One-dimensional field theories with odd-power self-interactions

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Classical solutions to nonlinear field theories are considered as model particles. Two fields are examined here, the $\lambda\phi^3$ field and a generalization of the sine-Gordon system. Each of these fields is in one space dimension and quantization is accomplished using the WKB method. Static solutions to the $\lambda\phi^3$ field are shown to represent objects with an internal structure resembling a dumbbell. The quantum mass of these objects is computed in the weak-coupling limit and an approximate expression for the classical force between two of these objects is obtained. This force seems to be attractive and constant at large separations. In the case of the generalized sine-Gordon field it is shown that classical solutions to the field equation may be obtained by a transformation from known solutions to the sine-Gordon equation. The behavior of this field is therefore similar to that of the sine-Gordon field.

I. INTRODUCTION

Recently there has been interest in constructing model particles from exact solutions to classical, nonlinear, partial differential equations. The procedure employed is to begin with a known solution to the classical field equation and to construct quantum states from it. The difficulty here is that there are not many exact solutions available for nonlinear field equations. Recently some static solutions have become available for a number of fields which have not previously been considered in the literature.[1] The purpose of this work is to examine some of these fields, in particular those which contain odd-power self-interaction terms. The fields to be considered here will be in one space dimension.

The behavior of both free and interacting particles will be examined at the classical level and first-order quantum corrections to the classical mass of these objects will be obtained using semiclassical methods developed for field theory by Dashen, Hasslacher, and Neveu.[2] It will be assumed that the reader is generally familiar with the application of these methods and with the behavior of other one-dimensional field theories. Units will be chosen such that $\hbar = c = 1$.

The fields to be considered here have a Lagrangian density of the form

$$\mathcal{L} = \tfrac{1}{2}(\phi_t)^2 - \tfrac{1}{2}(\phi_x)^2 + F(\phi)\,, \tag{1.1}$$

where $F(\phi)$ is a functional of the real, scalar field $\phi(x, t)$. The subscripts indicate differentiation. Two cases are to be considered here:

$$F(\phi) = -\tfrac{1}{2}m^2\phi^2 + \tfrac{1}{2}\lambda\phi^3\,, \tag{1.2a}$$

$$F(\phi) = \frac{m^4}{\lambda}\left(A\cos\frac{\sqrt{\lambda}\phi}{m} + B\sin\frac{\sqrt{\lambda}\phi}{m} - 1\right). \tag{1.2b}$$

The $\lambda\phi^3$ field, Eq. (1.2a), is the simplest example of a Lagrangian density which contains an odd-power self-interaction term. This field is usually not considered suitable for constructing model particles[3] because the Hamiltonian density is not positive definite and therefore a state of lowest energy may not exist. However, when quantization about a static solution to the classical field equation is carried out, we are dealing with a sector of Hilbert space separate from that which would be accessible by the use of conventional perturbation theory and a conservation law may exist which would prohibit transitions from one sector to the other. Therefore, even though states built around the classical vacuum might undergo radiative collapse, this would not be allowed for states constructed from static solutions to the classical field equation.

A static solution to the $\lambda\phi^3$ field equation will be shown to represent an apparently unstable particle with an internal structure resembling a dumbbell. First-order quantum contributions to the mass of this object will be computed in the weak-coupling limit and a calculation of the classical force between two of these objects will be made. The result obtained indicates that at large separations this force is attractive and constant.

In the second case, Eq. (1.2b), the Lagrangian density contains an arbitrary linear combination of sine and cosine functions. The sine function may be considered to be an infinite sum of odd-power self-interaction terms. For this case the Hamiltonian density of the system will be positive definite for all real ϕ. It will be shown that both static and time-dependent solutions to the equation of motion may be generated by a simple transformation from those of the well-known sine-Gordon field. The classical and quantum characteristics of solutions obtained in this manner will therefore be similar to those of the sine-Gordon field.

