Invariants and classification of gauge fields

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A classification of gauge fields in terms of spinors and the rank of a matrix of Lorentz invariants is proposed. A peeling theorem for the Yang-Mills field in the asymptotic limit is shown to follow from an earlier work of Roskies. An analogous theorem is proved for all gauge fields in the geometric optics limit.

I. INTRODUCTION

The realization by Poincaré and Einstein that the invariance group of the laws of electromagnetism contains the Lorentz group and not the Galilean group was a major step in the history of physics. For this implied, as made explicit by Minkowski, that the geometry of space-time must be determined by the Lorentz group. It is well known, however, that electromagnetism has in addition a U(1) invariance which has been referred to as an "internal symmetry." But the action of this U(1) group on the electromagnetic field $F_{\mu\nu}$ is always the identity transformation and hence can be ignored in the study of the geometry of $F_{\mu\nu}$.

The situation, however, is different in the case of gauge fields corresponding to non-Abelian gauge groups such as the Yang-Mills fields[1] which are used in the description of weak and strong interactions. The gauge field $F^i_{\mu\nu}$ corresponding to the gauge group G transforms under the Lorentz group L in the indices μ, ν, and under the adjoint representation $\tilde{G}$ of G in the index i. If G is non-Abelian then $\tilde{G}$ is nontrivial. Thus in this case, even for the study of the classical field it is necessary to extend Minkowski's idea and let the group $L\times G$ determine the geometry of the gauge field.

By the geometry of the gauge field we mean the set of properties of the field which are invariant under the invariance group. A study of these properties has led to a classification of the electromagnetic field into null and non-null types.[2] An analogous classification of the gravitational field $R_{\mu\nu\rho\sigma}$ in vacuum due to Petrov[3] and Pirani[4] is also well known. These classifications were subsequently refined and made more natural by the use of the spinor formalism.[4-7] Similarly, the use of spinors elucidates and extends the recent classification of the Yang-Mills field proposed by Roskies.[8] [Throughout this paper, the Yang-Mills field will refer to the gauge field with SU(2) or O(3) as the gauge group.]

The purpose of the present paper is primarily to investigate classification schemes for an arbitrary gauge field. In Sec. II, a brief introduction is given to spinor algebra[9] and gauge fields are presented in the spinor formalism. A spinorial classification of a general gauge field is then made. In Sec. III arguments are presented which give the number of independent invariants that can be formed from $F^i_{\mu\nu}$, provided $\tilde{G}$ defined above is a subgroup of an orthogonal group. For the Yang-Mills field, we describe the classification of Sec. II in terms of invariants of the field thereby showing its relationship to Roskies's classification. We remark that a peeling theorem for the Yang-Mills field follows from Roskies's work. A further refinement of the classification is made, for an arbitrary gauge field, by considering the rank of a matrix of Lorentz scalars. In Sec. IV we show that the classification from the peeling theorem arises for any gauge group more easily from a consideration of the geometric optics limit.

We hope to stress an underlying unity of all gauge fields by means of this classification. This is especially important in view of the attempts to construct unified theories of weak, strong, electromagnetic, and gravitational interactions. Also we hope to draw attention to the spinor formalism which has proved to be useful in the analysis of the electromagnetic and gravitational fields; it may likewise be useful in the study of the Yang-Mills field and other gauge fields apart from its use in classifying these fields which is demonstrated in this paper.

II. GAUGE FIELDS IN SPINOR FORMALISM

It is well known that the group of unimodular transformations of a two-dimensional vector space over the field of complex numbers C, denoted by SL(2, C), provides a double-valued representation of the proper orthochronous homogeneous Lorentz group. An element of this vector space, which can be represented by a pair of complex numbers α^A $(A=0,1)$, is called a contravariant spinor of rank 1. For any given Lorentz transformation, α^A transforms as

$$\alpha^A \to L^A{}_B\alpha^B\,, \tag{2.1}$$

where $L^A{}_B$ is a 2×2 complex matrix with unit determinant, which represents the given Lorentz transformation. On defining $\bar\alpha^{A'}=\alpha^{A*}$ and $\bar L^{A'}{}_{B'} = L^A{}_B{}^*$, the asterisk denoting complex conjugation, it follows from (2.1) that

$$\bar\alpha^{A'} \to \bar L^{A'}{}_{B'}\bar\alpha^{B'}. \tag{2.1$'$}$$

In general, any spinor that transforms according tp (2.1′) is denoted by a prime in the index. A convariant spinor ξ_A is defined as a pair of complex numbers that transform according to the inverse transpose of (2.1). A general spinor of arbitrary rank with covariant and contravariant indices primed or unprimed is defined in a manner analogous to tensors.

An important set of spinors are the Levi-Civita alternating symbols in two dimensions,

$$\epsilon^{AB}=\epsilon^{A'B'}=\epsilon_{A'B'}=\epsilon_{AB}=\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}. \tag{2.2}$$

It can be verified easily that the ϵ^{AB} and ϵ_{AB} are invariant under Lorentz transformations. Thus ϵ^{AB}, ϵ_{AB}, $\epsilon^{A'B'}$, $\epsilon_{A'B'}$ play a role in spinor algebra analogous to the role of the metric in tensor algebra. Given the contravariant spinors α^A, $\xi^{A'}$ we can form the covariant spinors

$$\alpha_A=\alpha^B\epsilon_{BA},\quad \xi_{A'}=\xi^{B'}\epsilon_{B'A'}. \tag{2.3}$$

Then since from (2.2) $\epsilon^{AC}\epsilon_{BC}=\delta^A{}_B$,

$$\alpha^A=\epsilon^{AB}\alpha_B,\quad \xi^{A'}=\epsilon^{A'B'}\xi_{B'}. \tag{2.3$'$}$$

It follows immediately that $\alpha_A\beta^A=-\alpha^A\beta_A$. Hence $\alpha^A\ \alpha_A=0$. Conversely if α_A and β_A are linearly independent, then $\alpha_A\beta^A\neq 0$.

