|x-R|}), \quad \text{as } |x-R| \to +\infty \qquad (2.4d)$$

(K_1 is the modified Bessel function). The corresponding gauge field strength $F_{\mu\nu}$ and the Euclidean action $S \equiv \int d^2x\, \mathcal{L}$ are given by[23]

$$F_{\mu\nu}^{(N)}(x-R) \simeq \epsilon_{\mu\nu} \frac{N}{e} m_W^2 K_0(m_W|x-R|), \quad \text{for } |x-R| \gtrsim 1/m_s, \qquad (2.4e)$$

$$S^{(N)} \equiv S[A_\mu^{(N)}, \phi^{(N)}] \simeq \pi \frac{N^2 m_W^2}{e^2} \ln\frac{m_s}{m_W} \equiv S_N. \qquad (2.5)$$

In Fig. 1, the functions $\rho_N(|x-R|)$ and $E^{(N)}(|x-R|) \equiv \frac{1}{2}\epsilon_{\mu\nu}F_{\mu\nu}^{(N)}(x-R)$ are sketched.

In quantum mechanics, the existence of a solution to the Euclidean equations of motion is associated with a tunneling process in which a particle penetrates through an energy barrier.[24] Analogously, instantons are interpreted as mediating the tunneling between two different gauge vacuums.[4,8,12] To be more precise, in the $A_4 = 0$ gauge,[25] the winding number is given by

$$q = n(+\infty) - n(-\infty) \qquad (2.6a)$$

where

$$n(x_4) = \frac{e}{2\pi} \int_{-\infty}^{+\infty} dx_1 A_1(x_4, x_1)\,. \qquad (2.6b)$$

By a suitable time independent gauge transformation, we may take $n(-\infty)$ to be an integer. The instanton thus describes a tunneling from a classical zero-energy configuration labeled by n to one labeled by $n+q$. As a result, the true ground state of the system is a linear superposition of gauge vacuums (denoted by $|n\rangle$) centered around the zero-energy configurations labeled by n. As was shown,[14] there exists a unitary operator

$$T = \exp\left(-\frac{\pi}{e}\left(E(x_4, x_1 = +\infty) + E(x_4, x_1 = -\infty)\right)\right) \quad (E \equiv \tfrac{1}{2}\epsilon_{\mu\nu}F_{\mu\nu})\,, \qquad (2.7)$$

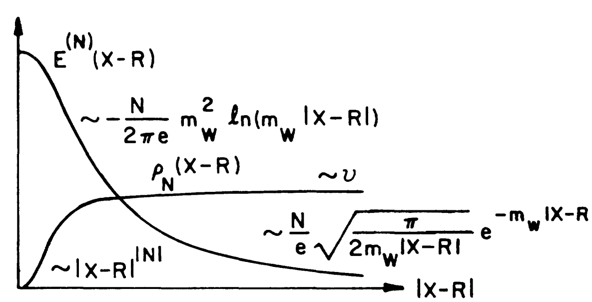

FIG. 1. Schematic plot of $\rho_N(|X-R|)$ and $E^{(N)}(|X-R|) \equiv \frac{1}{2}\epsilon_{\mu\nu}F_{\mu\nu}^{(N)}(X-R)$ for the case $e \ll \sqrt{\lambda}$.

which executes a global gauge transformation such that $T|n\rangle = |n+1\rangle$. By gauge invariance of the Hamiltonian H, we also have $[T,H]=0$. Thus, the particular linear superposition of n vacuums which simultaneously diagonalizes T and H is given by[12]

$$|\theta\rangle = \sum_{n=-\infty}^{+\infty} e^{-in\theta}|n\rangle , \tag{2.8}$$

with $T|\theta\rangle = e^{+i\theta}|\theta\rangle$. Note that the conventional perturbation-theory vacuum is obtained by expanding around any classical configuration labeled by n, neglecting the possibility of tunneling.

In the rest of this section, we explore in detail some properties of the theory in the θ vacuums via a semiclassical approximation. This is applicable if the tunneling amplitude is sufficiently small. This is given by $C_N e^{-S^{(N)}}/m_W^2$ [see (2.5)], where C_N is the result of Gaussian corrections about the classical configurations, and $1/m_W^2$ is roughly the volume in space-time over which tunneling takes place. We thus consider the range of parameters[22]

$$e \ll \sqrt{\lambda}, \quad v \gg 1 . \tag{2.9}$$

Unless C_N is extremely large this suffices to have $C_N e^{-S^{(N)}}/m_W^2 \ll 1$.

B. The gauge structure of the θ vacuums

In order to examine the physical consequences of tunneling, we compute the transition amplitude of the θ vacuums (from $x_4=-\infty$ to $x_4=+\infty$) in the presence of a conserved source J_μ coupled to A_μ. In terms of the Euclidean Feynman path integral, this is given by[12]

$$Z[J_\mu;\theta] = \int [dA_\mu][d\phi][d\phi^*] \times \exp\left(iq\theta - \int d^2x(\mathcal{L} + \mathcal{L}_{gf} + J_\mu A_\mu)\right) , \tag{2.10}$$

where q is the winding number defined by (2.3), and $\mathcal{L}_{gf} = (1/2\alpha)(\partial_\mu A_\mu)^2$ is the gauge fixing term. In the following, we work in the Landau gauge ($\alpha \to 0$).

As already remarked, we approximate the functional integral (2.10) using a semiclassical (saddle point) method. For this purpose, we need solutions of the classical equations of motion. For each value of $q=N$, the exact solution is given by (2.4). In addition, we have the following set of approximate solutions:

$$A_\mu^{(N_1\cdots N_k)}(x) = \sum_{i=1}^{k} A_\mu^{(N_i)}(x-R_i) , \tag{2.11a}$$

$$\phi^{(N_1\cdots N_k)}(x) = v \prod_{i=1}^{k} e^{i\delta_i} \frac{\phi^{(N_i)}(x-R_i)}{v} , \tag{2.11b}$$

where N_i are integers, and δ_i are real constants ($0 \le \delta_i < 2\pi$). This configuration has the following properties: (i) The winding number is $q=\sum_{i=1}^k N_i$. (ii) $|R_i - R_j| \sim \Delta \gg 1/m_W$ for any pair of i and j, it satisfies the equation of motion up to terms of order $e^{-m_W\Delta}$. The Euclidean action is given by

$$S[A_\mu^{(N_1\cdots N_k)}, \phi^{(N_1\cdots N_k)}] = \sum_{i=1}^{k} S_{N_i} + O(e^{-m_W\Delta}) . \tag{2.11c}$$

Thus, (2.11) represents an ensemble of k instantons interacting with each other through an exponentially damping potential.

The integrals over A_μ and ϕ are now carried out by computing the Gaussian fluctuations about the classical configurations $A_\mu^{(N_1\cdots N_k)}$, $\phi^{(N_1\cdots N_k)}$ for all $N_1,\ldots,N_k$. In doing this, we neglect the effects of interactions between the instantons. The positions of the instantons $R_1,\ldots,R_k$ are treated by the method of collective coordinates.[26,27] Up to an overall multiplicative constant, the result is given by

$$Z[J_\mu;\theta] \simeq \exp\left(-\sum_{N=-\infty}^{+\infty} W^{(N)}[J_\mu;\theta]\right) , \tag{2.12a}$$

where

$$\begin{aligned} W^{(N)}[J_\mu;\theta] &= -e^{iN\theta}\gamma_N \int d^2R \exp\left(-\int d^2x\, J_\mu(x)A_\mu^{(N)}(x-R) \right.\\ &\quad \left. +\tfrac{1}{2}\int d^2x\, d^2y\, J_\mu(x)[D_{\mu\nu}^{(N)}(x,y;R) - D_{\mu\nu}^{(0)}(x-y)]J_\nu(y)\right) , \quad \text{if } N \neq 0 , \\ &= -\tfrac{1}{2}\int d^2x\, d^2y\, J_\mu(x) D_{\mu\nu}^{(0)}(x-y) J_\nu(y) , \quad \text{if } N = 0 . \end{aligned} \tag{2.12b}$$

Here $D_{\mu\nu}^{(N)}$ is the Gaussian approximation to the gauge field propagator in the presence of an instanton with winding number $q=N$ and at position R ($N=0$ means no instanton), and $\gamma_N = C_N e^{-S_N}$ where C_N is a positive finite constant including finite Gaussian corrections and the normalization factors resulting from the extraction of the collective coordinates.

Recall that in deriving (2.12) the interaction between instantons has been neglected. This should be a good approximation if the average distance between instantons is much greater than $1/m_W$ so that their mutual overlap is exponentially small [see (2.11c)]. To check this, notice that the number density of instantons with winding number $q = N$ is given by

$$\frac{\gamma_N}{V}\frac{\partial}{\partial\gamma_N}\ln Z[J_\mu = 0, \theta] = 2\gamma_N\cos(N\theta)\,, \qquad (2.13)$$

with $V = \int d^2x$.[28] This is in fact quite small compared to $m_W{}^2$ due to our assumption (2.9). Our approximation may thus be termed a dilute-instanton-gas approximation. Notice also that γ_N decreases quite rapidly with increasing $|N|$ ($S_N \propto N^2$). From now on, we therefore keep only those terms having $N = 0, \pm 1$. (We have checked that retaining terms with $|N| \geq 2$ does not change the qualitative features of our results.)

Let us now discuss several consequences of (2.12).

(1) The energy density ϵ_θ of the θ vacuums normalized so that $\epsilon_\theta = 0$ for $\gamma_1 = 0$ is given by

$$\epsilon_\theta = -2\gamma_1\cos\theta\,. \qquad (2.14)$$

Clearly, the vacuum $|\theta = 0\rangle$ has the lowest energy among the θ vacuums. For this reason, we shall concentrate on the $\theta = 0$ vacuum in the rest of this section. The question of stability of the vacuum for general values of θ will be treated in Sec. IIC.

(2) We can examine the gauge structure of the $\theta = 0$ vacuum by computing the expectation value of $\phi(x)$. In the dilute-instanton-gas approximation, we easily find

$$\langle\theta = 0|\phi(x)|\theta = 0\rangle = v\exp\left[-\gamma_1\int d^2R\left(2 - \frac{\phi^{(1)}(x-R) + \phi^{(-1)}(x-R)}{v}\right)\right] = 0\,, \qquad (2.15)$$

since the integral of $\phi^{(\pm 1)}(x-R)$ over the angle of the vector R vanishes. Thus, the $\theta = 0$ vacuum is invariant under the global gauge transformations.[29] Intuitively, what has happened is this: In any one of the n vacuums, the phase of the scalar field $\phi(x)$ has an infinite correlation length. The instantons have destroyed this correlation by the rotating phase of $\phi^{(\pm 1)}$.

Since the vacuum is globally gauge invariant, we expect a massless pole in the gauge field propagator. In fact, with the help of (2.12), we find

$$\int d^2x\, e^{-ipx}\langle\theta = 0|T(A_\mu(x)A_\nu(0))|\theta = 0\rangle$$
$$= \left(\delta_{\mu\nu} - \frac{p_\mu p_\nu}{p^2}\right)\left[\frac{1}{p^2 + m_W{}^2} + \frac{8\pi^2}{e^2}\gamma_1\frac{1}{p^2}\left(\frac{m_W{}^2}{p^2 + m_W{}^2}\right)^2\right], \qquad (2.16)$$

where we have used (2.4) and have neglected the short-range piece $D^{(\pm 1)}_{\mu\nu} - D^{(0)}_{\mu\nu}$.