II. $\lambda\phi^3$ FIELD

The Lagrangian density corresponding to the choice of $F(\phi)$ in Eq. (1.2a) is

$$\mathcal{L} = \tfrac{1}{2}(\phi_t)^2 - \tfrac{1}{2}(\phi_x)^2 - \tfrac{1}{2}m^2\phi^2 + \tfrac{1}{2}\lambda\phi^3 \tag{2.1}$$

and the corresponding Hamiltonian density is

$$\mathcal{H} = \tfrac{1}{2}(\phi_t)^2 + \tfrac{1}{2}(\phi_x)^2 + \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{2}\lambda\phi^3 . \tag{2.2}$$

The equation of motion for this system will be

$$\phi_{tt} - \phi_{xx} + m^2\phi - \tfrac{3}{2}\lambda\phi^2 = 0 . \tag{2.3}$$

A. Constant solutions

There is a constant solution, $\phi = 0$, to Eq. (2.3), which is a minima of the potential energy functional $V(\phi)$,

$$V(\phi) = \tfrac{1}{2}(\phi_x)^2 + \tfrac{1}{2}m^2\phi^2 - \tfrac{1}{2}\lambda\phi^3 . \tag{2.4}$$

This solution is nondegenerate and represents the classical vacuum. However, this solution is not an absolute minima of $V(\phi)$ so that while the vacuum state will be stable classically, when the system is quantized there will be the possibility of decay of the vacuum and states around it via tunneling. In the weak-coupling limit $\lambda \to 0$, the barrier separating these states from those of the continuum becomes very high and wide, and so even though these states may not be absolutely stable they could be very long lived.

B. Static solutions

There are two static solutions to the equation of motion, Eq. (2.3), which are valid for all real λ.[1] They are

$$\phi = -\frac{m^2}{\lambda}\operatorname{csch}^2\frac{m(x-x_0)}{2} , \tag{2.5a}$$

$$\phi = +\frac{m^2}{\lambda}\operatorname{sech}^2\frac{m(x-x_0)}{2} . \tag{2.5b}$$

The corresponding Hamiltonian densities will be

$$\mathcal{H} = \frac{m^6}{\lambda^2}\operatorname{csch}^4\frac{m(x-x_0)}{2}\coth^2\frac{m(x-x_0)}{2} , \tag{2.6a}$$

$$\mathcal{H} = \frac{m^6}{\lambda^2}\operatorname{sech}^4\frac{m(x-x_0)}{2}\tanh^2\frac{m(x-x_0)}{2} . \tag{2.6b}$$

Each of these classical Hamiltonian densities will be positive definite for all real λ. In the first case, Eq. (2.5a), both the solution and its Hamiltonian density are singular at $x = x_0$, this leads to an infinite value of the classical energy for this solution. Since this behavior is not acceptable for a real particle, this particular solution will not be given further consideration.

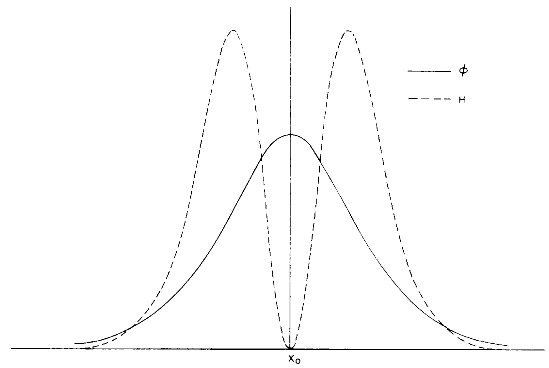

FIG. 1. The field ϕ and Hamiltonian density $\mathcal{H}$ corresponding to the static solution for the $\lambda\phi^3$ field. x_0 denotes the location of the center of the extended object.

The Hamiltonian density for the second solution, Eq. (2.5b), is bounded and therefore this solution will be interpreted as representing a stationary, extended object. An expression for a moving object could be obtained by a Lorentz transformation to another inertial frame, but this will not be necessary for purposes of this work.

A plot of $\phi(x)$ and its corresponding Hamiltonian density is given in Fig. 1. As can be seen from the Hamiltonian density this object has a dumbbell-like internal structure. At this point it should be noted that solutions of this type also exist for fields which contain higher-power, polynomial self-interaction terms.[1]

The classical stability of this solution may be examined by applying a small perturbation. Let

$$\phi(x,t) = \phi_0(x) + \eta(x)e^{i\omega t} , \tag{2.7}$$

and if we require that $\phi(x,t)$ satisfy the equation of motion, Eq. (2.3), then for small $\eta(x)$ we must have

$$\frac{d^2\eta}{dx^2} + \eta(3\lambda\phi_0 + \omega^2 - m^2) = 0 , \tag{2.8}$$

where $\phi_0(x)$ is as given in Eq. (2.5b). This is the same differential equation which will be obtained when computing quantum corrections to the mass of this object and it will be shown that there is one eigenvalue for which $\omega^2 < 0$. This result indicates that this particular static solution will be unstable. However, since $\eta(x)$ actually must satisfy a nonlinear differential equation, this conclusion is not certain.

