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Quantization of free field theory of massless antisymmetric tensor gauge fields of second rank

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We present the quantization program of the free field theory of massless antisymmetric tensor gauge fields of second rank introduced by Kalb and Ramond as carrying the interaction between closed relativistic strings. This has been done both on the constant-time surface and also on the null plane. Dirac's method of quantization for constrained Hamiltonian systems has been invoked.

I. INTRODUCTION

The study of the string picture of dual-resonance models has given birth to a large number of interesting problems. One of these is the action-at-a-distance theory of interacting strings[1,2] obtained in the manner of the action-at-a-distance formalism of interacting point particles.[3] The generalization from the point-particle mechanics to the string mechanics is ensured by the correspondence between the line element of the world line of the point particle and the surface element of the world sheet traversed by the string in time. In order to develop consistently both the Lagrangian and Hamiltonian formulations of the action-at-a-distance theory of interacting strings, we need to make a common time identification for all the strings in the same manner as that for the interacting point particles.[3] This is achieved by exploiting the parametrization invariance of the postulated free and interstring actions.[2]

The nature of the interstring forces, as shown by Kalb and Ramond, depends on the type of strings involved.[1] For the closed strings, which satisfy Lorentz-type equations of motion for the string bodies, these forces are mediated between the strings by a massless scalar field. On the other hand, for the open strings, the interaction is mediated by a massless vector field if we consider the end-point interactions and ignore the body interactions; whereas the presence of both body and end-point interactions requires a massive pseudovector field as the mediating field.

We shall consider here only the field that is responsible for the interaction between the closed strings. In analogy to the electromagnetic gauge field A_μ which corresponds to interactions between point particles, we have an antisymmetric gauge field $A_{\mu\nu}$ that mediates between the closed strings. The free field theory for these two fields can be obtained from the Lagrangian densities $\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ and $\frac{1}{12}F_{\mu\nu\alpha}F^{\mu\nu\alpha}$, respectively, where $F_{\mu\nu}$ is the usual antisymmetric field tensor of the electromagnetics and $F_{\mu\nu\alpha}$ can be given an analogical definition in terms of $A_{\mu\nu}$.[1,2,4] In a similar fashion, we may define the field $A_{\mu\nu\alpha}$ for the interaction between the closed shells and the corresponding field tensor $F_{\mu\nu\alpha\beta}$.[4] There are gauge invariances in all these cases.

In the following we shall study the quantization by Dirac's method[5] for the free field theory of $A_{\mu\nu}$. The corresponding problem for the electromagnetic gauge field has been done by Dirac, Anderson, and Bergmann, and Kundt.[6] Here we shall follow the program in close analogy to this electromagnetic gauge field theory as presented in the review by Hanson, Regge, and Teitelboim.[5]

The paper is divided into two parts. In Sec. II we shall describe the dynamics on constant-time surfaces and in Sec. III we shall describe the dynamics on the null plane.

Describing the dynamics on the constant-time surface, in Sec. II A we shall work out the Hamiltonian without the gauge constraints for our free field theory. In Sec. II B, choosing the radiation gauge, we shall work out the constant-time Dirac brackets of the canonical field variables. Section III C contains a discussion of quantization. Describing the dynamics on the null plane, in Sec. III A, we shall develop the Hamiltonian without gauge constraints. Choosing a null-plane radiation gauge, we shall work out the null-plane Dirac brackets in Sec. III B. Quantization will be discussed in Sec. III C. Section IV will contain some concluding remarks.

II. DYNAMICS ON CONSTANT-TIME SURFACE

Here we shall describe the dynamics of massless tensor gauge fields of second rank on constant-time surfaces.

A. Hamiltonian without gauge constraints

We shall start with the Lagrangian density as[1,4]

$$\mathcal{L} = -\tfrac{1}{12}F^{\alpha\mu\nu}F_{\alpha\mu\nu}, \tag{2.1}$$

where the antisymmetric field tensor is defined in terms of the antisymmetric tensor potentials $A^{\mu\nu}(x)$ as

$$F^{\mu\nu\alpha}(x)=\frac{\partial A^{\nu\alpha}(x)}{\partial x_\mu}+\frac{\partial A^{\alpha\mu}(x)}{\partial x_\nu}+\frac{\partial A^{\mu\nu}(x)}{\partial x_\alpha}. \qquad (2.2)$$

The Lagrangian density defined above is invariant under the gauge transformations

$$A^{\mu\nu}(x)\to A^{\mu\nu}(x)+\partial^\mu\Lambda^\nu(x)-\partial^\nu\Lambda^\mu(x). \qquad (2.3)$$

The Euler-Lagrangian equations of motion that follow from this Lagrangian are

$$\partial_\mu F^{\mu\nu\alpha}(x)=0. \qquad (2.4)$$

In order to write the Hamiltonian for our gauge field, we define the canonical momenta conjugate to $A_{\mu\nu}$ as

$$\Pi^{\mu\nu}(x)=\frac{\partial\mathcal{L}}{\partial\dot{A}_{\mu\nu}}=-F^{0\mu\nu}(x)=-\Pi^{\nu\mu}(x), \qquad (2.5)$$

where we intend to describe the dynamics on the constant-time surfaces. We note that this gives us weakly vanishing Π^{0i}, $i=1,2,3$, due to the antisymmetry of $F^{\alpha\mu\nu}$. Thus we have the primary constraints as

$$\Pi^{0i}\approx 0, \quad i=1,2,3. \qquad (2.6)$$

These constraints, along with the Euler-Lagrangian equations (2.4), imply another set of weak equations

$$\partial_i\Pi^{ij}\approx 0. \qquad (2.7)$$

These in fact are the secondary constraints, as will be shown later.

The canonical Poisson brackets between $\Pi^{\mu\nu}$ and $A^{\mu\nu}$ can be written as

$$[\Pi^{\mu\nu}(\vec{x},t),A^{\alpha\beta}(\vec{y},t)]_P = -(g^{\mu\alpha}g^{\nu\beta}-g^{\nu\alpha}g^{\mu\beta})\delta^3(\vec{x}-\vec{y}). \qquad (2.8)$$

The other Poisson brackets are zero. These, as expected, are incompatible with the primary constraints (2.6) and also Eqs. (2.7).

