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Spinorial classification of the SU(2) gauge field

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A classification based on the spinor method is presented for the SU(2) gauge field.

In the preceding paper[1] a classification of a gauge field corresponding to an arbitrary group G was given. If G is specified, however, a refinement of this classification can be obtained, by introducing additional types corresponding to conditions invariant under the particular group G. In the present paper we consider the case $G = \mathrm{SU}(2)$ or O(3).[2,3] We also make remarks on the generalization of the classification obtained, for other gauge groups. The notation of the preceding paper will be used in this paper.

The spinor form ϕ^k_{AB} of an arbitrary gauge field can be written as

$$\phi^k_{AB} = \alpha^k_{(A}\beta_{B)} + \chi^k\gamma_A\gamma_B\ , \qquad (1)$$

for some α^k_A, β_A, γ_A which are irreducible spinors under $\mathrm{SL}(2,C)$ and χ^k, in the present case, is an isovector. The classification arises by considering special features which are invariant under $\mathrm{SL}(2,C)\times \mathrm{O}(3)$:

Type I: General case. Type A: $\chi^k = 0$ in (1). Type B: A subclass of A for which $\alpha^k_A\alpha^k_B\beta^A\beta^B = 0$ (summation over repeated indices unless otherwise indicated). Type C: A subclass of B with[4] $\alpha^k_A\alpha^k_B = \lambda\beta_A\beta_B$. Type D: A subclass of A for which $\alpha^k_A = \chi^k\alpha_A$. Type N: $\beta_A = 0$ in (1). Types E and F: Subclasses of D and N, respectively, with $\chi^k\chi^k = 0$. (Recall that χ^k is a complex vector.)

To obtain a refinement of the above classification, define $\phi_{AB}{}^{PQ} \equiv \phi^k_{AB}\sigma^{kPQ}$, where σ^{kQ}_P are the Pauli spin matrices[3] ($k = 1,2,3$, $P,Q = 0,1$). Since $\sigma^k{}_P{}^Q$ are trace free, σ^{kPQ} is symmetric in P,Q where the index is raised using the alternating symbol ϵ^{PQ}. Thus $\phi_{AB}{}^{PQ}$ is symmetric in the $\mathrm{SL}(2,C)$ indices A, B and the SU(2) indices P,Q. A subclass of G of the general case can be obtained by requiring that $\phi_{AB}{}^{PQ}$ can be written as[5]

$$\phi_{AB}{}^{PQ} = \alpha_{(A}{}^{(P}\beta_{B)}{}^{Q)}\ . \qquad (2)$$

Further specialization gives a new Type H, which is of the form $\alpha_{(A}{}^{(P}\beta_{B)}\delta^{Q)}$. It can be shown that C defined above is a subclass of H satisfying $\beta^A\alpha_{A,P}\delta^P = 0$. Moreover, D is a subclass of H with $\alpha_A{}^P = \alpha_A\gamma^P$; E then being a subclass of D corresponding to $\gamma^P = \delta^P$. H can also be specified as a subclass of A satisfying the additional condition that there exists an isovector ξ^k such that $\xi^k\xi^k = 0$ and $\phi^k_{AB}\xi^k = 0$.

It follows from (3.12) of Ref. 1 that only three independent complex invariants (or six real invariants), which may be taken to be τ, $\mathrm{Tr}L$, and $\mathrm{Tr}(L^2)$, can be formed from the anti-self-dual field ϕ^i_{AB} (without complex conjugation), unlike the usual Yang-Mills field which has nine independent invariants. The fact that the characteristic equation of L,

$$\lambda^3 - (\mathrm{Tr}L)\lambda^2 + \tfrac{1}{2}[(\mathrm{Tr}L)^2 - \mathrm{Tr}(L^2)]\lambda - \det L\, I = 0\ , \qquad (3)$$