The classical mass of this object may be computed from the Hamiltonian density by

$$M = \int_{-\infty}^{\infty} \mathcal{H}(x)dx = \frac{8m^5}{15\lambda^2} . \tag{2.9}$$

As can be seen from this result the mass of this object will be large in the weak-coupling limit. This is typical behavior for these types of fields.

III. QUANTIZATION

First-order quantum corrections to the classical characteristics of this field may be computed using path-integral methods. Since the only classical solutions available are time independent, the results obtained will be valid only in the weak-coupling limit. For a Lagrangian density of the form

$$\mathcal{L} = \tfrac{1}{2}(\phi_t)^2 - \tfrac{1}{2}(\phi_x)^2 + F(\phi), \tag{3.1}$$

The propagator of the Schrödinger equation for the state vector of the system may be expressed as a path integral[4]:

$$K(b,a) = \int_{\text{all }\phi} D[\phi]\exp\left\{i\int_{t_a}^{t_b}\int_{-L/2}^{+L/2} dx\left[\tfrac{1}{2}(\phi_t)^2 - \tfrac{1}{2}(\phi_x)^2 + F(\phi)\right]\right\}. \tag{3.2}$$

the energy eigenvalues of the system may be obtained from the location of the poles of the Laplace transform of the propagator.

In general the above path integral cannot be evaluated exactly so it is necessary to make an approximation. The procedure[2,5] is to expand the field into normal modes about the classical solution, retain only those modes which lie near this solution and to treat each of these modes as a quantum harmonic oscillator.

In the weak-coupling limit, $\lambda \to 0$, the equation which must be satisfied by each normal mode is[2,5]

$$\frac{d^2u_i}{dx^2} + \left(\omega_i^2 + \frac{\partial^2 F}{\partial\phi^2}\bigg|_{\phi=\phi_0}\right)u_i = 0, \tag{3.3}$$

where $F(\phi)$ is from Eq. (3.1) and $\phi_0(x)$ is the classical solution to the field equation.

With these results the energy of the system becomes

$$E = E_c + \sum_i (N_i + \tfrac{1}{2})\omega_i, \tag{3.4a}$$

where

$$E_c = \int_{-\infty}^{+\infty} \mathcal{H}(x)dx. \tag{3.4b}$$

These results will now be applied to the $\lambda\phi^3$ field.

A. $\lambda\phi^3$ field: vacuum sector

The relevant classical solution here is $\phi = 0$, this gives $E_c = 0$, and

$$\frac{\partial^2 F}{\partial\phi^2}\bigg|_{\phi=\phi_0} = -m^2. \tag{3.5}$$

The equation for the normal modes becomes

$$\frac{d^2u_i}{dx^2} + (\omega_i^2 - m^2)u_i = 0. \tag{3.6}$$

Therefore, the normal modes of the vacuum sector will be plane waves of momentum $q_i = 2n\pi/L$, n = integer. The total energy of the vacuum sector will be

$$E_{\text{vac}} = \sum_i (n_i + \tfrac{1}{2})\omega_i, \tag{3.7a}$$

where

$$\omega_i = (q_i^2 + m^2)^{1/2}. \tag{3.7b}$$

And so quantization yields excitations associated with the classical vacuum. These excitations may be interpreted to be mesons of mass m. In the weak-coupling limit the mass of these mesons will be small as compared to the classical mass of the static solution. This is the sector of Hilbert space which would be accessible by conventional perturbation theory.