Given any tensor, one can form the spinor quantity corresponding to it by using the connecting quantities $\sigma^\mu_{AA'}$ introduced by Infeld and Van der Waerden.[10] $\sigma^0_{AA'}$ here is the identity matrix and $\sigma^\mu_{AA'}$ for $\mu=1,2,3$ are the Pauli spin matrices. A vector V_μ corresponds to the two-index spinor $\sigma^\mu_{AA'}V_\mu$. Given the Maxwell tensor $F_{\mu\nu}$, it can be shown using the skew symmetry and the reality of $F_{\mu\nu}$ that the corresponding four-index spinor may be written

$$\sigma^\mu_{AA'}\sigma^\nu_{BB'}F_{\mu\nu}=\epsilon_{AB}\bar\phi_{A'B'}+\phi_{AB}\epsilon_{A'B'}\,, \tag{2.4}$$

where ϕ_{AB} is a symmetric spinor and $\bar\phi_{A'B'}=\phi^*_{AB}$. ϕ_{AB} is also the spinor corresponding to the anti-self-dual field $F_{\mu\nu}+i{}^*F_{\mu\nu}$, where ${}^*F^{\mu\nu}=\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$, i.e.,

$$\sigma^\mu_{AA'}\sigma^\nu_{BB'}(F_{\mu\nu}+i{}^*F_{\mu\nu})=2\phi_{AB}\epsilon_{A'B'}\,. \tag{2.5}$$

Since ϕ_{AB} is symmetric, it can also be shown that[7]

$$\phi_{AB}=\alpha_{(A}\beta_{B)} \tag{2.6}$$

for some spinors α_A, β_B and the parentheses denotes symmetrization.

The only invariant that can be formed from ϕ_{AB} is the complex invariant (or two real invariants) $\phi_{AB}\phi^{AB}$. It follows that only two real invariants can be formed from $F_{\mu\nu}$. A classification of $F_{\mu\nu}$ in terms of its invariants can be obtained by noticing that $\phi_{AB}\phi^{AB}=0$ if and only if α_A and β_A in (2.6) are proportional. In this case both the real invariants of $F_{\mu\nu}$ ($F_{\mu\nu}F^{\mu\nu}$ and $F_{\mu\nu}{}^*F^{\mu\nu}$) vanish. So an obvious classification of $F_{\mu\nu}$ is into the following two types:

Type I: General. No relationship between α_A and β_A.

Type N: $\phi_{AB}=\alpha_A\alpha_B$ for some spinor α_A. Both invariants vanish. The above classification also arises immediately if one considers what algebraic condition on the spinors α_A, β_A is invariant under the transformation group.

To generalize this classification to an arbitrary gauge field $F^i_{\mu\nu}$, $i=1,\ldots,N$ with gauge group G, consider the spinor ϕ^i_{AB} corresponding to $F^i_{\mu\nu}$. Analogous to (2.6),

$$\phi^i_{AB}=\alpha^{\hat i}_{(A}\beta^{\hat i}_{B)}\ \text{(no summation over } \hat i) \tag{2.7}$$

for some spinors $\alpha^{\hat i}_A$, $\beta^{\hat i}_B$, where the index $\hat i$ is just a label and is not a gauge index. As before, a classification may be obtained by considering algebraic relations among these spinors which are *invariant under the entire invariance group* which is now L × G. If the adjoint representation $\tilde{\mathrm G}$ is irreducible (which is the case if and only if G is simple), we then obtain the following classification[11]:

Type I: General.

Type A:

$$\phi^i_{AB}=\alpha_{(A}\beta^i_{B)},\quad i=1,\ldots,N \tag{2.8}$$

for some spinors α_A, β^i_A.

Type D:

$$\phi^i_{AB}=\phi^i\alpha_{(A}\beta_{B)},\quad i=1,\ldots,N \tag{2.9}$$

for some spinors α_A, β_B and a vector ϕ^i with N complex components.

Type N:

$$\phi^i_{AB}=\phi^i\alpha_A\alpha_B,\quad i=1,\ldots,N \tag{2.10}$$

for some spinor α_A and a vector ϕ^i.

The above classification is also valid if $\tilde{\mathrm G}$ is reducible but not fully reducible. But in this case a refinement is possible by placing more conditions on the spinors, which are invariant under the transformation group. If $\tilde{\mathrm G}$ is fully reducible then $F^i_{\mu\nu}$ may be regarded as consisting of several gauge

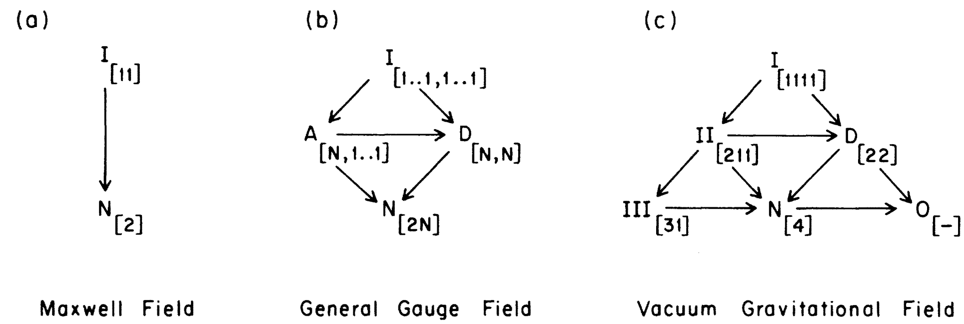

FIG. 1. A spinorial classification of a general gauge field (with an N-parameter gauge group) compared with the well-known classifications of the electromagnetic field into null and non-null types, and the Petrov-Pirani classification of the vacuum gravitational field. The partitions in square brackets represent the degeneracies of spinors, corresponding to each type. The arrows point in the direction of increasing specialization. (a) is a special case of (b) corresponding to $N=1$ for which I, A, and D become identical.

fields, each of which belongs to an irreducible or not fully reducible representation of G. Then each such field is subject to the above classification or its refinement as mentioned above. An example of $\tilde{G}$ being reducible is when G is Abelian. $\tilde{G}$ in this case consists only of the identity matrix and so the $F^i_{\mu\nu}$ may be regarded as N independent Maxwell fields which can be classified separately. (For the Maxwell field, types I, A, and D above may be regarded as identical).