In order to obtain the secondary constraints, we need to know the canonical Hamiltonian. To that effect, we first write down the canonical conserved currents:

$$T^{\mu\nu}=-\frac{1}{2}\frac{\partial\mathcal{L}}{\partial A_{\alpha\beta,\mu}}A_{\alpha\beta}{}^{,\nu}+g^{\mu\nu}\mathcal{L}$$
$$=\tfrac{1}{2}F^{\mu\alpha\beta}A_{\alpha\beta}{}^{,\nu}-\tfrac{1}{12}g^{\mu\nu}F^{\alpha\beta\delta}F_{\alpha\beta\delta} \qquad (2.9)$$

and

$$\mathfrak{M}^{\alpha\mu\nu}=x^\mu T^{\alpha\nu}-x^\nu T^{\alpha\mu}-\frac{\partial\mathcal{L}}{\partial A_{\mu\delta,\alpha}}A^\nu{}_\delta+\frac{\partial\mathcal{L}}{\partial A_{\nu\delta,\alpha}}A^\mu{}_\delta$$
$$=x^\mu T^{\alpha\nu}-x^\nu T^{\alpha\mu}+F^{\alpha\mu\delta}A^\nu{}_\delta-F^{\alpha\nu\delta}A^\mu{}_\delta, \qquad (2.10)$$

where we can easily verify, using equations of motion, that

$$\partial_\mu T^{\mu\nu}=0$$
and (2.11)
$$\partial_\alpha\mathfrak{M}^{\alpha\mu\nu}=0.$$

The corresponding conserved generators of the Poincaré group are

$$P^\mu=\int d^3x\,T^{0\mu} \text{ and } M^{\mu\nu}=\int d^3x\,\mathfrak{M}^{0\mu\nu}. \qquad (2.12)$$

The canonical Hamiltonian is

$$H_c\equiv P^0=\int d^3x\,T^{00}$$
$$=\int d^3x(\tfrac{1}{2}\Pi^{\alpha\beta}\dot{A}_{\alpha\beta}-\tfrac{1}{4}\Pi^{ij}\Pi_{ij}+\tfrac{1}{2}G^2), \qquad (2.13)$$

where $G^2=F^{123}F_{123}$.

Following Dirac, we add arbitrary multiples of primary constraints to this Hamiltonian so that our preliminary Hamiltonian becomes

$$H_0=H_c+\int d^3x\,v_i\Pi^{0i}(x)$$
$$=\int d^3x[\tfrac{1}{4}\Pi^{ij}\Pi_{ij}+\tfrac{1}{2}G^2-\tfrac{1}{2}\Pi^{ij}(\partial_iA_{j0}-\partial_jA_{i0})+v_i\Pi^{0i}], \qquad (2.14)$$

where we have used

$$\dot{A}_{ij}=\Pi_{ij}-\partial_iA_{j0}+\partial_jA_{i0}, \qquad (2.15)$$

which follows directly from the definition (2.5) of Π^{ij}.

Since the constraints have to hold for all times, we have from

$$\dot{\Pi}^{0i}=[\Pi^{0i},H_0]_P=\partial_k\Pi^{ki} \qquad (2.16)$$

the secondary constraints as

$$\partial_i\Pi^{ik}\approx 0, \qquad (2.17)$$

which is the same as Eq. (2.7). There are no more secondary constraints and, therefore, we have only two sets of constraints:

$$\phi_1^i\equiv\Pi^{0i}\approx 0,$$
$$\phi_2^k\equiv\partial_i\Pi^{ik}\approx 0, \qquad (2.18)$$

which are, as can be seen, first class.

To obtain the final Hamiltonian, we add arbitrary multiples of these first-class constraints to the canonical Hamiltonian H_c given above:

$$H=H_c+\int d^3x[v_i(x)\phi_1^i(x)+\omega_i(x)\phi_2^i(x)]$$
$$=\int d^3x[\tfrac{1}{4}\Pi^{ij}\Pi_{ij}+\tfrac{1}{2}G^2+v_i\Pi^{0i}+(\omega_j+A_{j0})\partial_i\Pi^{ij}], \qquad (2.19)$$

where we assume fields to vanish at infinity so that the surface term does not contribute. To evaluate the Lagrangian multipliers in (2.19), we note that

$$\dot{A}^{j0}=[A^{j0},H]_P=-v^j, \tag{2.20}$$

$$\dot{A}^{jk}=[A^{jk},H]_P=\Pi^{jk}-\partial_j(\omega_k+A_{k0})+\partial_k(\omega_j+A_{j0}), \tag{2.21}$$

and

$$\dot{\Pi}^{jk}=[\Pi^{jk},H]_P=F^{jkl}{}_{,l}. \tag{2.22}$$

Thus we may choose $v^j=-\dot{A}^{j0}$ and $\omega^j=0$ so that our Hamiltonian finally becomes

$$H=\int d^3x(\tfrac{1}{4}\Pi^{ij}\Pi_{ij}+\tfrac{1}{2}G^2+A_{j0}\partial_i\Pi^{ij}-\dot{A}_{0j}\Pi^{0i}), \tag{2.23}$$

where we have the arbitrary functions A_{j0}, which can be eliminated by imposing gauge constraints on our system. This we do in the next section.

B. Radiation gauge and Dirac brackets

To make the set of constraints second class, we add two kinds of gauge constraints to the already mentioned ones. In correspondence to the constraints $\Pi^{0i}\approx 0$, we choose $A^{0i}\approx 0$. This choice is ensured by the following gauge transformation:

$$A^{\mu\nu}(x)\to A^{\mu\nu\prime}(x)=A^{\mu\nu}(x)+\partial^\mu\int_0^{x^0}dt A^{0\nu}(\vec{x},t) -\partial^\nu\int_0^{x^0}dt A^{0\mu}(\vec{x},t). \tag{2.24}$$

From the Euler-Lagrangian equations $\partial_i F^{ij0\prime}=0$, this choice of gauge gives us

$$\partial_i\dot{A}^{ij\prime}\approx 0. \tag{2.25}$$

The other kind of gauge constraints we choose will have to fix A^{ij} for us. In analogy to the radiation gauge of electromagnetics, we therefore choose

$$\partial_i A^{ij\prime\prime}\approx 0, \tag{2.26}$$

which is ensured by the following gauge transformation:

$$A^{ij\prime}(x)\to A^{ij\prime\prime}(x)=A^{ij\prime}(x)+\partial_x^i\int d^3y G(\vec{x}-\vec{y})\partial_{y_k}A^{kj\prime}(\vec{y},x^0) -\partial_x^j\int d^3y\, G(\vec{x}-\vec{y})\partial_{y_k}A^{ki\prime}(\vec{y},x^0) \tag{2.27}$$

and

$$A^{0j\prime}(x)\to A^{0j\prime\prime}(x)=A^{0j\prime}(x)+\partial_x^0\int d^3y\, G(\vec{x}-\vec{y})\partial_{y_k}A^{kj\prime}(\vec{y},x^0) -\partial_x^j\int d^3y\, G(\vec{x}-\vec{y})\partial_{y_k}A^{k0\prime}(\vec{y},x^0), \tag{2.28}$$

where the right-hand side of (2.28) is weakly zero because of $A^{k0\prime}\approx 0$, and Eq. (2.25). The Green's function $G(\vec{x}-\vec{y})$ satisfies

$$\nabla^2 G(\vec{x}-\vec{y})=-\delta^3(\vec{x}-\vec{y}), \tag{2.29}$$

which has a solution $G(\vec{x}-\vec{y})=(4\pi|\vec{x}-\vec{y}|)^{-1}$.