B. $\lambda\phi^3$ field: heavy-particle sector

Here the relevant classical solution is given by Eq. (2.5b) and the classical mass, E_c, by Eq. (2.9). The equation to be satisfied by each normal mode in this case is

$$\frac{d^2u_i}{dx^2} + \left(\omega_i^2 - m^2 + 3m^2\operatorname{sech}^2\frac{mx}{2}\right)u_i = 0. \tag{3.8}$$

This equation is exactly soluable.[6] There are three discrete modes and a continuum. The eigenvalues and unnormalized modes are

$$\omega_0^2 = -\frac{5m^2}{4}, \quad u_0 = \operatorname{sech}^3\frac{mx}{2}, \tag{3.9a}$$

$$\omega_1^2 = 0, \quad u_1 = \tanh\frac{mx}{2}\operatorname{sech}^2\frac{mx}{2}, \tag{3.9b}$$

$$\omega_2^2 = \frac{3m^2}{4}, \quad u_2 = 4\tanh^2\frac{mx}{2}\operatorname{sech}\frac{mx}{2} - \operatorname{sech}^3\frac{mx}{2}, \tag{3.9c}$$

$$\omega_k^2 = \frac{m^2}{4}(k^2+4),$$
$$u_k = e^{ik(mx/2)}\operatorname{sech}^3\frac{mx}{2}(Ae^{-3mx/2} + Be^{-mx/2} + Ce^{+mx/2} + De^{+3mx/2}), \tag{3.9d}$$

where

$$
\begin{aligned}
A &= ik^3 - 11ik + 6k^2 - 6\,, \\
B &= 3ik^3 + 27ik + 6k^2 + 54\,, \\
C &= 3ik^3 + 27ik - 6k^2 - 54\,, \\
D &= ik^3 - 11ik - 6k^2 + 6\,.
\end{aligned} \tag{3.10}
$$

If $u_k(x)$ is to satisfy periodic boundary conditions as the length of the interval, L, becomes large, then we must have

$$
n\pi = \frac{kmL}{4} + \tan^{-1}\left(\frac{k(k^2-11)}{6(1-k^2)}\right). \tag{3.11}
$$

It should be noted that the lowest mode has an imaginary eigenvalue, $\omega_0^2 < 0$. This is a result of the apparent instability of the classical solution. Physically, this mode corresponds to a harmonic oscillator with a reversed sign on the potential energy. Considering each mode as a quantum harmonic oscillator, the total energy of the heavy particle will be given by

$$
E = \frac{8m^5}{15\lambda^2} - (N_0 + \tfrac{1}{2})\frac{i\sqrt{5}m}{2} + (N_2 + \tfrac{1}{2})\frac{\sqrt{3}m}{2} + \sum_n (N_n + \tfrac{1}{2})\frac{m}{2}(k_n^2 + 4)^{1/2}. \tag{3.12}
$$

As may be seen above, the lowest mode contributes a term to the energy which is imaginary. Since the wave function of the system may be written as

$$
\psi(x, t) = Ne^{-iEt}\Phi(x)\,, \tag{3.13}
$$

the wave function will decay with time and therefore this particle will be unstable.

The next mode has an eigenvalue which is equal to zero, $\omega_1 = 0$. This is the translation mode and is a consequence of working in the rest frame of the particle. If the computations had been performed in an inertial frame in which the particle had been moving, then the correct energy-momentum relationship would have been obtained here. The remaining discrete mode may be interpreted as an excited state of the particle.

Next come the continuum states which have been made countable by restricting $-L/2 < x < L/2$. These excitations can be interpreted as mesons, of mass m, which are associated with the presence of the heavy particle. These are not the same mesons as were obtained from the quantization of the vacuum sector.

The next quantity of interest to be computed is the quantum mass of this object. The mass will be given by the difference between the zero-point energy of the heavy particle and the zero-point energy of the vacuum. Therefore,

$$
M = \frac{8m^5}{15\lambda^2} - \frac{i\sqrt{5}m}{2} + \frac{\sqrt{3}m}{2} + \tfrac{1}{2}\sum_{k_n}\frac{m}{2}(k_n^2 + 4)^{1/2} - \tfrac{1}{2}\sum_{q_n}(q_n^2 + m^2)^{1/2}\,, \tag{3.14}
$$

where for large L we must have

$$
\frac{q_n L}{2} = n\pi = \frac{k_n mL}{4} + \tan^{-1}\left(\frac{k_n(k_n^2 - 11)}{6(1-k_n^2)}\right). \tag{3.15}
$$