(3) The presence of a massless pole in the gauge field propagator indicates that the energy $E(R)$ stored in a pair of charges g and $-g$ placed at a distance R has a Coulomb term which grows linearly with R. This can be checked by the Wilson formula[30]

$$\frac{\langle\theta = 0;\text{ out}|T\exp[-ig\oint_C dx_\mu A_\mu(x)]|\theta = 0;\text{ in}\rangle}{\langle\theta = 0;\text{ out}|\theta = 0;\text{ in}\rangle} \underset{\tau\to+\infty}{\sim} e^{-E(R)\tau}\,, \qquad (2.17)$$

where the contour integral is taken along the path C depected in Fig. 2. The left-hand side of (2.17) is easily evaluated by substituting

$$J_\mu(x) = ig\epsilon_{\mu\nu}\partial_\nu\theta_s(x) \qquad (2.18)$$

into (2.12) where S denotes the area of space-time enclosed in the contour C, and $\theta_s(x) = 1$ if $x \in S$ and $= 0$ if $x \notin S$. Neglecting terms independent of g and/or R, we find

$$E(R) = -\frac{g^2}{2m_W}e^{-m_W R} + 2\gamma_1\left[1 - \cos\left(2\pi\frac{g}{e}\right)\right]R + O(\gamma_1 R e^{-m_W R})\,, \qquad (2.19)$$

where the last term represents the correction to the second due to the finite instanton size.

This result has two important features:

(i) The energy $E(R)$ contains two terms, the first vanishing exponentially with a characteristic length

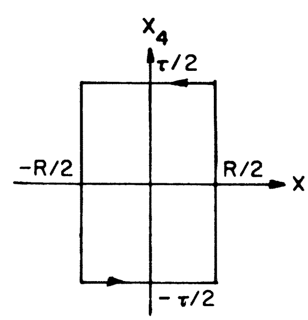

FIG. 2. The contour C for the Wilson formula (2.17).

scale $1/m_W$ and the second rising linearly with R and becoming appreciable for $R \gg \gamma_1^{-1/2}$. This may be interpreted as follows: Recall that the average distance between instantons is roughly given by $\gamma_1^{-1/2}$. Thus, on a length scale shorter than $\gamma_1^{-1/2}$, the gas of instantons is essentially nonexistent and the gauge symmetry remains spontaneously broken. Consequently, a pair of external charges feels a potential characteristic of a massive gauge field. On the other hand, on a length scale much greater than $\gamma_1^{-1/2}$, there are many instantons between two charges. These instantons restore the global gauge symmetry as we have seen in (2.15). As a result, the potential rises linearly with increasing separation.

(ii) The coefficient of the linearly rising piece is periodic in g/e vanishing for $g/e \in Z$. This is indicative of charge screening. In order to verify this conjecture, we compute the *total* charge in the presence of one external charge of magnitude g. Denoting such a state as $|g; \theta=0\rangle$, we obtain using (2.12) that

$$\begin{aligned}\langle g;\theta=0|Q_{\text{tot}}|g;\theta=0\rangle &= -i\left\langle g;\theta=0\left|\int_{-\infty}^{+\infty} dx_1 \partial_\mu F_{\mu 4}(x)\right|g;\theta=0\right\rangle \\ &= \frac{8\pi}{e}\gamma_1 \sin\left(\pi\frac{g}{e}\right). \qquad (2.20)\end{aligned}$$

Thus, the external charge is completely screened only if $g/e \in Z$. This is a quantum effect due to the creation of pairs of scalar particles (with charge $\pm e$) which are present under the guise of instantons.

C. Stability of the θ vacuums

Let us now go back to the general θ vacuum. A peculiar property of the θ vacuums $[\theta \neq 0 \ (\text{mod}\pi)]$ is that they have a nonvanishing background electric field. In fact, using (2.12), we find

$$\langle\theta|E(x)|\theta\rangle = i\frac{4\pi}{e}\gamma_1 \sin\theta \quad (E=\tfrac{1}{2}\epsilon_{\mu\nu}F_{\mu\nu}). \qquad (2.21)$$

This implies that the parity and time reversal invariances are spontaneously broken for $\theta \neq 0$ (mod π).[8,12] For this effect to be physically meaningful, however, the θ vacuums must be stable against gauge-invariant perturbations, in particular, charged pair production.

In order to examine the latter possibility, we compute (2.17) for an arbitrary value of θ and find

$$E_\theta(R) = -\frac{g^2}{2m_W}e^{-m_W R} + 2\gamma_1\left[\cos\theta - \cos\left(\theta - 2\pi\frac{g}{e}\right)\right]R. \qquad (2.22)$$

(If one reverses the direction of the contour C, $\theta - 2\pi g/e$ is replaced by $\theta + 2\pi g/e$.) We see that unless g/e and θ satisfy

$$\cos\theta \geq \cos\left(\theta - 2\pi\frac{g}{e}\right)$$

and (2.23)

$$\cos\theta \geq \cos\left(\theta + 2\pi\frac{g}{e}\right),$$

it is energetically favorable for the charged pair to recede from each other to spatial infinity, effectively changing the value of θ into $\theta \mp 2\pi g/e$. In such cases, the introduction of an additional field with charge g will induce the decay of the θ vacuums via charged pair production.[15]

Of course there is also the possibility that the creation of more than one charged pair decreases the energy density even though one charged pair may not. This means that the θ vacuums are unstable unless θ satisfies

$$\cos\theta \geq \cos\left(\theta - 2\pi n\frac{g}{e}\right)$$

and (2.24)

$$\cos\theta \geq \cos\left(\theta + 2\pi n\frac{g}{e}\right)$$

for any positive integer n. As an illustration, we plot in Fig. 3 the values of g/e and θ fulfilling (2.24) for $n=1$ and 2. It is easily seen that except for the lines $g/e \in Z$ the overlapping region of (2.24) for $n=1,2,\ldots,n_0$ becomes smaller as n_0 increases, and shrinks to the line $\theta=0$ (mod2π) in the limit $n_0 \to \infty$. We thus find that only the $\theta=0$ (mod2π) vacuum is stable. This does not mean, however, that the θ vacuums ($\theta \neq 0$) necessarily decay into an "effective" $\theta=0$ vacuum, since pair production can change θ only by an integer multiple of $2\pi g/e$. In order to determine the parity and time-reversal properties of the true ground state for $\theta \neq 0$, we have to introduce the additional field explicitly. This we shall do in Sec. III.

III. ADDING MASSIVE FERMIONS

In the preceding section, we have shown that the global gauge symmetry is restored in the θ vacuums. As a result, there exists a linearly rising potential between a pair of charges g and $-g$ which, however, vanishes for $g/e \in Z$. In order to study the spectrum which results from such a potential, we introduce a massive Fermi field ψ with charge g. The Lagrangian is given by

$$\mathcal{L}[A_\mu,\phi,\psi] = \mathcal{L}[A_\mu,\phi] + \bar\psi(\gamma_\mu\partial_\mu + m)\psi + ig\bar\psi\gamma_\mu\psi A_\mu. \qquad (3.1)$$

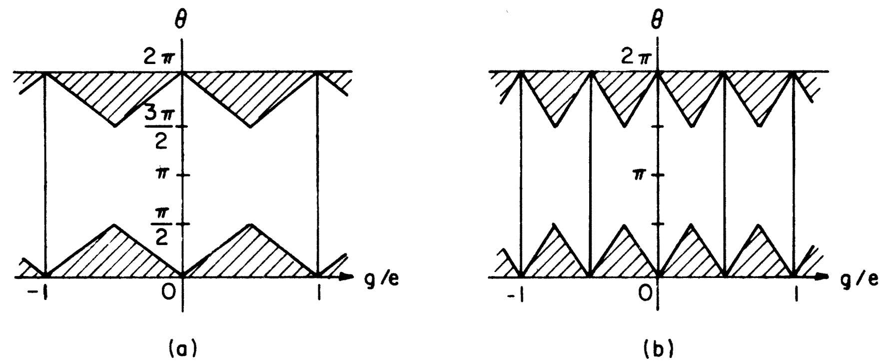

FIG. 3. The shaded region and the straight lines at $g/e = m/n$ ($m \in Z$) satisfy (2.24) for (a) $n=1$ and (b) $n=2$. The periodic extension $\theta \rightarrow \theta + 2\pi l$ ($l \in Z$) is understood.

A. Effective Lagrangian for the fermion

The spectrum in the fermion sector is most easily discussed in terms of an effective Lagrangian $\mathcal{L}_{\text{eff}}[\psi]$ which is obtained from the transition amplitude $\langle\theta;\text{out}|\theta;\text{in}\rangle$ by integrating out A_μ and ϕ within the dilute-instanton-gas approximation introduced in Sec. II.

In order to handle the θ vacuums correctly in the presence of fermions, we make a slight digression. To keep track of boundary conditions, we work in the $A_4=0$ gauge. The equations of motion for $E(x)=\frac{1}{2}\epsilon_{\mu\nu}F_{\mu\nu}(x)$ are given by

$$\partial_\mu(\epsilon_{\mu\nu}E) = igJ_\nu + iej_\nu, \tag{3.2}$$

with $J_\nu = \bar{\psi}\gamma_\nu\psi$ and $j_\nu = \phi^*\partial_\nu\phi - \partial_\nu\phi^*\phi - 2ieA_\nu\phi^*\phi$. Since J_ν and j_ν are separately conserved, one can write

$$J_\nu = -\frac{1}{\sqrt{\pi}}\epsilon_{\nu\mu}\partial_\mu\chi, \tag{3.3}$$

$$j_\nu = -\frac{1}{\sqrt{\pi}}\epsilon_{\nu\mu}\partial_\mu\zeta. \tag{3.4}$$

Combing (3.2)–(3.4), we obtain

$$E = i\frac{g}{\sqrt{\pi}}\chi + i\frac{e}{\sqrt{\pi}}\zeta, \tag{3.5}$$

where the integration constant has been absorbed in χ. In terms of χ and ζ, the unitary operator T defined by (2.7) is expressed as

$$T = \exp\left(-i2\sqrt{\pi}\frac{g}{e}\xi(x_4)\right)\exp[-i2\sqrt{\pi}\,\eta(x_4)] \equiv T_\xi T_\eta, \tag{3.6}$$

where

$$\xi(x_4) = \tfrac{1}{2}[\chi(x_4, x_1=+\infty) + \chi(x_4, x_1=-\infty)], \tag{3.7}$$

$$\eta(x_4) = \tfrac{1}{2}[\zeta(x_4, x_1=+\infty) + \zeta(x_4, x_1=-\infty)]. \tag{3.8}$$

The n vacuums are thus a tensor product of Fermi and scalar sectors, namely, $|n\rangle = T_\xi^n|n;\eta\rangle$, where $|n;\eta\rangle = T_\eta^n|0\rangle$ is the n vacuum in the absence of fermions. Since the operator T commutes with the Hamiltonian and $\eta(\pm\infty)$ is equal to zero for any instanton solution, the boundary condition for $\xi(x_4)$ is given by $\xi(+\infty)=\xi(-\infty)\,(\equiv\xi)$. Dropping an infinite multiplicative constant, we then find

$$\langle\theta;\text{out}|\theta;\text{in}\rangle = \int[d\psi][d\bar{\psi}]\exp\left(-\int d^2x\,\bar{\psi}(\gamma_\mu\partial_\mu + m)\psi\right) \times Z\left[igJ_\mu;\,\theta + 2\sqrt{\pi}\frac{g}{e}\xi\right], \tag{3.9}$$

where Z in the integrand is defined by (2.10), and we have returned to the Landau gauge. Replacing Z by its dilute-instanton-gas approximation (2.12) and ignoring the short-range terms $D_{\mu\nu}^{(0)}$ and $D_{\mu\nu}^{(\pm)} - D_{\mu\nu}^{(0)}$ (as well as terms with $|N|\geq 2$), we obtain

$$\langle\theta;\text{out}|\theta;\text{in}\rangle \simeq \int[d\psi][d\bar{\psi}]\exp\left(-\int d^2x\,\mathcal{L}_{\text{eff}}[\psi]\right), \tag{3.10a}$$

where

$$\mathcal{L}_{\text{eff}}[\psi] = \bar{\psi}(x)(\gamma_\mu\partial_\mu + m)\psi(x) - \gamma\cos\left(g\int d^2y\,J_\mu(y)A_\mu^{(1)}(y-x) - \theta - 2\sqrt{\pi}\frac{g}{e}\xi\right), \tag{3.10b}$$

with $\gamma \equiv 2\gamma_1$.