This classification can also be obtained by considering an eigenvector problem which does not require any knowledge of spinor formalism. Notice first that if the Maxwell field $F_{\mu\nu}$ is of type N it has exactly one real null eigenvector corresponding to eigenvalue zero. So for more general gauge fields it is natural to investigate the eigenvector equation

$$F^{i\mu}{}_{\nu}V^{\nu}=\lambda^i V^{\mu}. \tag{2.11}$$

We notice now that a null eigenvector V^{μ} corresponds to a common spinor of ϕ^i_{AB}. So we obtain a classification identical to the one above in the following way:

Type I: no real null eigenvector in general.

Type A: at least one real null eigenvector. $\lambda^i \neq 0$ in general.

Type D: two real null eigenvectors, distinct in general. $\lambda^i \neq 0$ in general.

Type N: One real null eigenvector. $\lambda^i=0$.

Figure 1 compares the above classification with the Petrov-Pirani classification of the vaccum gravitational field and the classification of the Maxwell field, by means of Penrose diagrams. We shall see in Sec. III that the three types of Yang-Mills field that occur in the asymptotic limit considered by Roskies,[8] are just the types I, A, and N above[12] for this special case. The situation here is analogous to the Petrov-Pirani classification where also the asymptotic limit does not give type D, which may be obtained instead through the spinor method. Type-D fields have physical relevance, since any time-symmetric field which is algebrically special must be type D. Because if such a field has one common null eigenvector [see (2.11)] then by time symmetry it must have two. This will be the case also if the field is symmetric with respect to a spacelike direction.

Our focus so far has been on gauge fields defined in Minkoswki space-time. Indeed (2.4) is valid only if $F_{\mu\nu}$ is a real field in Minkowski space-time, and the self-dual field corresponding to ϕ_{AB} as seen from (2.5) must then be complex. But it is possible to define $F^i_{\mu\nu}$ and ϕ^i_{AB} in the four-dimensional Euclidean space E_4. The self-dual field can then be real as in the instanton solutions.[13] Since the classification obtained in this section was in terms of ϕ^i_{AB}, it covers all these cases, i.e., it is a classification of $F^i_{\mu\nu}$ and the self-dual field both in Minkowski space-time and E_4. This is also true of the refinement of this classification that will be obtained in the next section by considering a matrix L of Lorentz invariants.

III. ON THE INVARIANTS OF GAUGE FIELDS

Let $F^i_{\mu\nu}$, $i=1,\ldots,N$ be the gauge field corresponding to an N-parameter gauge group G. An invariant of $F^i_{\mu\nu}$ is defined to be an algebraic function of $F^i_{\mu\nu}$ which is a scalar under L $\times$G. A systematic study of the invariants of $F^i_{\mu\nu}$ was done perhaps for the first time by Roskies[8] for the special case of an SU(2) gauge field. In this section we shall attempt to extend this analysis to more general gauge fields. For the purposes of this investigation it is useful to define the matrix

$$L^{ij}=2\phi^{iB}_{A}\phi^{jA}_{B}, \tag{3.1}$$

where ϕ^i_{AB} is the spinor corresponding to $F^i_{\mu\nu}$ defined in (2.4). It follows from (2.5) that

$$\phi^i_{AB}\phi^{jAB} = \tfrac{1}{8}(F^i_{\mu\nu} + i{}^*F^i_{\mu\nu})(F^{j\mu\nu} + i{}^*F^{j\mu\nu}).$$

Therefore,

$$L^{ij} = K^{ij} + iJ^{ij}, \tag{3.2}$$

where

$$K^{ij} = \tfrac{1}{2}F^i_{\mu\nu}F^{j\nu\mu}$$

and (3.3)

$$J^{ij} = \tfrac{1}{2}F^i_{\mu\nu}{}^*F^{j\nu\mu}.$$

The $N\times N$ matrices J, K, and L are all symmetric.

We shall now obtain the number of independent invariants that can be formed from a gauge field for which the adjoint representation of the gauge group

$$\tilde{G} \subseteq O(N), \tag{3.4}$$

where $O(N)$ is the orthogonal group in N dimensions. Since the arguments are rather subtle, the reader may omit them during a first reading. (3.4) includes all cases for which the gauge group is compact. For instance it includes the usual gauge fields such as the Yang-Mills field, for which $G = SU(N)$ where N is any positive integer. It is convenient to consider two cases.

Case a: $\tilde{G} = \{I\}$ where I is the $N\times N$ identity matrix. G is then an Abelian group and $F^i_{\mu\nu}$ may be regarded as N Maxwell fields. Of their $6N$ components, when $N = 1$ four components and when $N \geq 2$ six components, can always be made to vanish, using the freedom of the six-parameter group of Lorentz transformations.[8] This can be seen from the fact that $F^1_{\mu\nu}$ considered as a Maxwell field has a canonical Lorentz frame in which it has at most two nonzero components. In such a frame, four components of $F^1_{\mu\nu}$ are zero. We have however a two-parameter subgroup of the Lorentz group that will leave the remaining two components invariant. For instance if $F^1_{\mu\nu}$ is nonradiative we can go to a frame where its "$\vec{E}$ field" and the "$\vec{B}$ field" are parallel to the x axis. We can then rotate around or translate along the x axis while maintaining this situation. When $N \geq 2$, using this two-parameter freedom, two of the components of $F^2_{\mu\nu}$ can be made to vanish and we have no more freedom left. Therefore, in this canonical frame, six components always vanish when $N \geq 2$. Hence $6N - 6$ independent invariants can be formed from the $6N - 6$ independent components in this frame.