contains all these three complex invariants suggests a classification based on the matrix L. In particular, a classification into four main types I, II, III, and IV according to the number of zero eigenvalues (0, 1, 2, and 3, respectively) is obtained. II, III, and IV can also be described in terms of spinors: For II there must exist spinors γ_P, δ_Q such that $\phi_{AB}{}^{PQ}\gamma_P\delta_Q = 0$. III is a union of two classes A and L which have the forms $\alpha_{(A}{}^{PQ}\beta_{B)}$ and $\alpha_{AB}{}^{(P}\delta^{Q)}$. $\mathrm{IV} = \mathrm{B}\cup\mathrm{M}$ where B and M are, respectively, subclasses of A and L corresponding to the conditions $\alpha_A{}^{PQ}\alpha_{B,PQ}\beta^A\beta^B = 0$ and $\alpha_{AB,P}\alpha^{AB}{}_Q\delta^P\delta^Q = 0$.

The relationship between most of the types mentioned above and the associated conditions on the invariants for the usual and the anti-self-dual Yang-Mills field ϕ^i_{AB} is shown in Figs. 1(a) and 1(b), respectively. For completeness, type R which is the second type in Ref. 2, has been included in Fig. 1(a). [R has been omitted from Fig. 1(b) because its specification requires taking the complex conjuation of ϕ^i_{AB}.] Not all the above-mentioned types fit in naturally into either or both diagrams. Also we certainly have not exhausted all the types that are obtainable from the spinor formalism. For instance, two new types P and Q can be defined as follows: P is of the form $\alpha_{(A}{}^{(P}\beta_{B)}{}^{Q)}$ with $\alpha_A{}^P\overline{\alpha}_{B',P} = \beta_A{}^P\overline{\beta}_{B',P}$ and Q is of the form $\alpha_{(A}{}^{(P}\beta_{B)}\delta^{Q)}$ with $\beta^A\alpha_A{}^P\overline{\delta}_P = 0$. The number of inequivalent types cannot be exhausted since an infinite number of invariant conditions can be placed on the spinors leading to new and different types. Thus no classification into a finite number of types is ever "maximally detailed." Physical