In order to understand the significance of the cosine term of (3.10b), let us expand it in powers of g. Setting $\theta=\xi=0$ for simplicity, we find

$$\gamma g^2\tfrac{1}{2}\int d^2x\,d^2y\,J_\mu(x)\Delta_{\mu\nu}(x-y)J_\nu(y) + O(g^4), \tag{3.11}$$

with

$$\Delta_{\mu\nu}(x-y)=\int d^2zA_\mu^{(1)}(x-z)A_\nu^{(1)}(y-z)\underset{|x-y|\to+\infty}{\sim}\frac{4\pi^2}{e^2}\int\frac{d^2p}{(2\pi)^2}e^{ip(x-y)}\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\frac{1}{p^2}. \tag{3.12}$$

Thus, as is expected from the results of Sec. II, there is explicitly a linearly rising potential induced by instantons which couples to the fermion current. Of course we should not neglect terms higher order in g since we would then miss the charge screening effect which takes place for $g/e\in Z$.

The nonlocal Lagrangian (3.10b) can be simplified considerably if one uses an equivalent boson representation of the Fermi field[31]: The generating functional for a free Fermi field of mass m

$$Z_F[\mathcal{V}_\mu,\mathcal{A}_\mu]=\int[d\psi][d\bar\psi]\exp\left(-\int d^2x[\bar\psi(\gamma_\mu\partial_\mu+m)\psi+\bar\psi\gamma_\mu\psi\mathcal{V}_\mu+\bar\psi\gamma_\mu\gamma_5\psi\mathcal{A}_\mu]\right) \tag{3.13}$$

is equal, up to a constant, to that of a sine-Gordon field χ given by

$$Z_{sG}[\mathcal{V}_\mu,\mathcal{A}_\mu]=\int[d\chi]\exp\left[-\int d^2x\left(\tfrac{1}{2}\partial_\mu\chi\partial_\mu\chi-M^2\cos(2\sqrt{\pi}\chi)-\frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial_\nu\chi\mathcal{V}_\mu+\frac{1}{\sqrt{\pi}}\partial_\mu\chi\mathcal{A}_\mu\right)\right]. \tag{3.14}$$

In particular, we have the correspondence

$$J_\mu=\bar\psi\gamma_\mu\psi\longleftrightarrow-\frac{1}{\sqrt{\pi}}\epsilon_{\mu\nu}\partial_\nu\chi, \tag{3.15}$$

$$J_\mu{}^5=\bar\psi\gamma_\mu\gamma_5\psi\longleftrightarrow\frac{1}{\sqrt{\pi}}\partial_\mu\chi. \tag{3.16}$$

In the boson representation, a single fermion appears as a soliton having $\chi(x_1=+\infty)-\chi(x_1=-\infty)=\sqrt{\pi}$ and classical mass $m=4M/\sqrt{\pi}$. More generally, the fermion content of a state in the boson representation can be determined from its fermion number N_f and (fermionic) charge Q_f given by

$$N_f=\int_{-\infty}^{+\infty}dx_1J_4(x)=\frac{1}{\sqrt{\pi}}[\chi(x_1=+\infty)-\chi(x+=-\infty)], \tag{3.17}$$

$$Q_f=gN_f. \tag{3.18}$$

Expanding the right-hand side of (3.10a) in powers of γ, replacing each term by the equivalent representation through χ, and then resumming them, we obtain an equivalent boson representation of $\mathcal{L}_{eff}$ (Ref. 32):

$$\mathcal{L}_{eff}[\chi]=\tfrac{1}{2}\partial_\mu\chi(x)\partial_\mu\chi(x)-M^2\cos[2\sqrt{\pi}\chi(x)]-\gamma\cos\left(2\sqrt{\pi}\frac{g}{e}\int d^2y\,\chi(y)\rho^{(1)}(y-x)+\theta\right), \tag{3.19}$$

with

$$\rho^{(1)}(x)=\frac{e}{2\pi}\epsilon_{\mu\nu}\partial_\mu A_\nu^{(1)}(x)\simeq\int\frac{d^2p}{(2\pi)^2}e^{ipx}\frac{m_W{}^2}{p^2+m_W{}^2} \tag{3.20}$$

[see (2.4e)].

B. Spectrum in the Fermi sector

We now study the spectrum of states containing fermions in the $\theta=0$ vacuum using semiclassical techniques. Recall that the linearly rising potential becomes appreciable on a length scale $\sim\gamma^{-1/2}$ which is much greater than $1/m_W$. Thus, one may take the limit $m_W\to\infty$ in (3.19),[33] which gives

$$\mathcal{L}_{eff}[\chi]=\tfrac{1}{2}\partial_\mu\chi\partial_\mu\chi-M^2\cos(2\sqrt{\pi}\chi)-\gamma\cos\left(2\sqrt{\pi}\frac{g}{e}\chi\right). \tag{3.21}$$

The classical potential for χ is thus given by

$$V[\chi]=M^2[1-\cos(2\sqrt{\pi}\chi)]+\gamma\left[1-\cos\left(2\sqrt{\pi}\frac{g}{e}\chi\right)\right], \tag{3.22}$$

where we have explicitly subtracted the classical vacuum energy density.

Let us first study static solutions to the classical equation of motion. Such a solution $\chi_c(x_1)$ is obtained by solving

$$\frac{d^2\chi_c(x_1)}{dx_1{}^2}-\frac{\delta V[\chi_c]}{\delta\chi_c(x_1)}=0, \tag{3.23}$$

with the requirement that the classical energy

$$E_c=\int_{-\infty}^{+\infty}dx_1\left[\frac{1}{2}\left(\frac{d\chi_c(x_1)}{dx_1}\right)^2+V[\chi_c]\right] \tag{3.24}$$

is finite.

As an example, we consider the special case $g/e=\frac{1}{2}$. The potential $V[\chi]$ is plotted in Fig. 4(a). The classical solutions are easily found to be

$$\chi_c^\pm(x_1)=\pm\frac{2}{\sqrt{\pi}}\tan^{-1}\left[-\left(\frac{4M^2+\gamma}{\gamma}\right)^{1/2}\times\frac{1}{\sinh[\sqrt{\pi}(4M^2+\gamma)^{1/2}x_1]}\right] \tag{3.25}$$

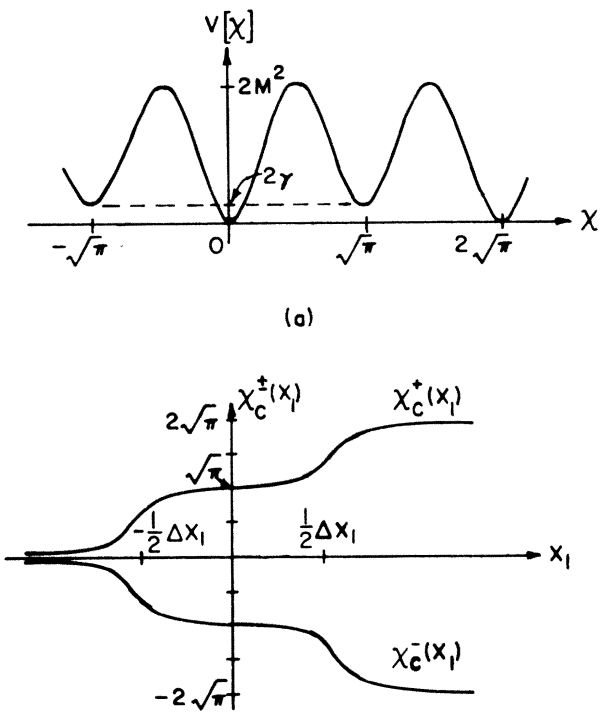

FIG. 4. (a) The shape of the potential $V[\chi]$ for $g/e=\frac{1}{2}$. (b) Two static solutions to the classical equation of motion for $g/e=\frac{1}{2}$.

The corresponding classical energy is given by

$$E_c=\frac{4}{\sqrt{\pi}}\left[(4M^2+\gamma)^{1/2}+\frac{\gamma}{2M}\ln\left(\frac{2M+(4M^2+\gamma)^{1/2}}{\sqrt{\gamma}}\right)\right]. \tag{3.26}$$

As is seen from Fig. 4(b), $\chi_c^+(\chi_c^-)$ has the structure of two bound fermions (or antifermions)separated by a distance

$$\Delta x_1=\frac{2}{\sqrt{\pi}}\frac{1}{\sqrt{4M^2+\gamma}}\ln\left[\left(\frac{4M^2+\gamma}{\gamma}\right)^{1/2}+\left(\frac{4M^2+2\gamma}{\gamma}\right)^{1/2}\right]. \tag{3.27}$$

If the fermion is very heavy ($m^2=16M^2/\pi\gg\gamma$), we have

$$\Delta x_1\sim\frac{4}{\pi m}\ln m\left(\frac{\pi}{\gamma}\right)^{1/2}$$

and $E_c\sim 2m+2\gamma\Delta x_1$. The latter shows that for $g/e=\frac{1}{2}$ there is an attractive linearly rising potential between a pair of static charges with the same sign.

For general values of g/e, the finiteness of the classical energy requires that χ_c satisfies $V[\chi_c(\pm\infty)]=0$. For nontrivial static solutions χ_c, this is possible if and only if g/e is equal to a rational number p/q ($p,q\in Z$). Assuming $\chi_c(-\infty)=0$ without loss of generality, one then has $\chi_c(+\infty)=\pm q\sqrt{\pi}$, and consequently, $N_f=\pm q$ and $Q_f=\pm pe$. This represents a bound state of $|q|$ fermions (or $|q|$ antifermions). There are several features worth mentioning. (i) The total fermionic charge is an integer multiple of e. This conforms with the result of Sec. II that only such states can have their charge completely screened [see (2.20)], and hence have finite energy. (ii) A single fermion state remains in the spectrum only if $q=1$, i.e., $g/e\in Z$. Its mass, however, is shifted by an amount proportional to γ. (iii) The linearly rising potential is attractive for a pair of charges of the *same* sign as well as that of the opposite sign.

So far, we have discussed the classical approximation to the lowest-lying states with fermion number $N_f=\pm q$. Clearly, there will also be excited states having l fermions and $\bar{l}$ antifermions such that $l-\bar{l}=\pm q$. In addition, for any value of g, there will be states having an equal number of fermions and antifermions. Classically, all these states will appear as periodic solutions to the equation of motion.

It would be interesting to quantize these solutions by the method of Dashen *et al.*[34] Since the binding of fermions occurs on a length scale $\sim\gamma^{-1/2}$, we would need solutions which represent large oscillations of χ. So far we have been unsuccessful in finding such solutions.

Another possible quantization scheme is to write χ as a linear superposition of basic solitons and antisolitons and construct an effective Hamiltonian for their positions and conjugate momenta.[26] This has offered us little progress, however, except that the periodicity in g of the linearly rising potential is correctly reproduced.

C. Stability of the θ vacuums–massive fermions

The classical potential for χ in an arbitrary θ vacuum is given by

$$V_\theta[\chi]=M^2[1-\cos(2\sqrt{\pi}\,\chi)]+\gamma\left[1-\cos\left(2\sqrt{\pi}\frac{g}{e}\chi+\theta\right)\right] \tag{3.28}$$

Barring large quantum fluctuations, the ground state of the system is determined by the absolute minimum of $V_\theta[\chi]$. Let us denote the corresponding value of χ as $\chi_{\min}$. If g/e is a rational number p/q ($p,q\in Z$), there are a countably infinite set of such minima connected to each other by shifting $\chi_{\min}$ by $nq\sqrt{\pi}$ ($n\in Z$). Of course they are all equivalent. For g/e not a rational number, there is only one absolute minimum.

Consider $\theta=0$ (mod2π), we have $\chi_{\min}=0$. Hence, the unperturbed Fermi vacuum ($\chi=0$) is stable.

In addition, for the set of points $\theta=\pi(\text{mod}2\pi)$ and $g/e=n$ ($n\in Z$), we also have $\chi_{\min}=0$ (if the relation $M^2>\gamma g^2/e^2$ is satisfied). [These points are what remain of the stability lines $g/e=n$ ($n\in Z$) in Fig. 3.] For all other values of θ, we have $\chi_{\min}\neq 0$ and thus in these cases the unperturbed Fermi vacuums are unstable to the production of charged fermion pairs as discussed in Sec. II.

