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Variational method for boson scattering

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The Hamiltonian for a scalar field interacting with a Schrödinger source is treated by coherent-state methods. It is shown that meson scattering can be calculated by a variational procedure.

I. INTRODUCTION

Consider a theory of bosons and fermions with Hamiltonian

$$H=\int\omega(k)a^\dagger(k)a(k)dk+\int t(p)\tilde{\psi}^\dagger(p)\tilde{\psi}(p)dp$$
$$+\int c(k)[\tilde{j}^\dagger(k)a(k)+a^\dagger(k)\tilde{j}(k)]dk\,, \qquad (1)$$
$$\tilde{j}(k)=\int e^{-ik\cdot r}j(r)dr\,,$$
$$j(r)=\psi^\dagger(r)\psi(r)\,.$$

That is, the bosons and fermions of momentum k have (bare) energies $\omega(k)$ and $t(k)$, respectively; the boson emission and absorption are by the fermion current $j(r)$. The Hamiltonian of Eq. (1) describes a prototype of theories of bosons and fermions with interaction of the Yukawa type. A realization of such a theory is the polaron; in that case $\omega(k)$ is a constant ω, $t(p)$ is $p^2/2m$, and $c(k)$ is $\gamma^{1/2}\omega^{3/4}/2\pi m^{1/4}k$, with m the electron mass and γ a dimensionless coupling constant related to the more usual coupling constant α by $\gamma=\sqrt{2}\,\alpha$. Besides this particular example, a set of theories with Hamiltonians similar to Eq. (1) has been given in Ref. 1; these theories can all be treated by methods similar to the one appropriate to the prototype Hamiltonian. Reference 2 describes an analogous situation involving isovector mesons.

The properties of the single-fermion states determined by H have been the subject of extensive investigation. In this paper, attention is focused on boson scattering by the fermion state. The translated-localized-state formalism of Ref. 3 gives a natural way of treating both the fermion ground state and boson scattering in the no-meson, one-meson, two-meson, ... approximations. In this paper it is shown that in the n-meson approximation a single function $Y_n(z)$ has zeros at the fermion states and phase along the scattering cut equal to the scattering phase shift. The function $Y_n(z)$ depends on the particular localized coherent state used as a zeroth approximation to the fermion state. Clearly the best localized coherent state is the one that minimizes the energy of the ground fermion state; this minimization determines a best $Y_n(z)$ and, thus, a best phase-shift function for the boson-fermion scattering in the n-meson approximation. This is the sense in which the scattering is variationally determined.

II. NO-MESON APPROXIMATION

The localized fermion states in the no-meson approximation are just the localized coherent states described in Ref. 3:

$$|x,b,f\rangle=W_x^\dagger\{b\}\int\psi^\dagger(r)f(r-x)dr\,|\Omega\rangle\,,$$
$$W_x^\dagger\{b\}=\exp\left[-\tfrac{1}{2}\int|b(k)|^2dk+\int b(k)a^\dagger(k)e^{-ik\cdot x}dk\right] \qquad (2)$$

where $|\Omega\rangle$ is the boson-fermion vacuum, and $b(k)$ and $f(r)$ are the two functions that determine the localized coherent state. It is easy to see that[3]

$$a(k)|x,b,f\rangle=e^{-ik\cdot x}b(k)|x,b,f\rangle\,,$$
$$e^{-i\hat{P}\cdot(x-y)}|y,b,f\rangle=|x,b,f\rangle\,, \qquad (3)$$

where $\hat{P}$ is the total momentum operator of the system. Now let

$$D_{bf}(y-x)\equiv\langle y,b,f|x,b,f\rangle\,,$$
$$A_{bf}(y-x)\equiv\langle y,b,f|H|x,b,f\rangle\,; \qquad (4)$$

then the translated localized states are

$$|K,b,f\rangle\equiv(2\pi)^{-3/2}\int e^{iK\cdot x}\,|x,b,f\rangle\,dx \qquad (5)$$

and

$$\langle K,b,f\,|Q,b,f\rangle=N_{bf}(K)\delta(K-Q)\,,$$
$$\langle K,b,f\,|H\,|Q,b,f\rangle=H_{bf}(K)\delta(K-Q)\,, \qquad (6)$$

with

$$N_{bf}(K)=\int e^{-iK\cdot x}D_{bf}(x)dx\,,$$
$$H_{bf}(K)=\int e^{-iK\cdot x}A_{bf}(x)dx\,. \qquad (7)$$