If the mass is computed here it will be found to be infinite. The reason for this is that up to this point in the computations, the classical form of the Lagrangian density has been used. This theory can be renormalized[7] by either working with the normal-ordered form of the Lagrangian density or by adding the appropriate counterterms to remove the ultraviolet divergences. The latter procedure will be followed here. Since

$$
\phi^2 = :\phi^2: - C_1'\,, \tag{3.16a}
$$

$$
\phi^3 = :\phi^3: - C_2'\,\phi\,, \tag{3.16b}
$$

we must add two counterterms to the Lagrangian density to cancel the divergences. C_1' and C_2' are divergent constants in momentum space.

With the addition of these counterterms the renormalized Lagrangian density becomes

$$
\mathcal{L} = \tfrac{1}{2}(\phi_t)^2 - \tfrac{1}{2}(\phi_x)^2 - \tfrac{1}{2}m^2\phi^2 + \tfrac{1}{2}\lambda\phi^3 + C_2\phi + C_1\,, \tag{3.17}
$$

where

$$
C_1 = \frac{1}{4\pi}\int_{-\Lambda_1}^{\Lambda_1}(q^2 + m^2)^{1/2}dq \tag{3.18a}
$$

and

$$
C_2 = -\frac{3\lambda}{8\pi}\int_{-\Lambda_2}^{\Lambda_2}\frac{k^4 dk}{(k^2+4)^{1/2}(k^2+1)(k^2+9)} - \frac{\lambda}{16}(\Lambda_2^2 + 4)^{1/2}. \tag{3.18b}
$$

The cutoffs Λ_1, Λ_2 are to be kept finite until all computations are performed and then the limits $\Lambda_1, \Lambda_2 \to \infty$ are to be taken. C_1 will remove the divergence from the zero-point energy of the vacuum and C_2 will remove the divergence from the mass of the heavy particle.

The mass of this object may now be computed, and the result is

$$
M = \frac{8m^5}{15\lambda^2} - \frac{i\sqrt{5}m}{2} + \frac{\sqrt{3}m}{2} - \frac{3m}{\pi} - \frac{\sqrt{3}m}{8} - \frac{61m}{16\sqrt{5}}\ln\left(\frac{3+\sqrt{5}}{3-\sqrt{5}}\right). \tag{3.19}
$$

And so in the weak-coupling limit the mass of this object is dominated by its classical value and the quantum corrections are relatively small. The imaginary part of the mass is due to the apparent instability of the classical solution.

IV. CLASSICAL INTERACTIONS

It would be interesting to know how two of these classical objects interact. However, for this field there are no exact solutions available which represent two interacting particles. Some idea of how two of these objects interact at the classical level may be obtained by using a method which was applied to the sine-Gordon equation by Rubenstein.[8] The idea is to represent the interaction between two of these objects by the solution of the initial-value problem corresponding to two, free, stationary particles placed some distance apart at $t=0$. The procedure is to obtain the first nonzero term in the Taylor series solution to this problem. This is then taken as an approximate solution which describes the interaction of two of these objects for small times.

The Lagrangian density will be given by Eq. (2.1). If the independent and dependent variables are rescaled according to

$$mt \to t, \quad mx \to x, \quad \frac{\lambda\phi}{m^2} \to \phi, \tag{4.1}$$

then in terms of the new variables, the Lagrangian density becomes

$$\mathcal{L} = \frac{m^6}{\lambda^2}\left[\tfrac{1}{2}(\phi_t)^2 - \tfrac{1}{2}(\phi_x)^2 - \tfrac{1}{2}\phi^2 + \tfrac{1}{2}\phi^3\right]. \tag{4.2}$$

The equation of motion will be

$$\phi_{tt} - \phi_{xx} + \phi - \tfrac{3}{2}\phi^2 = 0 \tag{4.3}$$

and this is to be solved subject to the initial conditions

$$\phi(x, 0) = \operatorname{sech}^2\frac{(x+x_0)}{2} + \operatorname{sech}^2\frac{(x-x_0)}{2}, \tag{4.4a}$$

$$\phi_t(x, 0) = 0. \tag{4.4b}$$

Make a Taylor series expansion of $\phi(x, t)$ about $t=0$ and we obtain

$$\phi(x, t) = \phi(x, 0) + \phi_t(x, 0)t + \tfrac{1}{2}\phi_{tt}(x, 0)t^2 + \cdots . \tag{4.5}$$