Finally, dropping all the primes, we have our set of constraints as

$$\begin{aligned}\chi_1^i&\equiv\Pi^{0i}\approx 0,\\ \chi_2^i&\equiv\partial_j\Pi^{ji}\approx 0,\\ \chi_3^i&\equiv A^{0i}\approx 0,\\ \chi_4^i&\equiv\partial_j A^{ji}\approx 0.\end{aligned} \tag{2.30}$$

The matrix of the Poisson brackets of these constraints is singular, which reflects that there are combinations of these constraints which are first class. In fact there are two such combinations. Equivalently, we may notice that only two each of the constraints χ_2^j and χ_4^j are independent. In particular, we may choose those with $j=1,2$ as the independent ones, then the matrix of the Poisson brackets of the independent second-class constraints becomes

$$([\chi_a^i(\vec{x},t),\chi_b^j(\vec{y},t)]_P)=\begin{pmatrix}0&0&g^{ij}&0\\0&0&0&g^{ij}\nabla^2-\partial^i\partial^j\\-g^{ij}&0&0&0\\0&-g^{ij}\nabla^2+\partial^i\partial^j&0&0\end{pmatrix}\delta^3(\vec{x}-\vec{y}), \tag{2.31}$$

where $i,j=1,2,3$ for constraints $a=1,3$ and $i,j=1,2$ for the constraints $a=2,4$. The inverse of this matrix exists, if we assume that the fields vanish at infinity, and it is

$$(\Delta_{ab}^{ij}(\vec{x},\vec{y})) = \begin{pmatrix} 0 & 0 & -g^{ij}\delta^3(\vec{x}-\vec{y}) & 0 \\ 0 & 0 & 0 & G^{ij}(\vec{x}-\vec{y}) \\ g^{ij}\delta^3(\vec{x}-\vec{y}) & 0 & 0 & 0 \\ 0 & -G^{ij}(\vec{x}-\vec{y}) & 0 & 0 \end{pmatrix}, \tag{2.32}$$

where again $i, j = 1, 2, 3$ for $a = 1, 3$ and $i, j = 1, 2$ only for $a = 2, 4$ and $G^{ij}(x-y)$ satisfy

$$(g^{ij}\nabla^2 - \partial^i\partial^j)G^{jk}(\vec{x}-\vec{y}) = -\delta^{ik}\delta^3(\vec{x}-\vec{y}), \tag{2.33}$$

where $i, j, k = 1, 2$. The solution to (2.33) can be written as

$$G^{11}(\vec{x}-\vec{y}) = \int \frac{d^3\omega}{(2\pi)^3} \frac{\omega_1^2+\omega_3^2}{\omega_3^2\omega^2} e^{i\vec{\omega}\cdot(\vec{x}-\vec{y})},$$

$$G^{22}(\vec{x}-\vec{y}) = \int \frac{d^3\omega}{(2\pi)^3} \frac{\omega_2^2+\omega_3^2}{\omega_3^2\omega^2} e^{i\vec{\omega}\cdot(\vec{x}-\vec{y})}, \tag{2.34}$$

$$G^{12}(\vec{x}-\vec{y}) = G^{21}(\vec{x}-\vec{y}) = \int \frac{d^3\omega}{(2\pi)^3} \frac{\omega_1\omega_2}{\omega_3^2\omega^2} e^{i\vec{\omega}\cdot(\vec{x}-\vec{y})}.$$

Now defining the Dirac bracket of two variables A and B as[5]

$$[A(\vec{x}), B(\vec{y})]_P^* = [A(\vec{x}), B(\vec{y})]_P - \int d^3x' \int d^3y' [A(\vec{x}), \chi_a^i(\vec{x}')]_P \Delta_{ab}^{ij}(\vec{x}', \vec{y}')[\chi_b^j(\vec{y}'), B(\vec{y})]_P \tag{2.35}$$

we have the Dirac brackets for $\Pi^{\mu\nu}$ and $A^{\alpha\beta}$ as

$$[\Pi^{\mu\nu}(\vec{x},t), A^{\alpha\beta}(\vec{y},t)]_P^* = -(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha})\delta^3(\vec{x}-\vec{y}) - (g^{\mu 0}g^{\nu i} - g^{\mu i}g^{\nu 0})(g^{0\alpha}g^{i\beta} - g^{0\beta}g^{i\alpha})\delta^3(\vec{x}-\vec{y})$$
$$- (g^{\mu i}g^{\nu k} - g^{\mu k}g^{\nu i})(g^{j\alpha}g^{l\beta} - g^{j\beta}g^{l\alpha})\partial_i\partial_j G_{lk}(\vec{x}-\vec{y}), \tag{2.36}$$

where in the last term $i, j = 1, 2, 3$ and $k, l = 1, 2$. The other Dirac brackets are

$$[\Pi^{\mu\nu}(\vec{x},t), \Pi^{\alpha\beta}(\vec{y},t)]_P^* = 0, \tag{2.37}$$

$$[A^{\mu\nu}(\vec{x},t), A^{\alpha\beta}(\vec{y},t)]_P^* = 0. \tag{2.38}$$

The last term on the right-hand side of (2.36) can be re-expressed in terms of the Green's function $G(x-y)$ which is a solution of Eq. (2.29) so that (2.36) becomes

$$[\Pi^{\mu\nu}(\vec{x},t), A^{\alpha\beta}(\vec{y},t)]_P^* = -(g^{\mu\alpha}g^{\nu\beta} - g^{\mu\beta}g^{\nu\alpha})\delta^3(\vec{x}-\vec{y}) - (g^{\mu 0}g^{\nu i} - g^{\mu i}g^{\nu 0})(g^{0\alpha}g^{i\beta} - g^{0\beta}g^{i\alpha})\delta^3(\vec{x}-\vec{y})$$
$$- (g^{\mu i}g^{\nu k} - g^{\mu k}g^{\nu i})(g^{j\alpha}g^{k\beta} - g^{j\beta}g^{k\alpha})\partial_i\partial_j G(\vec{x}-\vec{y}), \tag{2.39}$$

where now $i, j, k = 1, 2, 3$.

Now we may set the second-class constraints to be strongly equal to zero. The Dirac bracket relations given above are compatible with this.

In order to verify the Hamiltonian equations of motion, we note that our Hamiltonian has become

$$H = \int d^3x\, T^{00} = \int d^3x (\tfrac{1}{4}\Pi^{ij}\Pi_{ij} + \tfrac{1}{2}G^2) \tag{2.40}$$

and Hamiltonian equations of motion read

$$[A^{ij}, H]_P^* = \Pi^{ij} = -\partial^0 A^{ij} = \dot{A}^{ij} \tag{2.41}$$

and

$$[\Pi^{ij}, H]_P^* = \partial_k F^{kij} = \dot{\Pi}^{ij}, \tag{2.42}$$

where we have dropped the surface integrals as usual.