To summarize, for case a, the number of independent invariants are two when $N = 1$ and $6N - 6$ when $N \geq 2$.

Case b: $\{I\} \subset \tilde{G} \subseteq O(N)$. Let $\tilde{N}$ be the dimension of $\tilde{G}$ regarded as a Lie group. Then $\tilde{N} \leq N$. We shall now show that $6N - 6 - \tilde{N}$ independent invariants can be formed from $F^i_{\mu\nu}$.

Since $\tilde{G} \neq \{I\}$ by assumption, $N \neq 1$. Also it can be shown easily that $\tilde{G} \neq O(2)$ for any Lie group G. Hence $N \geq 3$. As in case a, in any given gauge, we can go to the canonical Lorentz frame in which 6 of the $6N$ components of $F^i_{\mu\nu}$ vanish. Consider now the tensor K^{ij} defined in (3.3). Since K^{ij} are Lorentz scalars we can use them to study the effect of G on $F^i_{\mu\nu}$ independently of the effect of L. Under a gauge transformation K transforms as

$$K \to GKG^T, \quad G \in \tilde{G} \subseteq O(N).$$

Since K is real symmetric, there exists $O \in O(N)$ such that OKO^T is diagonal.

Consider the set of all real $N\times N$ diagonal matrices K_D which are inequivalent (under similarity transformations) with respect to $O(N)$. These matrices are then inequivalent with respect to the subgroup $\tilde{G}$. Now let $\mathcal{G}$ be the Lie algebra of $\tilde{G}$ and $g_1, \ldots, g_{\tilde{N}}$ a basis for $\mathcal{G}$. Extend this to a basis $g_1, \ldots, g_{\tilde{N}}, h_{\tilde{N}+1}, \ldots, h_{N(N-1)/2}$ of the Lie algebra of $O(N)$. Let $\mathcal{H}$ be the subspace spanned by $h_{\tilde{N}+1}, \ldots, h_{N(N-1)/2}$. Then

$$\mathcal{G} \cap \mathcal{H} = \{0\}. \tag{3.5}$$

We shall restrict our considerations to infinitesimal transformations since this restriction does not alter the number of independent invariants. Also we consider the general case when the elements of each diagonal matrix K are unequal. The set

$$S = \{HK_DH^T;\quad H = e^h,\ h \in \mathcal{H}\}$$

is then easily shown to be the set of inequivalent matrices under $\tilde{G}$. For if $HK_DH^T = GH'K_DH'^TG^T$ ($G \in \tilde{G}$, $H = I + h$, $H' = I + h'$, $h, h' \in \mathcal{H}$), i.e., $H'^TG^THK_D = K_DH'^TG^TH$, then since the elements of K_D are unequal, $H'^TG^TH = I$. Therefore, $h'^T + g^T + h = 0$ or $g = h - h'$ which contradicts (3.5). The set of matrices K_D depend on N parameters. Therefore, the matrices belonging to S depend on $N + \frac{1}{2}N(N-1) - \tilde{N} = \frac{1}{2}N(N+1) - \tilde{N}$ parameters. But the set of all real symmetric matrices K depend on $\frac{1}{2}N(N+1)$ parameters. Hence for the set S of inequivalent K matrices under $\tilde{G}$, there must be $\tilde{N}$ independent relations on the components of K. These are also $\tilde{N}$ independent relations on the $6N - 6$ nonzero components of $F^i_{\mu\nu}$ in the canonical Lorentz frame mentioned above. Hence the set of fields inequivalent under $L\times G$ depend on $6N - 6 - \tilde{N}$ parameters. Therefore $6N - 6 - \tilde{N}$ independent invariants can be formed from $F^i_{\mu\nu}$ in case b.

Consider now the special case $G = SU(2)$ or $O(3)$. In this case $\tilde{N} = N = 3$ and the number of independent invariants that can be formed from $F^i_{\mu\nu}$ is $6N - 6 - \tilde{N} = 9$. A set of nine independent invariants in terms of the matrix L defined in (3.1) for this case is

$$\mathrm{Tr}L,\ \mathrm{Tr}(L^2),\ \mathrm{Tr}(L^3),\ \mathrm{Tr}(L\bar{L}),\ \mathrm{Tr}(L^2\bar{L}). \tag{3.6}$$

Since $\mathrm{Tr}(L\bar{L})$ is real and all other terms are complex, there are altogether nine real independent invariants in (3.6). By the Cayley-Hamilton theorem

$$L^3-(\mathrm{Tr}L)L^2+\tfrac{1}{2}[(\mathrm{Tr}L)^2-\mathrm{Tr}(L^2)]L-(\det L)I=0. \tag{3.7}$$

On taking the trace of (3.7) one obtains

$$\det L=\tfrac{1}{3}\mathrm{Tr}(L^3)-\tfrac{1}{2}(\mathrm{Tr}\,L)\mathrm{Tr}(L^2)+\tfrac{1}{6}(\mathrm{Tr}\,L)^3. \tag{3.8}$$