Let us now discuss the parity and time-reversal properties of the true vacuums. Two quantities which directly measure spontaneous violations of these symmetries are $\langle\theta|E|\theta\rangle$ ($E=\frac{1}{2}\epsilon_{\mu\nu}F_{\mu\nu}$), the background electric field, and $\langle\theta|m\bar\psi\gamma_5\psi|\theta\rangle$. In the classical approximation, these are given by

$$\langle\theta|E|\theta\rangle=i\frac{2\pi}{e}\gamma\sin\left(2\sqrt{\pi}\frac{g}{e}\chi_{\min}+\theta\right), \tag{3.29}$$

$$\langle\theta|m\bar\psi\gamma_5\psi|\theta\rangle=M^2\sin(2\sqrt{\pi}\chi_{\min}). \tag{3.30}$$

In general the θ vacuums are parity and time-reversal noninvariant. However, for those values of θ such that both expectation values vanish, namely,

$$2\sqrt{\pi}\frac{g}{e}\chi_{\min}+\theta=2m\pi, \tag{3.31}$$

$$2\sqrt{\pi}\chi_{\min}=2n\pi,\quad m,n\in Z \tag{3.32}$$

the ground state is both parity and time-reversal invariant.

Eliminating $\chi_{\min}$ from (3.31) and (3.32), we have

$$\theta=2m\pi-2n\pi g/e. \tag{3.33}$$

Intuitively, this may be understood as follows: The angle θ introduces into the theory an effective interaction $q\theta=(e/2\pi)\theta\int E(x)d^2x$ [see (2.10)]. This means that there are effective external charges $\pm e(\theta/2\pi)$ at spatial infinity $x_1=\pm\infty$. Such charges can be completely screened via the pair production of both fermions and scalars only if (3.33) is satisfied.

IV. MASSLESS FERMIONS

In the present section, we consider the limiting case in which the bare mass of the fermion vanishes. Two questions that immediately arise are the following: (1) How does the massless fermion modify the gauge structure of the theory studied in Sec. II? (2) What is the chiral structure of the theory?

The latter may need a bit more of an explanation. Recall that although the original Lagrangian density is formally invariant under the global chiral transformation $\psi(x)\to e^{\alpha\gamma_5}\psi(x)$, the gauge invariant axial-vector current J^5_μ is not conserved because of an anomaly:

$$\partial_\mu J^5_\mu=-i\frac{g}{2\pi}\epsilon_{\mu\nu}F_{\mu\nu}. \tag{4.1}$$

Nevertheless, since the right-hand side is a total divergence, one can define a conserved axial-vector current by

$$\hat J^5_\mu=J^5_\mu+i\frac{g}{\pi}\epsilon_{\mu\nu}A_\nu. \tag{4.2}$$

Although $\hat J^5_\mu$ itself is not gauge invariant, the corresponding charge

$$\hat Q_5=\int_{-\infty}^{+\infty}dx_1\hat J^5_4(x) \tag{4.3}$$

is invariant under local gauge transformations, i.e., those which have the same value at $x_1=\pm\infty$. Furthermore, $\hat Q_5$ generates the global chiral transformations and thus commutes with the Hamiltonian. The problem then is whether the chiral symmetry is spontaneously broken.

In order to study these problems we again use the dilute-instanton-gas approximation and the boson representation of fermions introduced in Secs. II and III. By taking the limit $m\to 0$ in (3.19), we obtain[35]

$$\mathcal{L}_{\text{eff}}[\chi]=\tfrac{1}{2}\partial_\mu\chi(x)\partial_\mu\chi(x)-\gamma\cos\left(2\sqrt{\pi}\frac{g}{e}\int d^2y\chi(y)\rho^{(1)}(y-x)+\theta\right). \tag{4.4}$$

The corresponding generating functional

$$Z=\int[d\chi]\exp\left(-\int d^2x\mathcal{L}_{\text{eff}}[\chi]\right) \tag{4.5}$$

is independent of θ since Z is invariant under the transformation $\chi\to\chi+\text{const.}$[36] For this reason, we set $\theta=0$ in the following.

A. Statistical-mechanical analogies

We shall analyze the system described by (4.5) using two statistical-mechanical analogies. One is the two-dimensional Coulomb gas analogy discovered by Callan *et al.*,[16] and the other is an analogy to the two-dimensional classical X-Y model.

1. Two-dimensional Coulomb gas analogy

Let us rewrite (4.5) as

$$Z=\lim_{\mu\to0}\int[d\chi]\exp\left(-\int d^2x\{\mathcal{L}_{\text{eff}}[\chi]+\tfrac{1}{2}\mu^2\chi(x)^2\}\right), \tag{4.6}$$

where μ is an infrared cutoff. Expanding the exponent in powers of γ and carrying out the χ integration term by term, we find up to an overall multiplicative constant that

$$Z=\lim_{\mu\to 0}\sum_{N_+,N_-=0}^{\infty}\frac{1}{N_+!N_-!}\left(\frac{\gamma}{2}\right)^{N_++N_-}\int\prod_{i=1}^{N_++N_-}d^2R_i\exp\left[-\frac{2\pi g^2}{e^2}\sum_{i,j=1}^{N_++N_-}q_iq_jV(R_i-R_j;\mu)\right], \tag{4.7}$$

where $q_i=1$ for $1\le i\le N_+$ and $q_i=-1$ for $N_++1\le i\le N_++N_-$, and[37]

$$V(R;\mu)\equiv\int\frac{d^2p}{(2\pi)^2}e^{ipR}\frac{1}{p^2+\mu^2}\left(\frac{m_W{}^2}{p^2+m_W{}^2}\right)^2. \tag{4.8}$$

[We have used (3.20).] Defining

$$V(R)\equiv\lim_{\mu\to 0}[V(R;\mu)-V(0;\mu)]$$

$$=\int\frac{d^2p}{(2\pi)^2}(e^{ipR}-1)\frac{1}{p^2}\left(\frac{m_W{}^2}{p^2+m_W{}^2}\right)^2, \tag{4.9}$$

we may rewrite the sum in the exponent of (4.8) as

$$\sum_{i\neq j}q_iq_jV(R_i-R_j)+(N_+-N_-)^2V(0;\mu). \tag{4.10}$$

Since $V(0;\mu)\sim-(1/2\pi)\ln(\mu/m_W)$ as $\mu\to 0$, only those terms of (4.7) satisfying $N_+=N_-$ remain in this limit. We thus obtain

$$Z=\sum_{N=0}^{\infty}\frac{1}{(N!)^2}\left(\frac{\gamma}{2}\right)^{2N}\int\prod_{i=1}^{2N}d^2R_i\exp[-\beta' H_{2N}], \tag{4.11}$$

where $\beta'\equiv 4\pi g^2/e^2$, and

$$H_{2N}=\frac{1}{2}\sum_{i,j=1}^{2N}q_iq_jV(R_i-R_j), \tag{4.12}$$

with q_i being defined as after (4.7) by setting $N_+=N_-=N$.

Notice that $V(R)$ behaves as

$$V(R)\to 0 \quad\text{as } R\to 0,$$

$$\sim-\frac{1}{2\pi}\ln(m_W|R|)\quad\text{as } |R|\to+\infty. \tag{4.13}$$

Hence (4.12) is identical to the grand canonical partition function for an (overall neutral) gas of particles with charges ± 1 interacting through the two-dimensional Coulomb potential with a soft core. The temperature is $T'=1/\beta'$ and $\gamma/2$ represents the fugacity.

As is easily seen, these charged particles can be identified with the instantons at positions R_i and with winding number q_i. Thus, the massless fermion has produced a logarithmic interaction between instantons. This long-range interaction has the initial effect of restricting the total winding number N_+-N_- equal to zero. Namely, the massless fermion prohibits tunneling in the global sense although it is still allowed locally.[38]

Returning to the Coulomb gas analogy, let us recall[39] that there is a phase transition at $\beta'=\beta_c'\simeq 8\pi$: Suppose the gas is confined within a sphere of radius L. Then the free energy f per particle is given by $f\sim(1/4\pi)\ln L-T'\ln L^2$ for $L\gg 1$. Hence, for $\beta'>\beta_c'\simeq 8\pi$, two oppositely signed charges are bound together to form a neutral dipole, whereas for $\beta'<\beta_c'$, these dipoles dissociate and the charges form a uniform plasma.

This picture has been confirmed by a detailed renormalization group calculation carried out by Kosterlitz.[18] Here we summarize his results relevant to our analysis: In terms of the effective variables $x=[\beta'(\tau)-8\pi]/4\pi$ and $y=2\pi\bar{\gamma}(\tau)\tau^2$ with τ being the scaling variable [in our case $\tau\ge 1/m_W$, $\bar{\beta}'(1/m_W)=\beta'$, $\bar{\gamma}(1/m_W)=\gamma$], his renormalization group equations read

$$dx=-\tfrac{1}{4}(x+2)^2y^2\frac{d\tau}{\tau}, \tag{4.14a}$$

$$dy=-xy\frac{d\tau}{\tau}, \tag{4.14b}$$

$$dF=\frac{1}{8\pi}y^2\frac{d\tau}{\tau^3}, \tag{4.14c}$$

where F is the free energy density defined by $F=-V^{-1}\ln Z$ (V is the volume of two-dimensional space occupied by the system). As long as γ is sufficiently small (dilute gas), these equations are expected to be valid for the entire range of β'. Linearizing (4.14) around the critical point, Kosterlitz obtained the following results: (1) The critical inverse temperature is given by $\beta_c'=8\pi+8\pi\gamma/m_W{}^2$. (2) The correlation length ξ is infinite for $\beta'>\beta_c'$ and grows as $\exp[b(\beta_c'-\beta')^{-1/2}]$ for $\beta'\to\beta_c'-0$ where b is a positive constant. (3) The free energy density has the behavior $F\sim\exp[-b'(\beta'-\beta_c')^{-1/2}]$ for $\beta'\to\beta_c'+0$ (b' is another positive constant) and $F\sim\exp[2b(\beta_c'-\beta')^{-1/2}]$ for $\beta'\to\beta_c'-0$.

Away from the critical temperature, (4.14) shows that the renormalization of temperature is small. Hence x may be regarded as constant. Integrating (4.14b) and (4.14c), we then obtain the following small-γ behavior of ξ and F:

$$\xi=+\infty\quad\text{for }\beta'>\beta_c',$$

$$\sim\gamma^{-\kappa(\beta')/2}\quad\text{for }\beta'<\beta_c', \tag{4.15}$$

$$F=\text{analytic in }\gamma\quad\text{for }\beta'>\beta_c',$$

$$\sim\gamma^{+\kappa(\beta')}\quad\text{for }\beta'<\beta_c', \tag{4.16}$$

with $\kappa(\beta')=8\pi/(8\pi-\beta')$.

These results show that the free energy density is analytic in γ and β' except at lines given by

$\gamma=0,\ 0\leqslant\beta'\leqslant 8\pi$ and $\beta'=\beta_c'(=8\pi+8\pi\gamma/m_W^2)$. In particular, the series expansion in powers of γ given by (4.11) will be valid only for $\beta'>\beta_c'$. This can easily be checked by examining the convergence property of the integral which appears in the γ^{2N} term of (4.11). Since the only source of possible divergence comes from the region in which the R_i's are large, we may replace $V(R)$ by its large $|R|$ asymptotic form (4.3). One then finds that the integral is completely convergent for arbitrary N if and only if $\beta'>8\pi$. Thus, barring a possible divergence of the series itself, the partition function is analytic at $\gamma=0$ for $\beta'>8\pi$. Of course, the convergence radius of the series is further restricted by the phase transition line $\beta'=\beta_c'$.

Although the Coulomb gas analogy provides a nice intuitive picture of the dynamics of instantons, it is not very suited to compute Green's functions of physical interest. This is because all fields have been integrated out. This leads us to study the second statistical-mechanical analogy, that of the classical X-Y model.