The Poincaré group generators are

$$P^k = \int d^3x\, T^{0k} = -\tfrac{1}{2}\int d^3x\, \Pi^{ij} A_{ij,}{}^k, \tag{2.43}$$

$$M^{ij}=\int d^3x\mathfrak{M}^{0ij}=\int d^3x[-\tfrac{1}{2}\Pi^{lk}(x^i\partial^j-x^j\partial^i)A_{lk}-\Pi^{ik}A^j{}_k+\Pi^{jk}A^i{}_k], \tag{2.44}$$

and

$$M^{0k}=\int d^3x\mathfrak{M}^{00k}=\int d^3x[-\tfrac{1}{2}x^0\Pi^{ij}A_{ij,}{}^k-x^k(\tfrac{1}{4}\Pi^{ij}\Pi_{ij}+\tfrac{1}{2}G^2)]. \tag{2.45}$$

It can be easily verified that P^μ and $M^{\mu\nu}$ obey the Poincaré algebra under the Dirac bracket operation.

The effect of P^k, M^{ij}, and M^{0k} on A^{lm} can be easily seen to be as follows:

$$[P^k,A^{lm}(x)]^*_P=\partial^kA^{lm}(x), \tag{2.46}$$

$$[M^{ij},A^{lm}(x)]^*_P=(x^i\partial^j-x^j\partial^i)A^{lm}(x)+[g^{il}A^{jm}(x)-g^{im}A^{jl}(x)]-[g^{jl}A^{im}(x)-g^{jm}A^{il}(x)], \tag{2.47}$$

and

$$[M^{0i},A^{lm}(x)]^*_P=(x^0\partial^i-x^i\partial^0)A^{lm}(x)-\int d^3y\,\partial^0A^{ij}(x)(g^{jm}\partial^l-g^{jl}\partial^m)G(\vec{x}-\vec{y}). \tag{2.48}$$

Although the choice of gauge that we have made above is not Lorentz invariant, the presence of the second term in the right-hand side of (2.48) ensures that the constraints are respected in the new Lorentz frame.

C. Quantization

With the choice of constraints as given in (2.30), we have only one each of $A^{\mu\nu}$ and $\Pi^{\mu\nu}$ as independent canonical variables. The quantization is achieved by replacing the Dirac brackets of the independent variables by the $(1/i)$ times the commutator of the corresponding operators. Thus the canonical commutation relations for A_{ij} and Π_{lk} are

$$[\Pi_{ij}(\vec{x},t),A_{lk}(\vec{y},t)]=-i(g_{il}g_{jk}-g_{ik}g_{jl})\delta^3(\vec{x}-\vec{y})-i(g_{im}g_{js}-g_{is}g_{jm})(g_{nl}g_{sk}-g_{nk}g_{sl})\partial_m\partial_nG(\vec{x}-\vec{y}), \tag{2.49}$$

and

$$[\Pi_{ij}(\vec{x},t),\Pi_{lk}(\vec{y},t)]=0, \tag{2.50}$$

$$[A_{ij}(\vec{x},t),A_{lk}(\vec{y},t)]=0. \tag{2.51}$$

Similarly, with this prescription we may translate the other Dirac bracket relations of the previous section into quantum-mechanical language.

III. DYNAMICS ON THE NULL PLANE

Instead of describing the dynamics on constant-time surfaces, we may also describe a Hamiltonian field theory for $A^{\mu\nu}$ on the null plane in the same manner as Kogut and Soper[7] have done in the case of free electromagnetic field theory.

The null-plane coordinates are defined as

$$x^\pm=\frac{1}{\sqrt{2}}(x^3\pm x^0)=x_\mp=\frac{1}{\sqrt{2}}(x_3\mp x_0),$$
$$\partial_\pm=\frac{1}{\sqrt{2}}(\partial_3\pm\partial_0)=\partial^\mp=\frac{1}{\sqrt{2}}(\partial^3\mp\partial^0), \tag{3.1}$$
$$x^i=x_i,\quad i=1,2.$$

The metric therefore reads $g^{+-}=g^{-+}=g^{11}=g^{22}=1$, others zero. We shall denote the two-dimensional vector (x^1,x^2) as $\underline{x}$.

Having established the notations, we shall now write down the Hamiltonian without gauge constraints.

A. Null-plane Hamiltonian without gauge constraints

Since the evolution of our system takes place on constant-x^+ surface, the canonical momentum conjugate to $A_{\mu\nu}$ may be defined as

$$\Pi^{\mu\nu}(x)=\frac{\partial\mathcal{L}}{\partial\partial_+A_{\mu\nu}}=-F^{+\mu\nu}(x)=-\Pi^{\nu\mu}(x), \tag{3.2}$$

where $\mathcal{L}$ is the same as in Sec. II. This yields the primary constraints to be

$$\Pi^{+i}\approx0,\quad i=1,2 \tag{3.3}$$

$$\Pi^{+-}\approx0, \tag{3.4}$$

$$\Pi^{12}+\partial_-A^{12}+\partial_1A^{2+}+\partial_2A^{+1}\approx0, \tag{3.5}$$

where the other two components of $\Pi^{\mu\nu}$ are given by

$$\Pi^{-i}+\partial_-A^{-i}+\partial_+A^{i+}+\partial_iA^{+-}\approx0,\quad i=1,2, \tag{3.6}$$

which are dynamical relations and not constraints because of presence of the x^+ derivative.

The canonical Poisson brackets on constant x^+ surface may be written as

$$[\Pi^{\mu\nu}(x^+,x^-,\underline{x}),A^{\alpha\beta}(x^+,y^-,\underline{y})]_P$$
$$=-(g^{\mu\alpha}g^{\nu\beta}-g^{\mu\beta}g^{\nu\alpha})\delta(x^--y^-)\delta^2(\underline{x}-\underline{y}). \tag{3.7}$$

The other Poisson brackets are zero. Here we may note that it is A^{-i} and not A^{+i} that is conjugate to Π^{+i}. Also, as expected, it may be noted that our constraints (3.3)–(3.5) are not compatible with these canonical Poisson-bracket relations.

The constraints (3.3) and (3.4) are first class, whereas constraint (3.5) is second class, as can be seen from

$$[(\Pi^{12}+\partial_-A^{12}+\partial_1A^{2+}+\partial_2A^{+1})(x)\times(\Pi^{12}+\partial_-A^{12}+\partial_1A^{2+}+\partial_2A^{+1})(y)]_P=2\partial_-\delta^3(x-y)\,, \tag{3.8}$$

where $\delta^3(x-y)\equiv\delta(x^- - y^-)\delta^2(\underline{x}-\underline{y})$.

In order to write conserved generators of our system, we note the conserved currents may be written as

$$T^{\mu\nu}=\tfrac{1}{2}F^{\mu\alpha\beta}A_{\alpha\beta,}{}^{\nu}-\tfrac{1}{12}g^{\mu\nu}F^{\alpha\beta\delta}F_{\alpha\beta\delta} \tag{3.9}$$

and

$$\mathfrak{M}^{\alpha\mu\nu}=x^\mu T^{\alpha\nu}-x^\nu T^{\alpha\mu}+F^{\alpha\mu\delta}A^{\nu}{}_{\delta}-F^{\alpha\nu\delta}A^{\mu}{}_{\delta}\,, \tag{3.10}$$

where the metric is as defined above following Eqs. (3.1).