Hence $\det L$ is not independent of the invariants (3.6) since it can be expressed in terms of them. It may appear at first sight that the invariant

$$\begin{aligned}\tau&=\tfrac{2}{3}\epsilon_{ijk}\phi_A^{iB}\phi_B^{jC}\phi_C^{kA}\\&=t+it',\end{aligned} \tag{3.9}$$

where[8]

$$\begin{aligned}t&=\tfrac{1}{6}\epsilon_{ijk}F^{i\nu}_{\mu}F^{j\rho}_{\nu}F^{k\mu}_{\rho},\\t'&=-\tfrac{1}{6}\epsilon_{ijk}{}^*F^{i\nu}_{\mu}{}^*F^{j\rho}_{\nu}{}^*F^{k\mu}_{\rho},\end{aligned} \tag{3.10}$$

is independent of (3.6). But it can be shown using (3.7), (3.8) of Ref. 8 and (3.8), (3.14) of the present paper that

$$\tau^2=-\det L. \tag{3.11}$$

Hence because of (3.8) τ is not independent of (3.6). We can then replace $\mathrm{Tr}\,L^3$ in (3.6) by τ and obtain as a set of independent invariants

$$\tau,\mathrm{Tr},L,\mathrm{Tr}(L^2),\mathrm{Tr}(L\bar{L}),\mathrm{Tr}(L^2\bar{L}). \tag{3.12}$$

(3.12) and their complex conjugates are equivalent to Roskies's fundamental invariants,

$$t,t',\mathrm{Tr}\,J,\mathrm{Tr}(J^2),\mathrm{Tr}K,\mathrm{Tr}(K^2),\mathrm{Tr}(JK),\det J,\det K. \tag{3.13}$$

This can be seen from the fact that the real and imaginary parts of (3.12) give the first six invariants of (3.13) and $\mathrm{Tr}(K^3+J^2K)$, $\mathrm{Tr}(J^3+K^2J)$. But the last two terms can be replaced by $\det J$ and $\det K$ because of (3.7), (3.8) of Ref. 8 and the following relations obtained from the Cayley-Hamilton theorem:

$$\begin{aligned}\mathrm{Tr}(J^3)&=3\det J+\tfrac{3}{2}\mathrm{Tr}\,J\,\mathrm{Tr}(J^2)-\tfrac{1}{2}(\mathrm{Tr}\,J)^3,\\\mathrm{Tr}(K^3)&=3\det K+\tfrac{3}{2}\mathrm{Tr}\,K\,\mathrm{Tr}(K^2)-\tfrac{1}{2}(\mathrm{Tr}\,K)^3.\end{aligned} \tag{3.14}$$

We now investigate what characterizes the L matrix for the four types in the spinor classification of Sec. II for the Yang-Mills field. This relates the spinor classification to Roskies's classification in terms of invariants for this case.

Type I: L is general; no relations among the invariants.

Type A or D: In this case we can write

$$L^{ij}=\alpha^i\alpha^j, \tag{3.15}$$

where

$$\alpha^i=\alpha_A\,\beta^{iA}.$$

Thus L has rank 1. Conversely, if L which is symmetric has rank 1 then its elements have the form of (3.15) and it is a simple matter to prove that the field is of type A or D. These two types cannot be distinguished on the basis of the L matrix alone. However, they can be distinguished by the fact that type A corresponds to two linearly independent ϕ^i_{AB} while type D has only one linearly independent ϕ^i_{AB}.

The statement that L is of rank 1 implies the six conditions on the invariants which defines Roskies's type II [(3.18)–(3.20) below]. To show this notice that if L has rank 1 then it has two zero eigenvalues, which means that from (3.7)

$$\det L=0, \tag{3.16}$$

$$\mathrm{Tr}(L^2)=(\mathrm{Tr}\,L)^2. \tag{3.17}$$

From (3.11) it follows that (3.16) is equivalent to

$$t=t'=0 \tag{3.18}$$

and from (3.2), (3.17) is equivalent to

$$\begin{aligned}&\mathrm{Tr}(J^2)-(\mathrm{Tr}\,J)^2=\mathrm{Tr}(K^2)-(\mathrm{Tr}\,K)^2,\\&\mathrm{Tr}(JK)=\mathrm{Tr}\,J\,\mathrm{Tr}\,K.\end{aligned} \tag{3.19}$$

Also, since L is of rank 1, $K=\tfrac{1}{2}(L+\bar{L})$ and $J=(1/2i)(L-\bar{L})$ can have at most rank 2. Therefore,

$$\det J=\det K=0. \tag{3.20}$$

However (3.18)–(3.20) do not imply that L is of rank 1, so that Roskies's type II includes our type A as a proper subset.[14]

Type N: Here L is the zero matrix; thus all the invariants vanish. Conversely, if all the invariants vanish then the matrix L must be zero and the field must be type N. (See the appendix and also Ref. 8.) This proves that our type N and Roskies's type I are identical.

The classification in Sec. II was obtained entirely by considering invariant conditions on irreducible spinors. But the important role played by the matrix L in the above discussion of the Yang-Mills field calls for an investigation of the classification of gauge fields in terms of the L matrix. For the Maxwell field L is a 1×1 matrix and $L=0$ for the null field and $L \neq 0$ if the field is non-null. Hence, the only classifications of gauge fields based on the L matrix that will give the known classification of Maxwell field as a special case is a classification according to the rank of L or according to the number of zero eigenvalues of L. The eigenvalues of L are invariant under the invariance group $\mathrm{L} \times \mathrm{G}$ if and only if G satisfies (3.4). But the rank of L is invariant regardless of the nature of G. We shall therefore obtain a classification of all gauge fields based on the rank of L keeping in mind that a further refinement is possible according the number of zero eigenvalues when G satisfies (3.4), which is the case for the Yang-Mills field for instance. Also we shall take into account the number of linearly independent ϕ^i_{AB} with respect to complex coefficients, in obtaining the classification, which as we saw earlier was needed to distinguish between types A and D.