2. *Two-dimensional classical X-Y model analogy*

Define a new field $\varphi(x)$ by

$$\varphi(x)\equiv 2\sqrt{\pi}\,\frac{g}{e}\int d^2y\ \chi(y)\rho^{(1)}(y-x). \tag{4.17}$$

Using (3.20), one can express $\chi(y)$ as

$$\chi(y)=\frac{e}{2\sqrt{\pi}\,g}\left(1-\frac{\Box}{m_W^2}\right)\varphi(y)\quad(\Box=\partial_\mu\partial_\mu). \tag{4.18}$$

In terms of φ, the generating functional Z of (4.5) is given (again up to a multiplicative constant) by

$$Z=\int[d\varphi]\exp[-\beta H], \tag{4.19}$$

where $\beta=e^2/4\pi g^2$ and

$$H=\int d^2x\left\{\tfrac{1}{2}\varphi(x)\left[-\Box\left(1-\frac{\Box}{m_W^2}\right)^2\right]\varphi(x)-\frac{\gamma}{\beta}\cos\varphi(x)\right\}. \tag{4.20}$$

We recognize H as the continuum limit of the two-dimensional classical X-Y model Hamiltonian. Namely, γ/β is the magnitude of a uniform external magnetic field and $\varphi(x)$ is the angle of a classical spin of unit length located at x measured from the direction of the external field. There is an effective large-momentum cutoff at $\sim m_W$. Z is the partition function of this spin system at temperature $T=1/\beta=4\pi g^2/e^2$.

The X-Y model is usually defined on a lattice.[18,39,40] In that case there are two kinds of important excitations, i.e., spin waves and vortices. In the continuum limit, however, the energy of the latter becomes infinite and hence they disappear. This suggests that the behavior of the system may be approximated at least for low temperature by an ensemble of spin waves. In order to make this idea more precise, we develop a simple mean-field theory based on a variational principle for path integrals due to Feynman[41]: Given two Hamiltonians H and H_0, the free energy density F for H defined by $F=-V^{-1}\ln Z$ (V is the volume of space occupied by the system) is bounded as

$$F\leqslant F_0+\frac{\beta}{V}\langle H-H_0\rangle_{H_0}, \tag{4.21}$$

where F_0 is the free energy density for H_0, and $\langle\cdot\rangle_{H_0}$ means the expectation value with respect to H_0. Of course H is given by (4.20). For H_0, we choose

$$H_0=\int d^2x\ \tfrac{1}{2}\varphi(x)\left[(-\Box+\mu^2)\left(1-\frac{\Box}{m_W^2}\right)^2\right]\varphi(x) \tag{4.22}$$

and determine μ^2 ($\geqslant 0$; the mass of the spin wave) by minimizing the right-hand side of (4.21). The validity of this choice will be discussed later together with the results obtained.

Computation of the right-hand side of (4.21) is straightforward. Denoting it as $\bar F(\mu^2,m_W^2,T)$, we find[42]

$$\bar F(\mu^2,m_W^2,T)=m_W^2 f(\sigma), \tag{4.23}$$

where $\sigma=\mu^2/m_W^2$ and

$$f(\sigma)=\frac{\sigma}{8\pi}-\frac{\gamma}{m_W^2}\exp\left[-\frac{T}{8\pi}\left(\frac{1}{\sigma-1}-\frac{1}{(\sigma-1)^2}\ln\sigma\right)\right]. \tag{4.24}$$

As is easily seen, $f(\sigma)$ has its absolute minimum at $\sigma=\sigma(T)$ with

$$\begin{aligned}\sigma(T)&=0\quad\text{for } T>8\pi\\ &\neq 0\quad\text{for } T<8\pi\end{aligned} \tag{4.25}$$

(see Fig. 5). Thus this approximation predicts a phase transition at $T=8\pi$. For $T\to 8\pi-0$ or $\gamma/m_W^2\to 0\,(T<8\pi)$, $\sigma(T)$ vanishes as

$$\sigma(T)\sim\left(T\frac{\gamma}{m_W^2}\right)^{\kappa(T)}, \tag{4.26}$$

with $\kappa(T)=8\pi/(8\pi-T)$. Hence, the spin wave mass $m_W\sigma(T)^{1/2}$ is nonzero for $T<8\pi$, continuously goes to zero as $T\to 8\pi-0$, and remains zero for $T>8\pi$. For the free energy, we find

$$f[\sigma(T)]\sim-\frac{1}{T\kappa(T)}\left(T\frac{\gamma}{m_W^2}\right)^{\kappa(T)}\quad\text{as } T\to 8\pi-0 \text{ or } \gamma/m_W^2\to 0\ (T<8\pi). \tag{4.27}$$

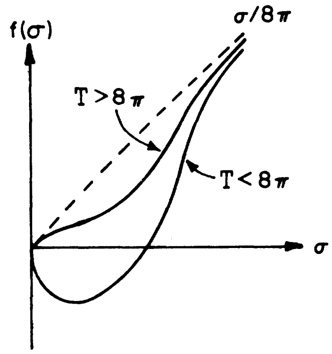

FIG. 5. Schematic plot of the function $f(\sigma)$ for $T<8\pi$ and $T>8\pi$.

The expectation value for the component $\cos\varphi(x)$ of the spin is given by $-\partial F/\partial\gamma$. Replacing F by $\overline{F}$, we obtain

$$\langle\cos\varphi(x)\rangle_{\overline{F}}=\alpha(T)\,, \tag{4.28}$$

where the subscript $\overline{F}$ indicates our approximation and

$$\alpha(T)=\exp\left[-\frac{T}{8\pi}\left(\frac{1}{\sigma(T)-1}-\frac{1}{[\sigma(T)-1]^2}\ln\sigma(T)\right)\right], \tag{4.29}$$

$$\alpha(T)\begin{cases}\neq 0 & \text{for } T<8\pi\\ \sim\left(T\dfrac{\gamma}{m_W^{\,2}}\right)^{\kappa(T)-1} & \text{for } T\to 8\pi-0\\ & \text{or } \gamma/m_W^{\,2}\to 0\ (T<8\pi),\\ =0 & \text{for } T>8\pi.\end{cases}$$

For the other component $\sin\varphi(x)$, we have without any approximation

$$\langle\sin\varphi(x)\rangle=0 \tag{4.30}$$

since the system is invariant under the transformation $\varphi(x)\to-\varphi(x)$.

We have also computed the second-order correction to (4.21) given by

$$-\frac{\beta^2}{V}\left[\langle(H-H_0)^2\rangle_{H_0}-\langle H-H_0\rangle^2{}_{H_0}\right]. \tag{4.31}$$

We have found that this correction, though not numerically small, does not qualitatively change the results given above.

Let us now examine the range of validity of the mean-field approximation. We shall do this by comparing its predictions with those of Kosterlitz's renormalization group equations (4.14) which apply to the X-Y model as well. (The correspondence of temperature is $\beta'=T=4\pi g^2/e^2$.) First of all, we observe that the mean-field approximation gives rather poor results near the critical point. In fact, it gives the critical temperature 8π compared to the renormalization group value $T_c=8\pi+8\pi\gamma/m_W^{\,2}$. It also gives a behavior for the correlation length $1/m_W\sigma(T)^{1/2}$ (to be identified with ξ) and the free energy density F different from those of the renormalization group. This, however, is not so surprising since H_0 of (4.22) suppresses large fluctuations of $\varphi(x)$ which in fact are quite important near T_c. Below and away from T_c, on the other hand, the mean-field approximation correctly describes the behavior of the system for sufficiently small γ, as may be seen by comparing (4.15) and (4.16) with (4.26) and (4.27). Finally, for $T>T_c$, our approximation becomes poor again: It ignores the effect of the external magnetic field. In particular, it predicts $\langle\cos\varphi(x)\rangle=0$, while in fact it is nonvanishing and has a series expansion in γ. Summarizing, it would be fair to say that our mean-field approximation provides a simple way of describing the system below and away from T_c.

B. The gauge structure

Let us go back to the field theory language and examine the gauge structure of the theory making use of the results and techniques developed above. Using (2.12) and (4.4) and (4.5), we obtain for the gauge field propagator[43]

$$\int d^2x\,e^{-ipx}\langle\theta\,|T(A_\mu(x)A_\nu(0))|\,\theta\rangle=\left(\delta_{\mu\nu}-\frac{p_\mu p_\nu}{p^2}\right)\left(\frac{1}{p^2+m_W^{\,2}}+\frac{1}{p^2}I(p^2)\right). \tag{4.32}$$

The first term in the square brackets comes from $D^{(0)}_{\mu\nu}$ in (2.12), $D^{(\pm1)}_{\mu\nu}-D^{(0)}_{\mu\nu}$ has been neglected, and

$$I(p^2)=-\int d^2x\,e^{-ipx}\left.\frac{\delta^2\ln Z[J]}{\delta J(x)\,\delta J(0)}\right|_{J=0}, \tag{4.33}$$

with

$$Z[J]=\int[d\chi]\exp\left(-\int d^2x\left\{\tfrac{1}{2}\partial_\mu\chi(x)\partial_\mu\chi(x)-\gamma\cos\left[\int d^2y\left(2\sqrt{\pi}\,\frac{g}{e}\,\chi(y)+\frac{2\pi}{e}J(y)\right)\rho^{(1)}(y-x)\right]\right\}\right) \tag{4.34}$$

Explicitly, $I(p^2)$ is expressed as

$$I(p^2) = -\gamma\langle\cos\varphi(0)\rangle_\chi + \gamma^2 \int d^2z e^{-ipz}\langle T[\sin\varphi(z)\sin\varphi(0)]\rangle_\chi\ , \tag{4.35}$$

where $\varphi(x)$ is defined in (4.15) and $\langle\circ\rangle_\chi$ means the expectation value with respect to $\mathcal{L}_{\text{eff}}[\chi]_{\theta=0}$ of (4.4).

For $g^2 > g_c^2 = 2e^2 + 2\pi e^2\gamma/m_W^2$, $I(p^2)$ can be computed by a perturbation expansion in γ. After a little algebra, we find

$$\begin{aligned} I(p^2) = {} & Z[J=0]^{-1}\frac{\gamma^2}{2}\int d^2z\,(e^{-ipz}-1)\exp\left(4\pi\frac{g^2}{e^2}V(z)\right) \\ & + Z[J=0]^{-1}\frac{\gamma^2}{2}\sum_{N=1}^{\infty}\frac{(\gamma/2)^{2N}}{(N+1)!N!}\int d^2z\int\prod_{i=1}^{2N}d^2R_i[(N+1)e^{-ipz}-Ne^{-ip\cdot R_{2N}}-1] \\ & \times\exp\left[4\pi\frac{g^2}{e^2}\left(V(z)+\sum_{i=1}^{2N}q_i[V(z-R_i)-V(R_i)]\right.\right. \\ & \left.\left.-\tfrac{1}{2}\sum_{i,j=1}^{2N}q_iq_jV(R_i-R_j)\right)\right]\ , \end{aligned} \tag{4.36}$$

where $q_i = +1$ for $1 \leq i \leq N$ and $= -1$ for $N+1 \leq i \leq 2N$, and $V(x)$ is defined by (4.9). Clearly $I(p^2)$ vanishes at $p^2 = 0$. Using Kosterlitz's renormalization group equations (4.14), we easily find the leading behavior $I(p^2) \sim (p^2)^{(g^2/e^2)-1}$. Thus the gauge field propagator does not have a massless pole and the gauge symmetry is spontaneously broken.