The generators of the Poincaré group thus read

$$P^\mu=\int dx^-d^2x\,T^{+\mu}(x),\qquad M^{\mu\nu}=\int dx^-d^2x\,\mathfrak{M}^{+\mu\nu}(x)\,. \tag{3.11}$$

The canonical Hamiltonian is given by

$$H_c=-P^-=-\int dx^-d^2x\,T^{+-}(x)=\int dx^-d^2x\left(\tfrac{1}{2}\Pi^{\alpha\beta}A_{\alpha\beta,}{}^{-}+\tfrac{1}{12}F^{\alpha\beta\delta}F_{\alpha\beta\delta}\right), \tag{3.12}$$

where we have chosen $H_c=-P^-$ and not $H_c=+P^-$ so that Hamiltonian is positive definite. The Hamiltonian can be re-expressed as

$$H_c=\int dx^-d^2x\left[\tfrac{1}{2}\Pi^{-i}\Pi^{-i}+\Pi^{-i}(\partial_-A_{+i}-\partial_iA_{+-})+\tfrac{1}{12}\Pi^{ij}(\partial_iA_{+j}-\partial_jA_{+i})\right]. \tag{3.13}$$

Given the canonical Hamiltonian as in Eq. (3.13), we obtain the preliminary Hamiltonian by adding arbitrary multiples of primary constraints to it:

$$H_0=H_c+\int dx^-d^2x\{v^i(x)\Pi^{+i}(x)+v^-(x)\Pi^{+-}(x)+\omega(x)[\Pi^{12}(x)+\partial_-A^{12}(x)+\partial_1A^{2+}(x)+\partial_2A^{+1}(x)]\}\,. \tag{3.14}$$

Since the constraints have to be valid for all times, the canonical Poisson brackets of the primary constraints with this Hamiltonian have to be zero. This gives the secondary constraints

$$\partial_+\Pi^{+i}=[\Pi^{+i},H_0]_P=\partial_-\Pi^{-i}+\partial_j\Pi^{ji}\approx0,\quad i=1,2 \tag{3.15}$$

$$\partial_+\Pi^{+-}=[\Pi^{+-},H_0]_P=-\partial_i\Pi^{-i}\approx0\,, \tag{3.16}$$

$$\begin{aligned}\partial_+(\Pi^{12}+\partial_-A^{12}+\partial_1A^{2+}+\partial_2A^{+1})&=[\Pi^{12}+\partial_-A^{12}+\partial_1A^{2+}+\partial_2A^{+1},H_0]_P\\&=2\partial_-\omega(x)-\partial_1\Pi^{-2}(x)+\partial_2\Pi^{-1}(x)\\&\approx0.\end{aligned} \tag{3.17}$$

Here we have used the fact that the fields vanish at infinity. Equations (3.15) and (3.16) give us genuine secondary constraints, but (3.17) is rather a condition on function $\omega(x)$. There are no more secondary constraints.

Thus finally we have two sets of constraints as follows:

$$\begin{aligned}&\phi_i\equiv\Pi^{+i}\approx0,\quad i=1,2\\&\phi_3\equiv\Pi^{+-}\approx0,\\&\phi_{3+i}\equiv\partial_-\Pi^{-i}+\partial_k\Pi^{ki}\approx0,\quad i=1,2\\&\phi_6\equiv\partial_i\Pi^{-i}\approx0,\end{aligned} \tag{3.18}$$

and

$$\chi\equiv\Pi^{12}+\partial_-A^{12}+\partial_1A^{2+}+\partial_2A^{+1}\approx0\,, \tag{3.19}$$

which are, respectively, first class and second class.

Now we define the inverse of the Poisson bracket $[\chi(x),\chi(y)]_P$ as $\psi(x,y)$:

$$\int dz^-d^2z[\chi(x),\chi(z)]_P\psi(z,y)=\delta(x^- - y^-)\delta^2(\underline{x}-\underline{y})\,, \tag{3.20}$$

wherefrom we note

$$\psi(x,y)=\tfrac{1}{4}\epsilon(x^- - y^-)\delta^2(\underline{x}-\underline{y})\,, \tag{3.21}$$

where $\epsilon(x^- - y^-)=\mathrm{sgn}(x^- - y^-)$. The first-class Hamiltonian may be written as

$$H'=H_c-\int dx^-d^2x\int dy^-d^2y[H_c,\chi(x)]\times\psi(x,y)\chi(y)\,. \tag{3.22}$$

Using (3.21), we compute the preliminary brackets between two variables $A(x)$, $B(y)$ as[5]

$$[A(x),B(y)]'_P=[A(x),B(y)]_P-\int dz^-d^2z\int dz'^-d^2z'[A(x),\chi(z)]_P\psi(z,z')[\chi(z'),B(y)]_P, \tag{3.23}$$

so that we obtain

$$[A^{\mu\nu}(x),A^{\alpha\beta}(y)]'_{P\,x^+=y^+}=(g^{\mu1}g^{\nu2}-g^{\mu2}g^{\nu1})(g^{\alpha1}g^{\beta2}-g^{\alpha2}g^{\beta1})\tfrac{1}{4}\epsilon(x^- - y^-)\delta^2(\underline{x}-\underline{y})\,, \tag{3.24}$$

$$[\Pi^{\mu\nu}(x),\Pi^{\alpha\beta}(y)]'_{\substack{P\\ x^+=y^+}}=-[(g^{\mu1}g^{\nu2}-g^{\mu2}g^{\nu1})\partial_-+(g^{\mu2}g^{\nu+}-g^{\mu+}g^{\nu2})\partial_1+(g^{\mu+}g^{\nu1}-g^{\mu1}g^{\nu+})\partial_2]$$
$$\times[(g^{\alpha1}g^{\beta2}-g^{\alpha2}g^{\beta1})\partial_-+(g^{\alpha2}g^{\beta+}-g^{\alpha+}g^{\beta2})\partial_1+(g^{\alpha+}g^{\beta1}-g^{\alpha1}g^{\beta+})\partial_2]\tfrac{1}{4}\epsilon(x^--y^-)\delta^2(\underline{x}-\underline{y}), \tag{3.25}$$

and

$$[\Pi^{\mu\nu}(x),A^{\alpha\beta}(y)]'_{\substack{P\\ x^+=y^+}}=-(g^{\mu\alpha}g^{\nu\beta}-g^{\mu\beta}g^{\nu\alpha})\delta^3(x-y)+[(g^{\mu1}g^{\nu2}-g^{\mu2}g^{\nu1})\partial_-+(g^{\mu2}g^{\nu+}-g^{\mu+}g^{\nu2})\partial_1+(g^{\mu+}g^{\nu1}-g^{\mu1}g^{\nu+})\partial_2]$$
$$\times(g^{\alpha1}g^{\beta2}-g^{\alpha2}g^{\beta1})\tfrac{1}{4}\epsilon(x^--y^-)\delta^2(\underline{x}-\underline{y}). \tag{3.26}$$