Since ϕ^i_{AB} is symmetric in A, B, there cannot be more than three linearly independent ϕ^i_{AB} for given i. Consequently, for any gauge field, the rank of L cannot be more than 3. This gives a classification of gauge fields into five nontrivial types, the four given in Sec. II and a fifth which we shall call type II corresponding two linearly independent ϕ^i_{AB} and L being of rank 2. We illustrate this scheme in Fig. 2. The proof for the nonexistent types in Fig. 2 is given in the Appendix. Since L is given by (3.2) and ϕ^i_{AB} corresponds to $F^i_{\mu\nu} + i^*F^i_{\mu\nu}$, this classification can also be described in terms of $F^i_{\mu\nu}$ without the need for the spinor formalism. For the Yang-Mills field, type II is given by the vanishing of $\det L$, with no other relations among the invariants.

We conclude this section with some remarks about the peeling theorem which illustrate the usefulness of the eigenvector problem (2.11). Roskies has shown that the asymptotic form of the Yang-Mills field is

$$\phi^i_{AB} = \frac{\phi^{(1)i}_{AB}}{r} + \frac{\phi^{(2)i}_{AB}}{r^2} + \frac{\phi^{(3)i}_{AB}}{r^3}, \tag{3.21}$$

where for $\phi^{(1)i}_{AB}$ all the invariants vanish and $\phi^{(3)i}_{AB}$ is general. Hence they must be of types N and I in our spinoral classification. For this to be analogous to the peeling theorem of general relatively, it is also necessary to prove that $\phi^{(2)i}_{AB}$ is of type A and moreover the repeated spinor of $\phi^{(1)i}_{AB}$ and the common spinor of the three $\phi^{(2)i}_{AB}$ are the same and are tangent to the radially outgoing null direction. This can be proved by using conditions (4.6) and (4.8) of Ref. 8, Eq. (2.11) of the present paper, and the remarks which follow this equation. With these observations, (3.21) may be stated as a peeling theorem for the Yang-Mills fields: The asymptotic form of the Yang-Mills field is

$$\phi^i_{AB} = \frac{1}{r}\phi^i\alpha_A\alpha_B + \frac{1}{r^2}\alpha_{(A}\beta^i_{B)} + \frac{1}{r^3} I^i_{AB}, \tag{3.22}$$

where α_A is tangent to the radially outgoing null direction.

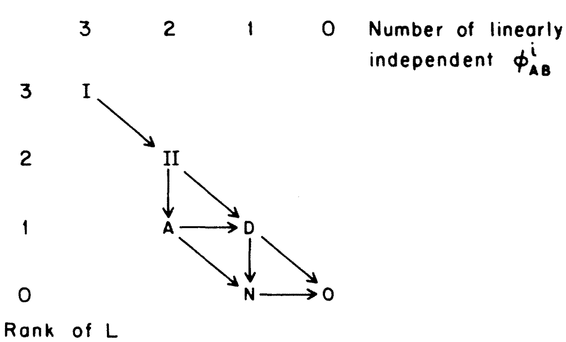

FIG. 2. The classification of a general gauge field using the matrix L. A refinement of Fig. 1(b) is obtained, with type II as the extra case. Type O is the trivial case for which the field is zero. ϕ^i_{AB} can also be replaced by $F^i_{\mu\nu} + i^*F^i_{\mu\nu}$.

IV. THE GEOMETRIC OPTICS LIMIT

In this section, we show how the methods of geometric optics applied to gauge fields lead to the classification in terms of types I, A, and N obtained in Sec. II. This may be regarded as a further indication of the physical significance of this classification. For simplicity we shall consider first the Maxwell field.

In geometric optics one considers special wavelike solutions of the field equations for which the amplitude and polarization are slowly varying in comparison with the phase. In other words one considers the limit when the dimensionless parameter $\epsilon = \lambda/L$ approaches zero, where λ is the average wavelength and L is the typical length over which the amplitude and the polarization vary.[15] This can be mathematically accomplished by writing the electromagnetic vector potential A^μ as $\mathrm{Re}(a_\mu e^{i\phi/\epsilon})$ and requiring that each succeeding derivative of a_μ is smaller than the preceding one by an order of magnitude ϵ. The latter requirement is a generalization of the requirement that a_μ be constant. Performing then a Taylor expansion of a_μ around any space-time point x, we can write

$$A_\mu = \mathrm{Re}[(a^{(0)}_\mu + \epsilon a^{(1)}_\mu + \epsilon^2 a^{(2)}_\mu + \cdots)e^{i\phi/\epsilon}], \tag{4.1}$$

where ϵ is a small constant. By the property of the Taylor expansion,

$$a^{(0)}_{\mu,\nu} = 0, \quad a^{(1)}_{\mu,\nu\rho} = 0, \quad a^{(2)}_{\mu,\nu\rho\sigma} = 0, \ldots . \tag{4.2}$$

On substituting (4.1) in the source-free Maxwell's equations

$$\Box A_\mu - \nabla_\mu(\nabla^\nu A_\nu) = 0\,, \tag{4.3}$$

and using (4.2), one obtains on equating to zero the coefficient of $1/\epsilon^2$,

$$-(k_\nu k^\nu)a^{(0)\mu} + (k^\nu a_\nu^{(0)})k^\mu = 0\,, \tag{4.4}$$

where

$$k_\mu = -\nabla_\mu \phi\,. \tag{4.5}$$

Now the Maxwell field $F_{\mu\nu} = A_{[\mu,\nu]}$ can be written using (4.1), (4.2), and (4.5) as

$$F_{\mu\nu} = \frac{1}{\epsilon} F_{\mu\nu}^{(0)} + F_{\mu\nu}^{(1)} + \epsilon F_{\mu\nu}^{(2)} + \cdots\,, \tag{4.6}$$

where

$$\begin{aligned} F_{\mu\nu}^{(0)} &= -\mathrm{Re}(ia_{[\mu}^{(0)}k_{\nu]}e^{i\phi/\epsilon})\,,\\ F_{\mu\nu}^{(1)} &= -\mathrm{Re}(ia_{[\mu}^{(1)}k_{\nu]}e^{i\phi/\epsilon})\,,\\ F_{\mu\nu}^{(2)} &= -\mathrm{Re}[(ia_{[\mu}^{(2)}k_{\nu]} - a_{[\mu,\nu]}^{(1)})e^{i\phi/\epsilon}]\,. \end{aligned} \tag{4.7}$$