In the region $g^2 < g_c^2$, the series expansion in powers of γ is no longer valid. Instead, we use the mean-field approximation developed for the X-Y model analogy. To calculate $I(p^2)$ (or more generally $Z[J]$), a slight modification is necessary. The Hamiltonian (4.20) is replaced by

$$H' = \int d^2x\left\{\tfrac{1}{2}\varphi(x)\left[-\Box\left(1-\frac{\Box}{m_W^2}\right)^2\right]\varphi(x) - \frac{e^2\gamma}{4\pi g^2}\cos[\varphi(x)+\varphi_{\text{ext}}(x)]\right\} \tag{4.37}$$

with

$$\varphi_{\text{ext}}(x) = \frac{2\pi}{e}\int d^2y\,J(y)\rho^{(1)}(y-x)\ , \tag{4.38}$$

and instead of H_0 of (4.22), we choose

$$H_0' = \int d^2x\left\{\tfrac{1}{2}\varphi(x)\left[-\Box\left(1-\frac{\Box}{m_W^2}\right)^2\right]\varphi(x) + \frac{\mu^2}{2}[\varphi(x)+\varphi_{\text{ext}}(x)]\left(1-\frac{\Box}{m_W^2}\right)^2[\varphi(x)+\varphi_{\text{ext}}(x)]\right\}. \tag{4.39}$$

This choice is motivated by the observation that for $g^2 < g_c^2$ the X-Y model spins are strongly aligned along the direction of the external magnetic field which, in the present case, is given by $\varphi_{\text{ext}}(x)$. The Feynman upper bound $\bar{W}[J]$ for $W[J] \equiv -\ln Z[J]$, after minimizing with respect to μ, is given by[42]

$$\bar{W}[J] = \int d^2x\,\frac{\mu(g)^2}{8\pi} + \frac{e^2}{8\pi g^2}\int d^2x\,d^2y\,J(x)D(x-y)J(y) - \gamma\bar{\alpha}(g)\int d^2x\cos\left(\int d^2y\,\Delta(x-y)J(y)\right), \tag{4.40}$$

where $\mu(g) \equiv m_W\sigma(4\pi g^2/e^2)^{1/2}$, $\bar{\alpha}(g) \equiv \alpha(4\pi g^2/e^2)$ [see (4.26) and (4.29)], and

$$D(x-y) = \frac{4\pi^2}{e^2}\mu(g)^4\int\frac{d^2p}{(2\pi)^2}e^{ip(x-y)}\frac{p^2}{[p^2+\mu(g)^2]^2}\ , \tag{4.41}$$

$$\Delta(x-y) = \frac{2\pi}{e}\int\frac{d^2p}{(2\pi)^2}e^{ip(x-y)}\frac{p^2}{p^2+\mu(g)^2}\ . \tag{4.42}$$

[We recall that for $g^2 < g_c^2$,

$$\mu(g) \sim m_W\left(4\pi\frac{g^2}{e^2}\frac{\gamma}{m_W^2}\right)^{\kappa(g)/2},$$

$$\bar{\alpha}(g) \sim \left(4\pi\frac{g^2}{e^2}\frac{\gamma}{m_W^2}\right)^{\kappa(g)-1}$$

with $\kappa(g) = 2e^2/(2e^2-g^2)$, and $\mu(g) = \bar{\alpha}(g) = 0$ for $g^2 > g_c^2$.]

Using (4.33) and (4.40)–(4.42), we find

$$I(p^2)_{\bar{F}} = \frac{4\pi^2}{e^2}\gamma\bar{\alpha}(g)p^2\,\frac{p^2+\mu(g)^4e^2/4\pi g^2\gamma\bar{\alpha}(g)}{[p^2+\mu(g)^2]^2}\ . \tag{4.43}$$

(The subscript $\bar{F}$ indicates our approximation.) Thus, for $g^2 < g_c^2$, the massless pole in (4.32) is explicitly canceled out and the gauge symmetry is still spontaneously broken. Several qualifications are in order. (1) Our approximation breaks down near g_c^2. Since the system is in the same phase up to g_c^2, however, the gauge symmetry will also be broken in this region. (2) The double pole of (4.41) at $p^2=-\mu(g)^2$ may be understood as follows: Our approximation is valid for small g^2/e^2. In this case, we have $4\pi g^2\gamma\bar{\alpha}(g)/e^2\mu(g)^2\simeq 1$. Hence, the double pole is approximately reduced to a single pole with residue $\pi\mu(g)^2/g^2$.

The mean-field approximation provides us with a stronger piece of evidence for the spontaneous breaking of the gauge symmetry. Namely, it gives us an upper bound for all values of g^2 for the static potential $E(R)$ between a pair of charges g' and $-g'$ separated by a distance R. We find that $E(R)$ damps exponentially with increasing R for any value of g^2 and g'^2. To show this, we once again resort to the Wilson formula (2.17). Choosing $J(x)=g'\theta(x_1+R/2)\theta(R/2-x_1)$ in (4.40), recalling that $D_{\mu\nu}^{(0)}$ of (2.12) gives rise to the potential $-(g'^2/2m_W)e^{-m_W R}$, and neglecting terms independent of g,g' and/or R, we find

$$E(R)\leqslant-\frac{g'^2}{2m_W}e^{-m_W R}+\pi\left(\frac{g'}{2g}\right)^2\mu(g)^2\,\mathrm{Re}^{-\mu(g)R}+\gamma\bar{\alpha}(g)\int_{-\infty}^{+\infty}dx_1[1-\cos(\tilde{J}(x_1))]\,, \qquad (4.44)$$

where

$$\tilde{J}(x_1)=\pi\frac{g'}{e}\left[\mathrm{sgn}\left(x_1+\frac{R}{2}\right)\exp\left(-\mu(g)\left|x_1+\frac{R}{2}\right|\right)-\mathrm{sgn}\left(x_1-\frac{R}{2}\right)\exp\left(-\mu(g)\left|x_1-\frac{R}{2}\right|\right)\right]. \qquad (4.45)$$

An interesting feature of the above result is that the characteristic length scale beyond which the gauge symmetry is spontaneously broken depends on the value of g. It is given by $1/m_W$ for $g^2>g_c^2$, and as g^2 becomes smaller than g_c^2, it changes smoothly into $1/\mu(g)$ ($\gg 1/m_W$). In terms of the Coulomb gas analogy, the interpretation is clear[44]: For $g^2>g_c^2$, the instantons (Coulomb charges) are pairwise bound into dipoles with net winding number equal to zero. Hence, as far as the gauge symmetry is concerned, the structure of the θ vacuums is essentially identical to that of the perturbation theory vacuums. Thus, the gauge symmetry is spontaneously broken beyond a length scale $\sim 1/m_W$. For $g^2<g_c^2$, the instantons form a uniform plasma which exhibits a Debye-Hückel type screening with the screening distance $\sim 1/\mu(g)$. Beyond this length scale, the gauge symmetrizing effect of instantons we saw in Sec. II is cutoff and the gauge symmetry remains spontaneously broken.

C. The Chiral Structure

Let us now turn our attention to the chiral structure of the theory. Note first that the field χ is shifted by a constant under the global chiral transformation: $e^{\alpha\hat{Q}_5}\chi(x)e^{-\alpha\hat{Q}_5}=\chi(x)-\alpha/\sqrt{\pi}$. This leads to $e^{\alpha\hat{Q}_5}Te^{-\alpha\hat{Q}_5}=\exp[2i(g/e)\alpha]T$ where the operator T is defined by (2.7), and consequently $\hat{Q}_5|n\rangle=2ni(g/e)|n\rangle$ and $e^{\alpha\hat{Q}_5}|\theta\rangle=|\theta-2\alpha g/e\rangle$. Both $|n\rangle$ and $|\theta\rangle$ form a degenerate set of vacuums, the former as a result of the absence of tunneling in the global sense, and the latter because the Hamiltonian commutes with $\hat{Q}_5$. In order to decide which are the correct vacuums, let us calculate in the θ vacuums the expectation value of the operator $\cos\varphi(x)$ which is not invariant under the chiral transformation $[e^{\alpha\hat{Q}_5}\varphi(x)e^{-\alpha\hat{Q}_5}=\varphi(x)-2\alpha g/e]$. Since the θ vacuums are degenerate, it is sufficient to consider the case $\theta=0$. As has been shown in Sec. IV A, $\langle\theta=0|\cos\varphi(x)|\theta=0\rangle=\langle\cos\varphi(x)\rangle_\chi$, where $\langle\cdot\rangle_\chi$ means the expectation value with respect to $\mathcal{L}_{\mathrm{eff}}[\chi]_{\theta=0}$ of (4.4), is nonzero for any value of g^2. For $g^2>g_c^2$, it has a series expansion in γ, and for $g^2<g_c^2$, its leading term in the limit $\gamma\to 0$ is given by (4.29). This result has two consequences. (1) The θ vacuums are the correct vacuums since they alone satisfy the cluster decomposition property.[17] [This can be deduced from the following: By chiral invariance $\langle n|e^{i\varphi(x)}|m\rangle=\delta_{n+1,m}\langle m-1|e^{i\varphi(x)}|m\rangle$ and hence $\langle\theta=0|\cos\varphi(x)|\theta=0\rangle\neq 0$ implies that $\langle m\pm 1|\cos\varphi(x)|m\rangle\neq 0$.] (2) The chiral symmetry is spontaneously broken.[10]

The corresponding Nambu-Goldstone pole appears in the Green's function $\langle\theta=0|T^*[\hat{J}_\mu^{\,5}(x)\sin\varphi(y)]|\theta=0\rangle$. After integrating out A_μ and ϕ within the dilute-instanton-gas approximation, this becomes

$$\int d^2x\,e^{-ipx}\langle\theta=0|T^*[\hat{J}_\mu^{\,5}(x)\sin\varphi(0)]|\theta=0\rangle=i\frac{p_\mu}{\sqrt{\pi}}\int d^2x\,e^{-ipx}\langle T[\chi(x)\sin\varphi(0)]\rangle_\chi+i\frac{2g}{e}\gamma\frac{p_\mu}{p^2}\int d^2x\,e^{-ipx}\langle T[\sin\varphi(x)\sin\varphi(0)]\rangle_\chi\,, \qquad (4.46)$$

where the massless pole of the second term comes from $A_\nu^{(1)}(x)$ [recall (4.2)].[13] Using (4.35), we can rewrite the second term as follows:

$$i\frac{2g}{e}\frac{p_\mu}{p^2}\left(\langle\cos\varphi(0)\rangle_\chi+\frac{1}{\gamma}I(p^2)\right). \qquad (4.47)$$

As we have seen, $I(p^2)$ vanishes at $p^2=0$. In addition, the first term of (4.46) behaves as p_μ times a constant for $g^2<g_c^2$ since χ is a massive field (see below), and as $p_\rho\,(p^2)^{(g^2/e^2)-2}$ for $g^2>g_c^2$ which is obtained using Kosterlitz's equations (4.14).

The first term of (4.47) thus represents the Nambu-Goldstone pole. Note that the residue of the massless pole has exactly the form expected. Namely, it is equal to

$$\lim_{p\to 0} p_\mu \int d^2x\, e^{-ipx}\langle\theta=0\,|\,T^*[\hat{J}_\mu{}^5(x)\sin\varphi(0)]\,|\,\theta=0\rangle$$

$$= i\frac{2g}{e}\langle\theta=0\,|\cos\varphi(0)|\,\theta=0\rangle. \quad (4.48)$$

Observe also that this Nambu-Goldstone pole decouples from the gauge-invariant sector of the theory. In fact, Green's functions of gauge-invariant operators contain $A_\mu^{(1)}$ only in terms of $F_{\mu\nu}^{(1)}$ which does not give rise to a massless pole. We thus have one more example of spontaneous symmetry breaking without the appearance of a Nambu-Goldstone boson in the physical spectrum. The mechanism is exactly that of "seizing" proposed by Kogut and Susskind[20] as a way out of the U(1) problem.

It is important to observe that the chiral symmetry is actually broken down to a discrete symmetry $\chi(x)\to\chi(x)+\sqrt{\pi}(e/g)n$ $(n\in Z)$. Since the mass operator is given by $\bar{\psi}\psi\sim\cos(2\sqrt{\pi}\chi)$,[45] the discrete symmetry still prohibits a nonvanishing fermion mass unless $g/e=1/n$ $(n\in Z)$ or the symmetry is spontaneously broken.[21] To examine the latter possibility, let us consider the vacuum expectation value

$$\langle\theta=0\,|\,\bar{\psi}\psi(x)\,|\,\theta=0\rangle\sim\langle\cos[2\sqrt{\pi}\chi(x)]\rangle_\chi. \quad (4.49)$$

For $g^2>g_c{}^2$, we can evaluate the right-hand side by a series expansion in γ and we find that it vanishes order by order. Thus the discrete symmetry remains unbroken and the fermion is massless. For $g^2<g_c{}^2$, we compute (4.49) by replacing $\mathcal{L}_{\rm eff}[\chi]_{\theta=0}$ of (4.4) by $\frac{1}{2}\partial_\mu\chi\partial_\mu\chi+\frac{1}{2}\mu(g)^2\chi^2$ where $\mu(g)$ is given after (4.42). This crude approximation gives

$$\langle\cos(2\sqrt{\pi}\chi)\rangle_\chi\simeq\mu(g)/\mu_0\sim\frac{m_W}{\mu_0}\left(4\pi\frac{g^2}{e^2}\frac{\gamma}{m_W{}^2}\right)^{e^2/(2e^2-g^2)}$$

Thus, below $g^2=g_c{}^2$, the discrete symmetry is spontaneously broken and the fermion will acquire a finite mass.