All other relevant equations may now be obtained by evaluating preliminary brackets. These preliminary brackets are compatible with the second-class constraint (3.19). Working with the prime brackets, we have to use the Hamiltonian to which arbitrary multiples of first-class constraints have been added:

$$H=H_c+\int d^2x\,dx^-\{v^i(x)\Pi^{+i}(x)+v^-(x)\Pi^{+-}(x)+u^i(x)[\partial_-\Pi^{-i}(x)+\partial_k\Pi^{ki}(x)]+u^-(x)[\partial_i\Pi^{-i}(x)]\}, \tag{3.27}$$

where H_c is given by Eq. (3.13). The Hamiltonian equations read as follows:

$$\begin{aligned}
&[A^{-i}(x),H]'_P=v^i(x),\\
&[A^{ij}(x),H]'_P=\partial_+A^{ij}(x)-\partial^iu^j(x)+\partial^ju^i(x),\\
&[A^{+-}(x),H]'_P=-v^-(x),\\
&[A^{+i}(x),H]'_P=\partial_+A^{+i}(x)-\partial_iu^-(x),\\
&[\Pi^{+-}(x),H]'_P=-\partial_i\Pi^{-i}(x)\approx 0,\\
&[\Pi^{+i}(x),H]'_P=\partial_-\Pi^{-i}(x)+\partial_j\Pi^{ji}(x)\approx 0,\\
&[\Pi^{-i}(x),H]'_P=\partial_+\Pi^{-i}(x),\\
&[\Pi^{ij}(x),H]'_P=\partial_+\Pi^{ij}(x).
\end{aligned} \tag{3.28}$$

Thus, we may choose

$$\begin{aligned}
&v^i=\partial_+A^{-i},\quad v^-=-\partial_+A^{+-},\\
&u^j=0,\quad u^-=0.
\end{aligned} \tag{3.29}$$

The arbitrary functions present in our theory due to the first-class constraints have to be removed. This is done by imposing gauge constraints on the system.

B. Null-plane radiation gauge and Dirac brackets

Analogous to the gauge we have chosen in Sec. II B, we choose conjugates of Π^{+i},Π^{+-} to be zero,

$$\begin{aligned}
&A^{-i}\approx 0,\quad i=1,2\\
&A^{-+}\approx 0,
\end{aligned} \tag{3.30}$$

which is ensured by the following gauge transformation:

$$A^{\mu\nu}\to A^{\mu\nu\prime}=A^{\mu\nu}-\partial^\mu\int_0^{x^+}dt\,A^{-\nu}(t,x^-,\underline{x})+\partial^\nu\int_0^{x^+}dt\,A^{-\mu}(t,x^-,\underline{x}). \tag{3.31}$$

This choice, using the Euler-Lagrangian equation

$$F^{-\mu\nu\prime}{}_{,\mu}=0,$$

gives us

$$\partial_+(\Pi^{-\nu}+\partial_iA^{i\nu\prime})\approx 0. \tag{3.32}$$

Thus the other set of gauge constraints we choose are

$$\begin{aligned}
&\Pi^{-j}+\partial_iA^{ij\prime\prime}\approx 0,\quad j=1,2\\
&\partial_iA^{i+\prime\prime}\approx 0,
\end{aligned} \tag{3.33}$$

which are ensured by the following gauge transformation:

$$A^{\mu\nu\prime}\to A^{\mu\nu\prime\prime}=A^{\mu\nu\prime}-\partial_x^\mu\int d^2y\,G(\underline{x}-\underline{y})[\partial_y^lA^{l\nu\prime}(y)+\Pi^{-\nu}(y)]+\partial_x^\nu\int d^2y\,G(\underline{x}-\underline{y})[\partial_y^lA^{l\mu\prime}(y)+\Pi^{-\mu}(y)], \tag{3.34}$$

where Green's function $G(\underline{x}-\underline{y})$ satisfies

$$\partial^k\partial^kG(\underline{x}-\underline{y})=\delta^2(\underline{x}-\underline{y}),$$

which has a solution $G(\underline{x}-\underline{y})=\ln(\underline{x}-\underline{y})^2/4\pi$.

Thus, finally, our set of constraints consists of the constraints (3.18), (3.30), and (3.33). However, all of these constraints are not independent. Dropping all the primes we may choose the following set of constraints as independent:

$$\begin{aligned}
&\chi_i\equiv\Pi^{+i}\approx 0,\quad i=1,2\\
&\chi_3\equiv\Pi^{+-}\approx 0,\\
&\chi_4\equiv\partial_-\Pi^{-1}+\partial_2\Pi^{21}\approx 0,\\
&\chi_{4+i}\equiv A^{-i}\approx 0,\quad i=1,2\\
&\chi_7\equiv A^{-+}\approx 0,\\
&\chi_{7+i}\equiv\Pi^{-i}+\partial_jA^{ji}\approx 0,\quad i=1,2\\
&\chi_{10}\equiv\partial_kA^{k+}\approx 0.
\end{aligned} \tag{3.35}$$

The two constraints from (3.18) that we have dropped from this final list of constraints can be expressed in terms of these. The matrix of preliminary brackets of these constraints $C_{rs}(x,y) \equiv [\chi_r(x),\chi_s(y)]'_P$ has the following nonvanishing elements:

$$
\begin{aligned}
C_{15}(x,y) &= C_{26}(x,y) = C_{37}(x,y) = -C_{51}(x,y) \\
&= -C_{62}(x,y) = -C_{73}(x,y) = -\delta^3(x-y)\,,\\
C_{48}(x,y) &= -C_{84}(x,y) = \partial_2{}^2\delta^3(x-y)\,,\\
C_{49}(x,y) &= -C_{94}(x,y) = -\partial_1\partial_2\delta^3(x-y)\,,\\
C_{410}(x,y) &= -C_{104}(x,y) = -\partial_1\partial_-\delta^3(x-y)\,,\\
C_{810}(x,y) &= -C_{108}(x,y) = -\partial_1\delta^3(x-y)\,,\\
C_{910}(x,y) &= -C_{109}(x,y) = -\partial_2\delta^3(x-y)\,.
\end{aligned}
\tag{3.36}
$$