But from (4.4) either $a^{(0)\mu} \propto k^\mu$ or $k_\nu k^\nu = 0$. In the former case $F_{\mu\nu}^{(0)} = 0$ and in the latter case $k^\nu a_\nu^{(0)} = 0$. Hence $F_{\mu\nu}^{(0)}$ is of type N. $F_{\mu\nu}^{(1)}$, $F_{\mu\nu}^{(2)}, \ldots$ are all of type I, but $F_{\mu\nu}^{(1)}$ differs from the succeeding ones in that it always has k^μ as an eigenvector corresponding to an eigenvalue which is in general nonzero.

We apply now essentially the same procedure to any gauge field corresponding to a gauge group G. The field equations in the absence of sources are then

$$F^{i\mu\nu}{}_{,\nu} + C_{ijk}A_\nu^j F^{k\mu\nu} = 0\,, \tag{4.8}$$

where C_{ijk} are the structure constants of G and

$$F_{\mu\nu}^i = A_{[\mu,\nu]}^i + C_{ijk}A_\mu^j A_\nu^k\,. \tag{4.9}$$

We expand

$$A_\mu^i = \mathrm{Re}[(a_\mu^{i(0)} + \epsilon a_\mu^{i(1)} + \epsilon^2 a_\mu^{i(2)} \cdots)e^{i\phi/\epsilon}]\,, \tag{4.10}$$

where $a_\mu^{i(r)}, r = 0, 1, 2, \ldots$ satisfy

$$a_{\mu,\nu}^{i(0)} = 0, \quad a_{\mu,\nu\rho}^{i(1)} = 0, \quad a_{\mu,\nu\rho\sigma}^{i(2)} = 0, \ldots\,. \tag{4.11}$$

Substitute (4.10) into (4.8) and using (4.5) obtain to the lowest order in ϵ,

$$-(k_\nu k^\nu)a_\mu^{i(0)} + (k^\nu a_\nu^{i(0)})k_\mu = 0\,. \tag{4.12}$$

From (4.9), (4.10), and (4.11),

$$F_{\mu\nu}^i = \frac{1}{\epsilon} F_{\mu\nu}^{i(0)} + F_{\mu\nu}^{i(1)} + \epsilon F_{\mu\nu}^{i(2)} + \cdots\,, \tag{4.13}$$

where

$$\begin{aligned} F_{\mu\nu}^{i(0)} &= -\mathrm{Re}(ia_{[\mu}^{i(0)}k_{\nu]}e^{i\phi/\epsilon})\,,\\ F_{\mu\nu}^{i(1)} &= -\mathrm{Re}[(a_{[\mu}^{i(1)}k_{\nu]}\\ &\qquad + C_{ijk}a_\mu^{j(0)}a_\nu^{k(0)})e^{i\phi/\epsilon}]\,. \end{aligned} \tag{4.14}$$

It follows from (4.12) that either $a_\mu^{i(0)} \propto k_\mu$ or $k_\nu k^\nu = 0 = k^\nu a_\nu^{i(0)}$. Therefore, for $F_{\mu\nu}^{i(0)}$ the J and K matrices defined in (3.3) are both zero. Hence $F_{\mu\nu}^{i(0)}$ is of type N. Also it is easily checked that when $F_{\mu\nu}^{i(0)} \neq 0$, $F_\nu^{i\mu(1)}$ has the null vector k^μ as the common eigenvector corresponding to a nonzero eigenvalue in general. It follows from the remarks after (2.11) that $F_{\mu\nu}^{i(1)}$ is of type A. It can also be verified that $F_{\mu\nu}^{i(2)}$ and all other succeeding fields in (4.13) are of the general type I. Hence the geometric optics limit provides a classification of all gauge fields into types I, A, and N. Also we have proved an analog of the peeling theorem for geometric optics which in term of spinors, with obvious notation is

$$\phi_{AB}^i = \frac{1}{\epsilon}\phi^i\alpha_A\alpha_B + \alpha_{(A}\beta_{B)}^i + \epsilon I_{AB}^i\,. \tag{4.15}$$

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We wish to thank Ralph Roskies for drawing our attention to this problem and for useful discussions. The work of K.P.T. was supported in part by a National Science Foundation Grant.

APPENDIX: PROOF OF THE NONEXISTENCE OF THE TYPES OMITTED IN FIG. 2

Let $\phi_{AB}^i, i = 1, \ldots, N$ represent an arbitrary gauge field and L^{ij} the $N \times N$ matrix defined in (3.1). Then clearly

$$\phi_{AB}^i X^i = 0 \Rightarrow L^{ij}X^j = 0\,, \tag{A1}$$

where X^i are N complex numbers. Therefore since at most three ϕ_{AB}^i can be linearly independent, rank $L \leq 3$. The fact that there are no types above the diagonal in Fig. 2 also follows from the implication (A1).