This result is confirmed by the following observation. Taking the limit $m_W\to\infty$ in $\mathcal{L}_{\rm eff}[\chi]_{\theta=0}$ of (4.4), we obtain

$$\mathcal{L}_{\rm eff}[\chi]_{\theta=0}\underset{m_W\to\infty}{=}\tfrac{1}{2}\partial_\mu\chi(x)\partial_\mu\chi(x)-\gamma\cos\left(2\sqrt{\pi}\frac{g}{e}\chi(x)\right). \quad (4.50)$$

The right-hand side is nothing but the sine-Gordon Lagrangian which has been studied in great detail.[31,34,46] In particular, it has been shown that the sine-Gordon system is well defined for $g^2<2e^2$ (which justifies our approximation) and the spectrum consists of solitons and soliton-antisoliton bound states. The soliton has a mass $\tilde{m}=4(\gamma/\pi)^{1/2}(2e^2-g^2)/eg$ and satisfies $\chi(x_4;x_1=+\infty)-\chi(x_4,x_1=-\infty)=e\sqrt{\pi}/g$. It thus has fermion number $N_f=e/g$ and fermionic charge $Q_f=e$. This is the massive fermion state anticipated above. Of course the new fermion is not a one-particle state of the original fermion. Rather, it is a composite state of original fermions and a cloud of instantons surrounding them. The existence of a single fermion state with $Q_F=e$ for $g^2<2e^2$, in particular for $g/e\neq$ rational number, is consistent with the spontaneous breaking of the gauge symmetry discussed in Sec. IV B.

There are two additional features worth mentioning. (1) The spontaneously generated mass $\tilde{m}$ vanishes linearly as $g^2\to 2e^2-0$. However, this behavior near $g^2=2e^2$ is not a reliable description of the theory for finite m_W. (2) For $g/e=1/n$ $(n\in Z)$, the limit in which the bare fermion mass goes to zero is smooth. Namely, the n massive fermion bound state which we have shown to exist in Sec. III smoothly goes over into the new fermion of the sine-Gordon system. An example is given by the case $g/e=\frac{1}{2}$ which was worked out in (3.25)–(3.27).

V. DISCUSSION

In this article we have studied the physics of instantons in a U(1) gauge theory in 1+1 dimensions. The primary objective has been to clarify how instantons change the gauge and chiral structure of naive perturbation theory. The results we have obtained have already been summarized in Sec. I. Here we want to add a few remarks.

(1) In order to make our formulas explicit, we have restricted the range of parameters in the original Lagrangian (2.1) by (2.9). However, the range of validity of our results is only restricted by that of semiclassical approximations. Namely, the average distance between instantons must be large as compared to their sizes. It would certainly be important to find a description of the theory in the opposite limit in which the instanton density becomes large.

(2) We have shown that the instantons symmetrize an otherwise spontaneously broken gauge vacuum. This mechanism becomes effective on a length scale characteristic of the mean instanton separation. A similar symmetrization of the gauge symmetry might very likely occur in unified theories of weak and electromagnetic interactions, such as the Weinberg-Salam model, which possess instantons. Fortunately, the average distance between instantons in such a case is fantastically large $\sim 10^{140}$ parsecs (see G.'t Hooft, Ref. 8).

(3) For $\theta\neq 0$ (modπ), the θ vacuums are both par-

ity and time-reversal noninvariant. When massless fermions are present, however, the resulting chiral symmetry makes all θ vacuums degenerate allowing us to rotate θ away. This of course is true independent of the dimensionality of space-time and the nature of the gauge group.[14]

We have also seen that even if fermions have nonvanishing bare mass it is still possible to have a parity and time-reversal invariant theory for some values of θ. These are the values such that the background electric field is completely screened by the creation of charged fermion pairs. For non-Abelian gauge theories in 3+1 dimensions, nonzero θ implies a background $F^a_{\mu\nu}\tilde{F}^a_{\mu\nu}$ field which may be screened by a similar mechanism.[47]

(4) It has been suggested that instantons might solve the U(1) problem of quantum chromodynamics by realizing the seizing mechanism of Kogut and Susskind.[20] We have explicitly shown that this indeed occurs in our model: The instantons spontaneously break the chiral symmetry, but the accompanying Nambu-Goldstone pole decouples from the gauge-invariant sector of the theory. The chiral structure of our model is further complicated by the fact that there still may remain a discrete chiral symmetry. We have shown that this symmetry is also spontaneously broken for $g^2<g_c^2$ and as a result the fermion acquires a nonvanishing mass.

(5) Let us compare our results with the recent work of Callan *et al.*[16] They introduce N massless fermions $\psi_a (a=1,\ldots,N)$, all with charge e, and discuss the chiral structure as a function of N. In terms of a set of equivalent Bose fields χ^a $(a=1,\ldots,N)$, the effective Lagrangian for fermions is given by

$$\mathcal{L}_{\text{eff}}=\frac{1}{2}\sum_{b=1}^{N-1}\partial_\mu\Phi^b(x)\partial_\mu\Phi^b(x)+\tfrac{1}{2}\partial_\mu\Phi(x)\partial_\mu\Phi(x) - \gamma\cos\left(2\sqrt{\pi N}\int d^2y\,\Phi(y)\rho^{(1)}(y-x)+\theta\right) \tag{5.1}$$

where $\Phi=N^{-1/2}\sum_{a=1}^N\chi^a$, and the Φ^b's are related to χ^a's through $\chi^a=N^{-1/2}\Phi+\sum_{b=1}^{N-1}D_{ab}\Phi^b$ with $\sum_{a=1}^N D_{ab}=0$ and $\sum_{a=1}^N D_{ab}D_{ab'}=\delta_{bb'}$.[48] As is easily seen, the U(1) chiral transformation shifts Φ by a constant while leaving Φ^b's unchanged. Thus, the U(1) chiral symmetry is broken down to a discrete symmetry $\Phi\to\Phi+n\sqrt{\pi/N}$ $(n\in Z)$. Furthermore, comparing (5.1) with (4.4), we see that, with the identification $N\leftrightarrow g^2/e^2$, our analysis of Sec. IV can be reinterpreted to describe this model. There is a phase transition at $N=N_c=2+2\gamma/m_W{}^2$. For $N<N_c$, the discrete symmetry is spontaneously broken, while it remains unbroken for $N>N_c$. Notice, however, that the spontaneous breaking of the discrete symmetry does not mean a nonzero vacuum expectation value for $\bar{\psi}_a\psi_a$. In fact, using the formula[49]

$$\bar{\psi}_a\psi_a\sim\cos\left[2\left(\frac{\pi}{N}\right)^{1/2}\Phi+2\sqrt{\pi}\sum_{b=1}^{N-1}D_{ab}\Phi_b\right]$$

and the fact that the Φ_b's are massless, one easily finds $\langle\theta|\bar{\psi}_a\psi_a|\theta\rangle=0$. [Another way to understand this is the following. A nonzero vacuum expectation value of $\bar{\psi}_a\psi_a$ means a spontaneous breaking of the $\mathrm{SU}(N)\otimes\mathrm{SU}(N)$ symmetry, which is impossible in 1+1 dimensions due to the theorem by Coleman.[10]] This result has the following consequence regarding the mass generation for fermions. It does not in itself imply that a mass term is absent in the propagator $\langle\psi_a(x)\bar{\psi}_b(y)\rangle$. If, however, such a term were generated in the Hartree-Fock approximation, as discussed in Ref. 16, this would imply within the same approximation that $\langle\bar{\psi}_a(x)\psi_a(x)\rangle\neq0$. We thus claim that the *original* fermions remain massless for all $N>1$.

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[2]K. G. Wilson, Phys. Rev. D 10, 2445 (1974); J. Kogut and L. Susskind, *ibid.* 11, 395 (1975); A. A. Migdal, Zh. Eksp. Teor. Fiz. 69, 810 (1975); 69, 1457 (1975), [Sov. Phys. -JETP 42, 413 (1975)]; 42, 743 (1975)] A. M. Polyakov, report (unpublished).

[3]For Abelian gauge theories, this problem has recently been examined by the so-called dual transformation: T. Banks, R. Myerson, and J. Kogut, Nucl. Phys. B129, 493 (1977).

[4]A. M. Polyakov, Phys. Lett. 59B, 82 (1975); Nucl. Phys.

B120, 429 (1977).

[5]A. M. Polyakov, Zh. Eskp. Teor. Fiz. 68, 1975 (1975) [Sov. Phys.—JETP 41, 988 (1976)]; G. 't Hooft, Nucl. Phys. B79, 276 (1974).

[6]C. G. Callan *et al*., Phys. Lett. 66B, 375 (1977).

[7]A. A. Belavin *et al.*, Phys. Lett. 59B, 85 (1975).

[8]G. 't Hooft, Phys. Rev. Lett. 37, 8 (1976); Phys. Rev. D 14, 3432 (1976); D. G. Caldi, Phys. Rev. Lett. 39, 121 (1977).

[8a]This has been criticized by R. J. Crewther, Phys. Lett. 70B, 349 (1977).

[9]We use the Euclidean formulation of the theory throughout this article. The notations are as follows: Greek indices run over 1 and 4 ($x_4=-it$ is the Euclidean time). The metric is $\delta_{\mu\nu}$. $\epsilon_{\mu\nu}$ is the totally antisymmetric tensor with $\epsilon_{14}=1$. Dirac matrices are

$$\gamma_1=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \gamma_4=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},$$

$$\gamma_5=\gamma_1\gamma_4=\begin{pmatrix}-i & 0\\ 0 & i\end{pmatrix}$$

and satisfy $\{\gamma_\mu,\gamma_\nu\}=2\delta_{\mu\nu}$.

[10]The theorem due to S. Coleman [Commun. Math. Phys. 31, 259 (1973)] which states that there cannot be a spontaneous breakdown of a continuous symmetry in (1+1)-dimensional space-time is based on two assumptions, manifest Lorentz convariance and positivity of the relevant sector of Hilbert space. For gauge symmetries it is impossible to satisfy both of these simultaneously, and hence the theorem does not apply. Neither does it apply to the chiral U(1) symmetry because of the anomaly. On the other hand, for SU(N) $\otimes$SU(N) flavor symmetry ($N\geq 2$), the theorem does apply and prohibits its spontaneous breaking (see Sec. V).

[11]In spite of the presence of an anomaly, one can still define a conserved axial vector current. The corresponding charge generates global chiral transformations and is invariant under gauge transformations which take the same value at $x_1=\pm\infty$. See Sec. IV for details.

[12]C. G. Callan *et al*., Phys. Lett. 63B, 334 (1976); R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37, 172 (1976).

[13]We calculate in the $\partial_\mu A_\mu=0$ gauge throughout this article. The description of any gauge dependent object such as a massless pole in the gauge field propagator is understood in this gauge.

[14]C. G. Callan *et al.*, Ref. 12.

[15]A similar observation has been made by E. H. Monsay and T. N. Tudron, Phys. Rev. D 16, 3503 (1977).

[16]C. G. Callan *et al*., Phys. Rev. D 16, 2526 (1977).

[17]One still has to use the θ vacuums to ensure the cluster decomposition property. See C. G. Callan *et al*., Ref. 12.

[18]J. M. Kosterlitz, J. Phys. C 7, 1046 (1974).