Assuming that the fields vanish at infinity, the inverse of this matrix can be defined. The nonvanishing elements of this inverse matrix $\Delta_{rs}(x,y)$ are as follows:

$$
\begin{aligned}
&\Delta_{15}(x,y) = \Delta_{26}(x,y) = \Delta_{37}(x,y) = -\Delta_{51}(x,y)\\
&\qquad = -\Delta_{62}(x,y) = -\Delta_{73}(x,y) = \delta^3(x-y)\,,\\
&\Delta_{44}=G_{11},\quad \Delta_{48}=G_{12},\quad \Delta_{49}=G_{13},\quad \Delta_{4,10}=G_{14}\,,\\
&\Delta_{84}=G_{21},\quad \Delta_{88}=G_{22},\quad \Delta_{89}=G_{23},\quad \Delta_{810}=G_{24}\,,\\
&\Delta_{94}=G_{31},\quad \Delta_{98}=G_{32},\quad \Delta_{99}=G_{33},\quad \Delta_{910}=G_{34}\,,\\
&\Delta_{104}=G_{41},\quad \Delta_{108}=G_{42},\quad \Delta_{109}=G_{43},\quad \Delta_{1010}=G_{44}\,,
\end{aligned}
\tag{3.37}
$$

where the 4×4 matrix $G_{mn}(x,y)$ is given as

$$G_{mn}(x,y) = \int \frac{d^2\omega\, d\omega^-}{(2\pi)^3}\, G_{mn}(\omega)\exp[i\underline{\omega}\cdot(\underline{x}-\underline{y}) + i\omega_+(x^- - y^-)] \tag{3.38}$$

and

$$
(G_{mn}(\omega)) = \begin{pmatrix}
0 & \dfrac{1}{\omega_1{}^2+\omega_2{}^2} & \dfrac{-\omega_1}{\omega_2(\omega_1{}^2+\omega_2{}^2)} & 0\\[2ex]
\dfrac{-1}{\omega_1{}^2+\omega_2{}^2} & \dfrac{2i\omega_1{}^2\omega_+}{(\omega_1{}^2+\omega_2{}^2)^2} & \dfrac{i\omega_+\omega_1(\omega_2{}^2-\omega_1{}^2)}{\omega_2(\omega_1{}^2+\omega_2{}^2)^2} & \dfrac{i\omega_1}{\omega_1{}^2+\omega_2{}^2}\\[2ex]
\dfrac{\omega_1}{\omega_2(\omega_1{}^2+\omega_2{}^2)} & \dfrac{i\omega_+\omega_1(\omega_2{}^2-\omega_1{}^2)}{\omega_2(\omega_1{}^2+\omega_2{}^2)^2} & \dfrac{-2i\omega_1{}^2\omega_+}{(\omega_1{}^2+\omega_2{}^2)^2} & \dfrac{i\omega_2}{\omega_1{}^2+\omega_2{}^2}\\[2ex]
0 & \dfrac{i\omega_1}{\omega_1{}^2+\omega_2{}^2} & \dfrac{i\omega_2}{\omega_1{}^2+\omega_2{}^2} & 0
\end{pmatrix}. \tag{3.39}
$$

Thus, defining the Dirac brackets of two variables $A(x), B(y)$ as[5]

$$[A(x),B(y)]^*_P = [A(x),B(y)]'_P - \int dz^- d^2z \int dz^{-\prime} d^2z' [A(x),\chi_r(z)]'_P\,\Delta_{rs}(z,z')[\chi_s(z'),B(y)]'_P\,,$$

we have the equal-x^+ Dirac brackets for $A^{\mu\nu}(x^+,x^-,\underline{x})$, and $\Pi^{\mu\nu}(x^+,y^-,\underline{y})$ after some straightforward calculations as

$$
\begin{aligned}
&[A^{\mu\nu}(\underline{x}^+,x^-,\underline{x}),A^{\alpha\beta}(x^+,y^-,\underline{y})]^*_P\\
&\quad = \tfrac{1}{2}(g^{\mu 1}g^{\nu 2}-g^{\mu 2}g^{\nu 1})(g^{\alpha 1}g^{\beta 2}-g^{\alpha 2}g^{\beta 1})F(x-y)\\
&\qquad + 2[(g^{-\mu}g^{1\nu}-g^{-\nu}g^{1\mu})\partial_2+(g^{-\nu}g^{2\mu}-g^{-\mu}g^{2\nu})\partial_1][(g^{-\alpha}g^{1\beta}-g^{-\beta}g^{1\alpha})\partial_2+(g^{-\beta}g^{2\alpha}-g^{-\alpha}g^{2\beta})\partial_1]\partial_-H(x-y)\\
&\qquad -(g^{1\mu}g^{2\nu}-g^{2\mu}g^{1\nu})[(g^{-\alpha}g^{1\beta}-g^{-\beta}g^{1\alpha})\partial_2+(g^{-\beta}g^{2\alpha}-g^{-\alpha}g^{2\beta})\partial_1]G(x-y)\\
&\qquad -(g^{1\alpha}g^{2\beta}-g^{2\alpha}g^{1\beta})[(g^{-\mu}g^{1\nu}-g^{-\nu}g^{1\mu})\partial_2+(g^{-\nu}g^{2\mu}-g^{-\mu}g^{2\nu})\partial_1]G(x-y)\,,
\end{aligned}
\tag{3.40}
$$

$$
\begin{aligned}
&[\Pi^{\mu\nu}(x^+,x^-,\underline{x}),\Pi^{\alpha\beta}(x^+,y^-,\underline{y})]^*_P\\
&\quad = -\tfrac{1}{2}[(g^{\mu 1}g^{\nu 2}-g^{\mu 2}g^{\nu 1})\partial_- + (g^{\mu 2}g^{\nu +}-g^{\mu +}g^{\nu 2})\partial_1 + (g^{\mu +}g^{\nu 1}-g^{\mu 1}g^{\nu +})\partial_2]\\
&\qquad \times[(g^{\alpha 1}g^{\beta 2}-g^{\alpha 2}g^{\beta 1})\partial_- + (g^{\alpha 2}g^{\beta +}-g^{\alpha +}g^{\beta 2})\partial_1 + (g^{\alpha +}g^{\beta 1}-g^{\alpha 1}g^{\beta +})\partial_2]F(x-y)\,,
\end{aligned}
\tag{3.41}
$$

$$
\begin{aligned}
&[\Pi^{\mu\nu}(x^+,x^-,\underline{x}),A^{\alpha\beta}(x^+,y^-,\underline{y})]^*_P\\
&\quad = -[(g^{\mu\alpha}g^{\nu\beta}-g^{\mu\beta}g^{\nu\alpha}) - \tfrac{3}{2}(g^{\mu 1}g^{\nu 2}-g^{\mu 2}g^{\nu 1})(g^{\alpha 1}g^{\beta 2}-g^{\beta 1}g^{\alpha 2}) - (g^{\mu -}g^{\nu 1}-g^{\mu 1}g^{\nu -})(g^{+\alpha}g^{1\beta}-g^{+\beta}g^{1\alpha})\\
&\qquad - (g^{\mu -}g^{\nu 2}-g^{\mu 2}g^{\nu -})(g^{+\alpha}g^{2\beta}-g^{+\beta}g^{2\alpha}) - (g^{\mu -}g^{\nu +}-g^{\mu +}g^{\nu -})(g^{+\alpha}g^{-\beta}-g^{+\beta}g^{-\alpha})]\,\delta^3(x-y)\\
&\qquad + \tfrac{1}{2}[(g^{\mu 2}g^{\nu +}-g^{\mu +}g^{\nu 2})\partial_1 + (g^{\mu +}g^{\nu 1}-g^{\mu 1}g^{\nu +})\partial_2]\,(g^{\alpha 1}g^{\beta 2}-g^{\alpha 2}g^{\beta 1})F(x-y)\\
&\qquad - (g^{\mu 1}g^{\nu 2}-g^{\mu 2}g^{\nu 1})[(g^{-\alpha}g^{1\beta}-g^{-\beta}g^{1\alpha})\partial_2 + (g^{-\beta}g^{2\alpha}-g^{-\alpha}g^{2\beta})\partial_1]\partial_-G(x-y)\\
&\qquad - (g^{\mu i}g^{\nu +}-g^{\mu +}g^{\nu i})(g^{-\alpha}g^{j\beta}-g^{-\beta}g^{j\alpha})\partial_i\partial_j G(x-y)\,,
\end{aligned}
\tag{3.42}
$$