In the no-meson approximation, the best translated localized state is the one with functions b and f chosen so as to minimize the functional F_{TLS}

$\{b,f\}$,

$$F_{\mathrm{TLS}}\{b,f\} = H_{bf}(0)/N_{bf}(0)\,. \tag{8}$$

It also follows from (3) and (5) that

$$a(k)\,|K,b,f\rangle = b(k)\,|K-k,b,f\rangle\,. \tag{9}$$

This equation shows that in the one-meson approximation only the states $a^\dagger(k)\,|K,b,f\rangle$ and the states $|K,b,f\rangle$ need be considered; because the no-meson states are coherent states, the states $a(k)\,|K,b,f\rangle$ are automatically included at all stages.

In the following, it is convenient to let

$$W_{bf}(K) = H_{bf}(K)/N_{bf}(K)\,. \tag{10}$$

[As was noted in Ref. 3, it is not a good approximation to determine the effective mass from the small-K behavior of $W_{bf}(K)$.] Also, in the following, $H(K)$, $N(K)$, and $W(K)$ will be written without subscript; in all cases, subscript bf is implied.

III. ONE-MESON APPROXIMATION AND SCATTERING

The one-meson subspace is defined as the space spanned by the states $|K,b,f\rangle$ and $a^\dagger(k)\,|K,b,f\rangle$. From Eqs. (6) and (9),

$$\langle P,b,f\,|a^\dagger(q)\,|Q-q,b,f\rangle = b^*(q)N(P-q)\delta(P-Q)\,, \tag{11}$$

so that the one-meson states orthogonal to the no-meson states are

$$|Q-q,q,b,f\rangle \equiv a^\dagger(q)\,|Q-q,b,f\rangle - b^*(q)\,\frac{N(Q-q)}{N(Q)}\,|Q,b,f\rangle\,. \tag{12}$$

Then

$$\langle P-p,p,b,f\,|Q-q,q,b,f\rangle = \delta(P-Q)\left\{N(P-p)\delta(p-q) + b(p)b^*(q)\left[N(P-p-q) - \frac{N(P-p)N(Q-q)}{N(P)}\right]\right\}. \tag{13}$$

In the matrix elements of H, the matrix element of $\tilde{j}(k)$ is specified by

$$\langle K,b,f\,|\tilde{j}(k)\,|Q,b,f\rangle = \delta(K+k-Q)J_{bf}(k,Q)\,; \tag{14}$$

then some algebra gives

$$\begin{aligned}
\langle P,b,f\,|H\,|Q-q,q,b,f\rangle &= \delta(P-Q)\{b^*(q)N(P-q)[W(P-q)-W(P)+\omega(q)] + c(q)J^*(q,P\}\,,\\
\langle P-p,p,b,f\,|H\,|Q-q,q,b,f\rangle &= \delta(P-Q)\Big[\delta(p-q)N(P-p)[W(P-p)+\omega(p)]\\
&\quad + b(p)b^*(q)\Big(N(P-p-q)[W(P-p-q)+\omega(p)+\omega(q)]\\
&\quad - \frac{N(P-p)N(P-q)}{N(P)}\,[W(P-p)+W(P-q)\\
&\quad - W(P)+\omega(p)+\omega(q)]\Big)\\
&\quad + c(p)b^*(q)\Big(J(p,P-q) - \frac{N(P-q)}{N(P)}\,J(p,P)\Big)\\
&\quad + c(q)b^*(p)\Big(J^*(q,P-p) - \frac{N(P-p)}{N(P)}\,J^*(q,P)\Big)\Big]\,.
\end{aligned} \tag{15}$$

Now consider a superposition of no- and one-boson states

$$|P,Z,F,b,f\rangle = Z\,|P,b,f\rangle + \int F(p)\,|P-p,p,b,f\rangle\,dp\,. \tag{16}$$