[19]These phase transitions have the unusual character that conventional long-range order as indicated by nonvanishing order parameters is absent. That this must be so has been proved by N. D. Mermin and H. Wagner, Phys. Rev. Lett. 22, 1133 (1966). See also S. Coleman, Ref. 10.

[20]J. Kogut and L. Susskind, Phys. Rev. D 11, 3594 (1975).

[21]A similar situation has been encountered by Callan *et al*. (Ref. 16).

[22]In superconductor language, this corresponds to the case of extreme type II. Instantons also exist for the case $e\gtrsim\sqrt{\lambda}$. All the qualitative features of the theory which we shall derive explicitly for $e\ll\sqrt{\lambda}$ would remain valid in this case, provided the parameters are in the range such that the average distance between the instantons is large compared to their sizes (i.e., $\exp S^{(N)}\gg 1$).

[23]In the limit $|X-R|\to 0$, $A_N(|X-R|)$ vanishes as $|X-R|^2$ and $F_{\mu\nu}^{(N)}$ goes to a constant. To simplify calculations, however, we shall use (2.4c) and (2.4e) for the entire range of $|X-R|$. This means that we underestimate the short-distance cutoff of the instanton solution. This approximation does not affect our results since they are the consequences of the long-distance properties of the instantons.

[24]K. Freed, J. Chem. Phys. 56, 692 (1972); D. McLaughlin, J. Math. Phys. 13, 1099 (1972); S. Coleman, Phys. Rev. D 15, 2929 (1977).

[25]This is an ideal gauge for studying the time development of the system since the only allowable gauge transformations are time independent ones.

[26]See, for example, J.-L. Gervais *et al*., Phys. Rep. 23C, 237 (1976); A. Hosoya and K. Kikkawa, Nucl. Phys. B101, 271 (1975).

[27]Strictly speaking, double counting of multi-instanton configurations should be avoided by excluding the region such that $|R_i-R_j|\ll 1/m_W$. This complication can be ignored, however, without any qualitative change in the large distance properties of the system.

[28]For those values of θ such that $\cos(N\theta)<0$, the interpretation of (2.13) as the instanton number density apparently breaks down. We regard this pathology as one of the manifestations of the fact that the θ vacuums are unstable for those values of θ. See Sec. IIC.

[29]This is a gauge independent statement. Consider the operator $\Phi(x)=\phi(x)\exp[ie\int d^2y A_\mu(y)\partial_\mu\eta(y)]$ where the c-number function η satisfies $\partial_\mu\partial_\mu\eta(y)=\delta(y-x)-\delta(y-x_\infty)$ and the limit $|x_\infty|\to\infty$ is understood. This operator is locally gauge invariant but globally gauge dependent. In the Landau gauge, $\Phi(x)$ reduces to $\phi(x)$.

[30]K. G. Wilson, Ref. 2.

[31]See, for example, S. Coleman, Phys. Rev. D 11, 2088 (1975); S. Mandelstam, *ibid*. 11, 3026 (1975).

[32]We have carried out an integration by parts in the cosine term of (3.10b). The resulting surface term cancels $-2\sqrt{\pi}(g/e)\xi$.

[33]In the full-fledged quantum theory, one cannot use this approximation for $g^2>2e^2$ since the ground-state energy of (3.21) is then unbounded from below (see S. Coleman, Ref. 31). Retaining the short-distance cutoff, however, controls the quantum fluctuations sufficiently to solve the above problem. We then expect that the qualitative features of the spectrum obtained by a classical approximation to (3.21) will remain valid in the full quantum-mechanical treatment of (3.20) since they are basically determined by the large-distance properties of the theory.

[34]R. F. Dashen *et al*., Phys. Rev. D 11, 3424 (1975).

[35]In contrast to the original Lagrangian (3.1) (with $m=0$), the effective Lagrangian (4.4) is not invariant under the global chiral transformation since $e^{\alpha\hat{Q}_5}\chi(x)e^{-\alpha\hat{Q}_5}=\chi(x)-\alpha/\sqrt{\pi}$. This is because $\mathcal{L}_{\rm eff}[\chi]$

has been constructed from $\langle\theta;\,\mathrm{out}|\theta;\mathrm{in}\rangle$ and $|\theta\rangle$ is not invariant under the chiral transformation, i.e., $e^{\alpha\hat{Q}_5}|\theta\rangle = |\theta - 2(g/e)\alpha\rangle$.

[36]This means that in the presence of a U(1) chiral symmetry one can always redefine fields so that the vacuum respects both parity and time-reversal invariance. A similar observation for non-Abelain gauge theories in 3+1 dimensions has been made by R. D. Peccei and H. R. Quinn, Phys. Rev. Lett. 38, 1440 (1977).

[37]The particular form of the large-momentum cutoff in $V(R;\mu)$ is the consequence of our approximation (3.20) (recall Ref. 23). As is easily seen, however, the critical behavior of the system does not depend on the detailed form of the cutoff.

[38]In the framework of canonical quantization, this may be understood as a consequence of the fact that $\hat{Q}_5$ commutes with the Hamiltonian, but not with the operator T defined by (2.7). See G. 't Hooft, Ref. 8; C. G. Callan *et al.*, Ref. 12.

[39]J. M. Kosterlitz and D. J. Thouless, J. Phys. C 6, 1181 (1973).

[40]V. L. Berezinski, Zh. Eksp. Teor. Fiz. 59, 907 (1970) [Sov. Phys.—JETP 32, 493 (1971)]; 61, 1144 (1971) [34, 610 (1972)]; J. Zittartz, Z. Phys. B23, 55 (1976); B23, 63 (1976); J. Villain, J. Phys. (Paris) 36, 581 ((1975); J. V. Jose *et al.*, Phys. Rev. B 16, 1217 (1977); R. Savit, *ibid.* 17, 1340 (1978).

[41]R. P. Feynman, *Statistical Mechanics* (Benjamin, Reading, Mass., 1972).

[42]We have added counterterms

$$\delta E = -\frac{1}{2}\int \frac{d^2p}{(2\pi)^2}\ln p^2$$

and

$$\delta E_0 = -\frac{1}{2}\int \frac{d^2p}{(2\pi)^2}\ln(p^2+\mu^2)$$

to H and H_0, respectively. These are sufficient to remove all ultraviolet divergences of the theory.

[43]From now on, T (T^*) denotes (covariant) time ordering.

[44]In the linear response formulation, the dielectric response function $\epsilon(p)$ of the Coulomb gas is given by $\epsilon(p) = [1-(g^2/\pi)(1/p^2)I(p^2)]^{-1}$. In the mean field approximation (4.40)

$$I(p^2) \simeq \frac{\pi\mu(g)^2}{g^2}\,p^2/[p^2+\mu(g)^2],$$

we find $\epsilon(p) \simeq 1+\mu(g)^2/p^2$.

[45]The normal ordering of $\cos(2\sqrt{\pi}\,\chi)$ at some fixed mass μ_0 is understood.

[46]V. E. Korepin and L. D. Faddeev, Teor. Mat. Fiz. 25, 147 (1975) [Theor. Math. Phys. 25, 1039 (1976)] and references cited therein. A. Luther, Phys. Rev. B 14, 2153 (1976).

[47]M. Lüsher, DESY Report No. DESY 77/16 (unpublished).

[48]See, for example, M. B. Halpern, Phys. Rev. D 12, 1684 (1975); 13, 337 (1976).

[49]Remark similar to Ref. 45 applies here, too.

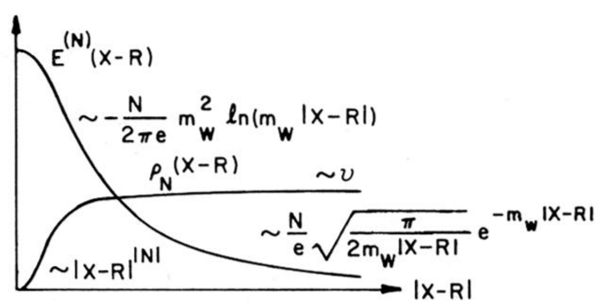

FIG. 1. Schematic plot of $\rho_N(|X-R|)$ and $E^{(N)}(|X-R|) \equiv \frac{1}{2}\epsilon_{\mu\nu} F^{(N)}_{\mu\nu}(X-R)$ for the case $e << \sqrt{\lambda}$.

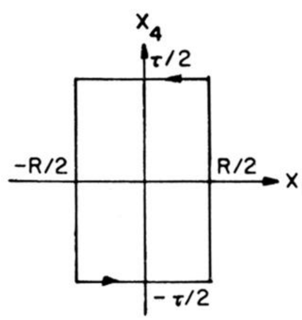

FIG. 2. The contour C for the Wilson formula (2.17).

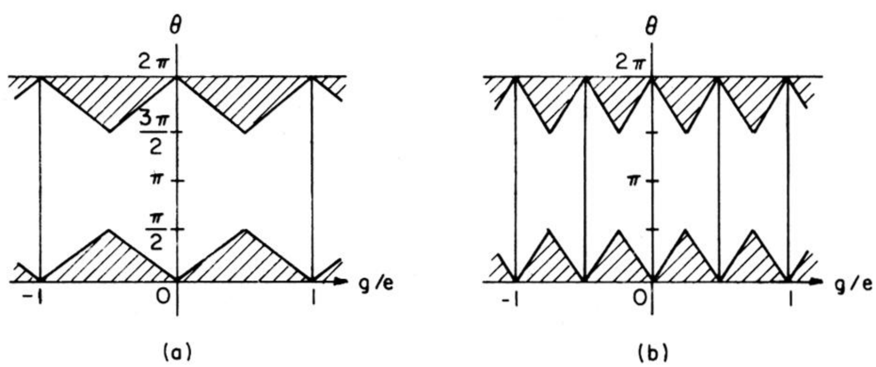

FIG. 3. The shaded region and the straight lines at $g/e = m/n$ ($m \in Z$) satisfy (2.24) for (a) $n = 1$ and (b) $n = 2$. The periodic extension $\theta \rightarrow \theta + 2\pi l$ ($l \in Z$) is understood.

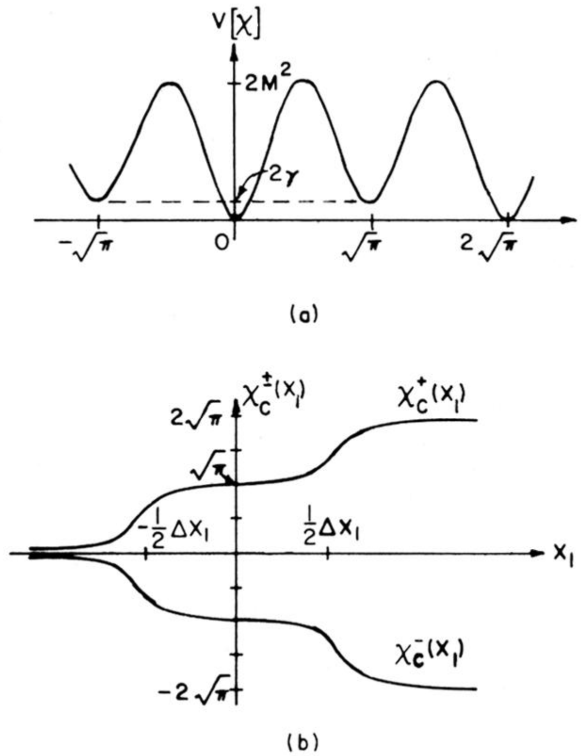

FIG. 4. (a) The shape of the potential $V[\chi]$ for $g/e = \frac{1}{2}$. (b) Two static solutions to the classical equation of motion for $g/e = \frac{1}{2}$.

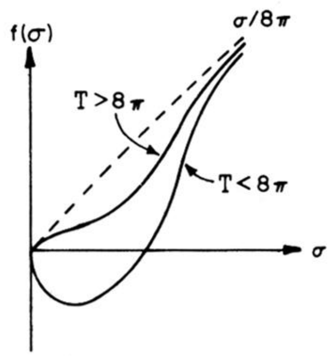

FIG. 5. Schematic plot of the function $f(\sigma)$ for $T < 8\pi$ and $T > 8\pi$.