If rank $L \leq 2$ then there exists a nonsingular $N \times N$ matrix P such that $L' \equiv PLP^T$ satisfies $L^{11'} = L^{12'} = L^{13'} = 0$. We can write $L^{i'j'} = 2\phi_A^{i'B}\phi_B^{j'A}$, where $\phi_{AB}^{i'} = P^{ij}\phi_{AB}^j$. Then $\phi_{AB}^{1'} = \alpha_A\alpha_B$, $\phi_{AB}^{2'} = \alpha_{(A}\beta_{B)}$, $\phi_{AB}^{3'} = \alpha_{(A}\gamma_{B)}$ for some spinors $\alpha_A, \beta_A, \gamma_A$. Since at most two of the spinors $\alpha_A, \beta_A, \gamma_A$ can be linearly independent, at most two $\phi_{AB}^{i'}$ and consequently at most two ϕ_{AB}^i can be linearly independent.

Finally we note that rank $L = 0$ is equivalent to $L^{ij} = 0$. Then ϕ_{AB}^i must be of the form $\chi^i\alpha_A\alpha_B$. Hence it is not possible to have more than one linearly independent ϕ_{AB}^i in this case.

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[1]C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954).

[2]J. L. Synge, *Relativity: The Special Theory* (North-Holland, Amsterdam, 1958), Chap. 9.

[3]A. Z. Petrov, Sci. Not. Kazan Univ. 114, 55 (1954).

[4]F. A. E. Pirani, Phys. Rev. 105, 1089 (1957).

[5]L. Witten, Phys. Rev. 115, 206 (1959).

[6]R. Penrose, Ann. Phys. (N.Y.) 10, 171 (1960).

[7]See also F. A. E. Pirani, in *Lectures on General Relativity, Brandeis Summer Institute in Theoretical Physics*, edited by S. Deser and K. W. Ford (Prentice-Hall, Englewood Cliffs, N.J., 1965), Vol. I, Chap. 3.

[8]R. Roskies, Phys. Rev. D 15, 1722 (1977).

[9]For a more detailed treatment of spinors, see for instance Ref. 7.

[10]L. Infeld and B. L. Van der Waerden, Sb. Preuss, Akad. Wiss. Phys.—Mat. Kl. 380 (1933).

[11]An approach similar to ours has been independently adopted for the special case of the SU(2) gauge field by M. Carmeli in a recent letter—Phys. Rev. Lett. 39, 523 (1977). However, there are some errors in this letter which will be pointed out elsewhere: J. Anandan and R. Roskies, following paper, Phys. Rev. D 18, 1152 (1978).

[12]Roskies's type I is our type N in this special case of the Yang-Mills field. We have changed his notation so as to show the similarity with the Petrov-Pirani classification. (See Fig. 1.)

[13]A. A. Belavin, A. M. Polyakov, A. S. Schwartz, and Yu. S. Tyupkin, Phys. Lett. 59B, 85 (1975).

[14]For instance, the field defined by $\phi^1_{AB}=\alpha_A\alpha_B$, $\phi^2_{AB}=-(1+i)\beta_A\beta_B$, and $\phi^3_{AB}=-i\alpha_A\alpha_B$ with $\alpha_A\beta^A=1$, for which

$$L=2\begin{bmatrix}0 & 1+i & 0\\ 1+i & 0 & 1-i\\ 0 & 1-i & 0\end{bmatrix},$$

is not type A but is Roskies's type II.

[15]For a very similar but not identical approach to geometric optics, see for instance C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1970), Sec. 22.5.

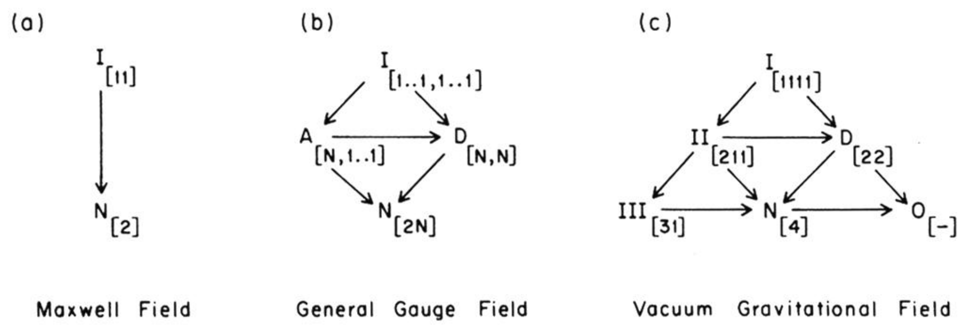

FIG. 1. A spinorial classification of a general gauge field (with an N-parameter gauge group) compared with the well-known classifications of the electromagnetic field into null and non-null types, and the Petrov-Pirani classification of the vacuum gravitational field. The partitions in square brackets represent the degeneracies of spinors, corresponding to each type. The arrows point in the direction of increasing specialization. (a) is a special case of (b) corresponding to $N=1$ for which I, A, and D become identical.

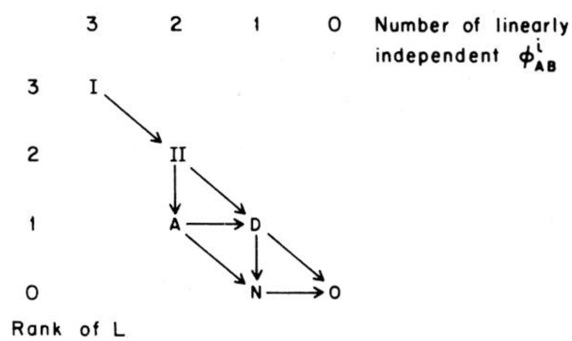

FIG. 2. The classification of a general gauge field using the matrix L. A refinement of Fig. 1(b) is obtained, with type II as the extra case. Type O is the trivial case for which the field is zero. $\phi_{A}{}^{i}{}_{B}$ can also be replaced by $F_{\mu\nu}^{\ i} + i{}^{*}F_{\mu\nu}^{\ i}$.