Then

$$\begin{aligned}
\langle P,Z,F,b,f\,|Q,Z,F,b,f\rangle &= \delta(P-Q)N_F(P)\,,\\
\langle P,Z,F,b,f\,|H\,|\,Q,Z,F,b,f\rangle &= \delta(P-Q)H_F(P)\,.
\end{aligned} \tag{17}$$

Only $N_F(0)$ and $H_F(0)$ are of interest here; these are given by

$$N_F(0)=N_{FS}+N_{FN},$$

$$N_{FS}=|Z|^2N(0)+\int N(p)\,|F(p)|^2dp\,,$$

$$N_{FN}=\int b(p)F^*(p)\left[N(p+q)-\frac{N(p)N(q)}{N(0)}\right]b^*(q)F(q)dp\,dq\,,$$

$$H_F(0)=H_{FS}+H_{FN}\,, \tag{18}$$

$$H_{FS}=|Z|^2H(0)+Z^*\int V^*(p)N(p)F(p)dp+Z\int F^*(p)N(p)V(p)\,dp+\int[W(p)+\omega(p)]N(p)\,|F(p)|^2\,dp\,,$$

$$H_{FN}=\int F^*(p)\Big[b(p)b^*(q)\Big([W(P+q)+\omega(p)+\omega(q)]N(p+q)-\frac{N(p)N(q)}{N(0)}\,[W(p)+\omega(p)+W(q)+\omega(q)-W(0)]\Big)$$
$$+c(p)b^*(q)\Big(J(p,-q)-\frac{N(q)}{N(0)}\,J(p,0)\Big)+c(q)b(p)\Big(J^*(q,-p)-\frac{N(p)}{N(0)}\,J(q,0)\Big)\Big]F(q)dp\,dq\,,$$

where

$$V_{bf}(p)=[W_{bf}(p)-W_{bf}(0)+\omega(p)]b(p)+c(p)J_{bf}(p,0)/N_{bf}(p) \tag{19}$$

and H and N are assumed to be even functions. In (18), S and N are for "separable" and "nonseparable," respectively.

Consider first the case that N_{FN} and H_{FN} can be neglected. Then it is a standard calculation to show that if $Y(z)$ is defined by (z complex)

$$Y(z)=W(0)-z+\frac{1}{N(0)}\int\frac{|V(p)|^2N(p)dp}{z-[W(p)+\omega(p)]}\,, \tag{20}$$

then the fermion state is at energy E, given by

$$Y(E)=0,\quad E<W(0) \tag{21}$$

and the boson-fermion scattering phase shift at energy E is given by

$$e^{-2i\delta(E)}=Y(E+i\,0)/Y(E-i\,0)\,, \tag{22}$$

that is, $\delta(E)$ is the phase of $Y(E-i0)$.

Clearly, in the one-meson approximation, the best coherent-state functions b and f are the ones that minimize the root E of Eq. (21). Once b and f are fixed, $Y(z)$ is completely determined, and, hence, so is $\delta(E)$.

Inclusion of effects due to N_{FN} and H_{FN} can be accomplished by using separable approximations for $N(p+q)$, $W(p+q)$, and $J(p,q)$; for example, for the spherical average of $N(p+q)$, the M-term separable approximation is

$$\int d\Omega_p\,d\Omega_q\,N(p+q)\cong\sum_{i=1}^{M}\lambda_i n_i(p)n_i(q)\,, \tag{23}$$

where Ref. 4 gives the technique for determining the λ_i and $n_i(p)$. With the separable approximations, it is again possible to find algebraically the function $Y(z)$ that has zeros at the (possibly several) fermion states and phase $\delta(E)$ along the lower side of the elastic cut. Again the one-meson coherent-state functions b and f are variationally determined, and, hence, so is $\delta(E)$ in the one-meson approximation.

IV. REMARKS AND SUMMARY

It is evident that the procedure can be carried over to the two-, three-, and n-meson subspaces. In each case, the energies and phase shifts are (implicit) functionals of the coherent-state functions b and f. Once b and f are variationally determined by minimizing the energy of the lowest state, then the entire scattering matrix is determined in the n-meson approximation.

Thus, the translated coherent states provide a systematic framework for computing bound and scattering states of the Hamiltonian of Eq. (1). Since the procedure is variational at every stage, both weak and strong coupling can be treated in the same way.

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