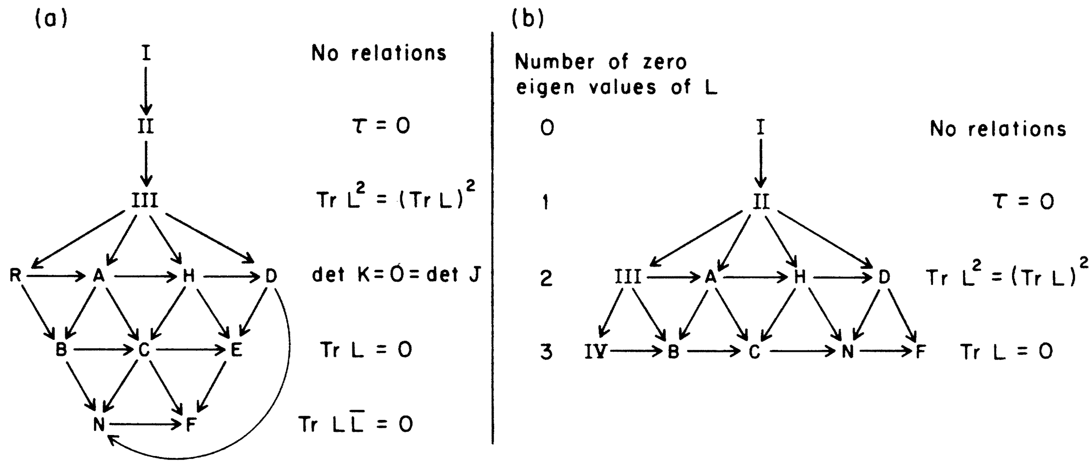

FIG. 1. A classification of the usual Yang-Mills field (a) and the self-dual Yang-Mills field (b) arranged so that each successive row, going down, corresponds to the vanishing of additional invariants shown in the column on the right of each figure. The arrows point toward increasing specialization. I is the general type while II, III, and IV are classes of fields for which the matrix L has 1, 2, and 3 zero eigenvalues, respectively. $A=\alpha^k_{(A}\beta_{B)}$, $B=\alpha^k_{(A}\beta_{B)}$ with $\alpha^k_A\alpha^k_B\beta^A\beta^B=0$, $C=\alpha^k_{(A}\beta_{B)}$ with $\alpha^k_A\alpha^k_B=\lambda\beta_A\beta_B$ for some complex number λ, $D=\xi^k\alpha_{(A}\beta_{B)}$, $N=\xi^k\beta_A\beta_B$, $E=\xi^k\alpha_{(A}\beta_{B)}$ with $\xi^k\xi^k=0$, $F=\xi^k\beta_A\beta_B$ with $\xi^k\xi^k=0$, $H=\alpha_{(A}{}^{(P}\beta_{B)}\delta^{Q)}$, R is the most general type satisfying the conditions of the invariants corresponding to the row to which it belongs.

and aesthetic considerations have to be used in choosing a finite classification.

Our present classification is valid whenever the adjoint representation of the gauge group is a subgroup of an orthogonal group. For instance, the classification is valid for any SU(n) gauge field.

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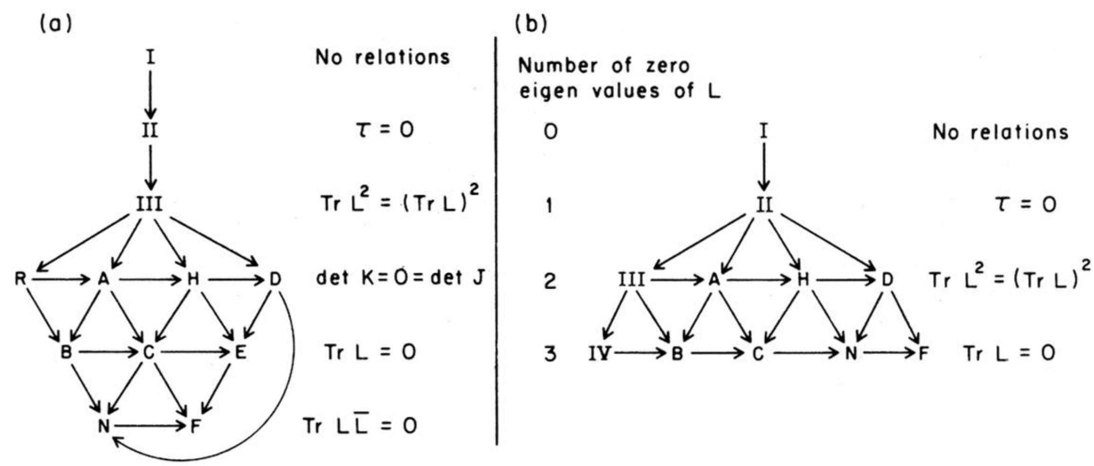
[1] J. Anandan and K. P. Tod, preceding paper, Phys. Rev. D 18, 1144 (1978).

[2] R. Roskies, Phys. Rev. D 15, 1722 (1977).

[3] M. Carmeli, Phys. Rev. Lett. 39, 523 (1977). Several errors in this paper are pointed out below.

[4] Type C is a slight generalization of Carmeli's III_v (Ref. 3). We have introduced a λ in the defining equation so that F can be obtained from C by specialization ($\lambda=0$). It is not the case that all invariants of III_v vanish, since such a field is necessarily of type N. (See Refs. 1 and 2.)

[5] Carmeli's type I_S (Ref. 3), is a subclass of type G. On using the tracelessness of Pauli spin matrices, or the symmetry of $\phi_{AB}{}^{PQ}$ in P, Q, we find also that Carmeli's $\mathrm{II}_s=\mathrm{D}_s$ = our type E and III_s = N = our type F. [Carmeli has an additional condition $\alpha_{AM}\alpha_{BM}=\alpha_A\alpha_B$ for III_s, which, however, is not invariant under SU(2), and therefore inadmissible.] Also the invariants Q and T given by Carmeli are not independent as claimed, because of (3.11) of Ref. 1.



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