where

$$F(x-y)=\tfrac{1}{2}\epsilon(x^--y^-)\delta^2(\underline{x}-\underline{y}),$$
$$G(x-y)=\frac{1}{4\pi}\ln(\underline{x}-\underline{y})^2\delta(x^--y^-),$$
$$\partial_-F(x-y)=\delta^3(x-y), \tag{3.43}$$
$$(\partial_1{}^2+\partial_2{}^2)G(x-y)=\delta^3(x-y),$$
$$(\partial_1{}^2+\partial_2{}^2)H(x-y)=G(x-y).$$

The second-class constraints (3.35) may now be put equal to zero strongly. These Dirac brackets are compatible with these constraints. The Dirac brackets for the transverse components may be written as

$$\begin{aligned}
&[A^{ij}(x^+,x^-,\underline{x}),A^{lk}(x^+,y^-,\underline{y})]^*_P\\
&\quad=\tfrac{1}{2}(g^{i1}g^{j2}-g^{i2}g^{j1})(g^{l1}g^{k2}-g^{l2}g^{k1})F(x-y),\\
&[\Pi^{ij}(x^+,x^-,\underline{x}),\Pi^{lk}(x^+,y^-,\underline{y})]^*_P\\
&\quad=-\tfrac{1}{2}(g^{i1}g^{j2}-g^{i2}g^{j1})(g^{l1}g^{k2}-g^{l2}g^{k1})\partial_-\delta^3(x-y),\\
&[\Pi^{ij}(x^+,x^-,\underline{x}),A^{lk}(x^+,y^-,\underline{y})]^*_P\\
&\quad=\tfrac{1}{2}(g^{i1}g^{j2}-g^{i2}g^{j1})(g^{l1}g^{k2}-g^{l2}g^{k1})\delta^3(x-y).
\end{aligned} \tag{3.44}$$

In order to obtain the Hamiltonian equations of motion, we notice that our Hamiltonian, putting second-class constraints strongly equal to zero, has become

$$H=\int d^2x\,dx^-\tfrac{1}{2}[(\partial_2A^{21})^2+(\partial_1A^{12})^2]. \tag{3.45}$$

This gives

$$\begin{aligned}
\partial_+A^{ij}(x)&=[A^{ij}(x),H]^*_P\\
&=-\tfrac{1}{4}(g^{i1}g^{j2}-g^{i2}g^{j1})(\partial_1{}^2+\partial_2{}^2)\\
&\quad\times\int dy^-A^{12}(x^+,y^-,\underline{x})\epsilon(x^--y^-),
\end{aligned} \tag{3.46}$$

which when differentiated with respect to x^- yields the Klein-Gordon equation

$$\begin{aligned}
\partial_-\partial_+A^{ij}(x)&=[\partial_-A^{ij}(x),H]^*_P\\
&=-\tfrac{1}{2}(\partial_1{}^2+\partial_2{}^2)A^{ij}(x)
\end{aligned}$$

or (3.47)

$$\partial^\mu\partial_\mu A^{ij}(x)=0.$$

Thus Eq. (3.46) is the x^- integral of the Klein-Gordon equation (3.47).

The other Poincaré group generators can also be obtained in the same manner as in Sec. II. These have regular Dirac bracket properties with A^{ij}. An additional term in the same manner as in the right-hand side of Eq. (2.48) is needed when Dirac brackets of the Lorentz boost generators with A^{ij} are taken. This term ensures that the constraints are valid in the new Lorentz frame.

C. Null-plane quantization

The constraints (3.35) and (3.19) leave only one of the components of $A^{\mu\nu}$ independent. The Dirac brackets for this independent component, $A^{12}(x)$, given in Eq. (3.44) are the starting point of the quantum theory. In quantum theory these Dirac brackets are replaced by commutators, $[A,B]^*_P\to(1/i)[A_{op},B_{op}]$. Thus the canonical commutation relation in this case are

$$\begin{aligned}
&[A^{ij}(x^+,x^-,\underline{x}),A^{lk}(x^+,y^-,\underline{y})]\\
&\quad=-\frac{i}{2}(g^{i1}g^{j2}-g^{i2}g^{j1})(g^{l1}g^{k2}-g^{l2}g^{k1})F(x-y),\\
&[\Pi^{ij}(x^+,x^-,\underline{x}),\Pi^{lk}(x^+,y^-,\underline{y})]\\
&\quad=\frac{i}{2}(g^{i1}g^{j2}-g^{i2}g^{j1})(g^{l1}g^{k2}-g^{l2}g^{k1})\partial_-\delta^3(x-y),\\
&[\Pi^{ij}(x^+,x^-,\underline{x}),A^{lk}(x^+,y^-,\underline{y})]\\
&\quad=-\frac{i}{2}(g^{i1}g^{j2}-g^{i2}g^{j1})(g^{l1}g^{k2}-g^{l2}g^{k1})\delta^3(x-y).
\end{aligned} \tag{3.48}$$

IV. CONCLUDING REMARKS

We have presented the quantization program on the constant-time surface and also on the null plane for a massless antisymmetric tensor gauge fields of second rank by invoking Dirac's method[5] of quantization of constrained Hamiltonian systems. The presence of arbitrary functions, reflected by the first-class constraints of the theory, is disposed of by imposing gauge constraints. These gauge constraints are ensured by invariance of the free-field Lagrangian (2.1) under the gauge transformations (2.3). The gauge constraints make the set of constraints second class, which are put strongly equal to zero, once the Dirac brackets are obtained. Eliminating the redundant components of the gauge field $A^{\mu\nu}$ and their conjugate momenta $\Pi^{\mu\nu}$, we are left with one each of $A^{\mu\nu}$ and $\Pi^{\mu\nu}$ as independent variables in the case of quantization on the constant-time surfaces and only one (A^{12}) in the case of the null plane quantization. The quantization is achieved by replacing the Dirac brackets on these independent components by $\frac{1}{2}$ times their commutators.

In a similar fashion, as above, we can develop the quantization in other gauges, for example, in the axial gauge, which has been studied in the context of electromagnetic field theory by Arnowitt and Fickler.[8]

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