Owing to the initial conditions, the linear term will be absent. If we retain only the first nonvanishing term, then $\phi(x, t)$ will be given by

$$\phi(x, t) \simeq \phi(x, 0) + \tfrac{1}{2}\phi_{tt}(x, 0)t^2, \tag{4.6}$$

which will be valid at small times. Now $\phi_{tt}(x, 0)$ may be obtained from the equation of motion, Eq. (4.3). So

$$\phi_{tt}(x, 0) = \phi_{xx}(x, 0) - \phi(x, 0) + \tfrac{3}{2}\phi^2(x, 0), \tag{4.7}$$

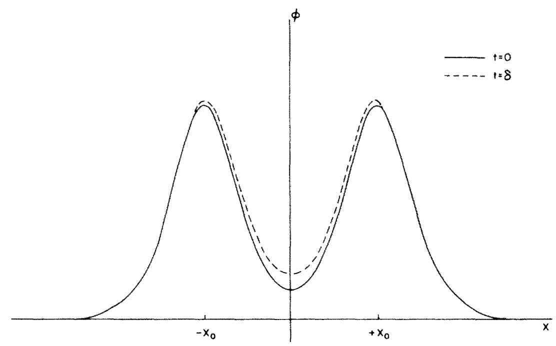

FIG. 2. The field ϕ, as a function of position x, at two times, $t=0$ and $t=\delta$, for the case of two extended objects initially located at $x=+x_0$ and $x=+x_0$.

which gives

$$\phi_{tt}(x, 0) = 3\operatorname{sech}^2\frac{(x+x_0)}{2}\operatorname{sech}^2\frac{(x-x_0)}{2}. \tag{4.8}$$

And so for small times $\phi(x, t)$ will be given approximately by

$$\phi(x, t) \simeq \operatorname{sech}^2\frac{(x+x_0)}{2} + \operatorname{sech}^2\frac{(x-x_0)}{2} + \tfrac{3}{2}t^2\operatorname{sech}^2\frac{(x+x_0)}{2}\operatorname{sech}^2\frac{(x-x_0)}{2}. \tag{4.9}$$

A plot of $\phi(x, t)$ is given in Fig. 2. From the graph it appears that there will be an attractive force between two of these objects.

If we assume that at large separations the change in $\phi(x, t)$ can be accounted for by a rigid displacement of each of the two objects then it is possible to obtain an approximate expression for the force between these objects. For the object at $x=x_0$ a rigid displacement gives

$$\delta\phi\,|_{x_0} = \frac{\partial\phi}{\partial x_0}\bigg|_{x_0}\delta x_0 = -\operatorname{sech}^2 x_0 \tanh x_0\,\delta x_0, \tag{4.10}$$

but we also must have

$$\delta\phi\,|_{x_0} = [\phi(x, t) - \phi(x, 0)]_{x_0} = \tfrac{3}{2}t^2\operatorname{sech}^2 x_0. \tag{4.11}$$

Equating these two expression gives

$$\delta x_0 = -\tfrac{3}{2}t^2(\tanh x_0)^{-1}. \tag{4.12}$$

The distance between the two objects, r, is equal to $2x_0$, and so the force on the particle located at $x=x_0$ will be

$$F = M(\delta x_0)_{tt} = -\frac{24m^5}{\lambda^2}\coth\frac{r}{2} \tag{4.13}$$

and the corresponding potential energy function will be

$$V(r) = +\frac{48}{15}\frac{m^5}{\lambda^2}\ln\left(\sinh\frac{r}{2}\right). \tag{4.14}$$

For large r these expressions become

$$F = -\frac{24}{15}\frac{m^5}{\lambda^2}, \tag{4.15a}$$

$$V = +\frac{48}{15}\frac{m^5}{\lambda^2}\left(\frac{r}{2} - \ln 2\right). \tag{4.15b}$$

These results indicate that at large separations the force between two of these classical objects is attractive and constant. The fact that this force is attractive suggests the possibility that a bound state of two particles may exist. If so, it might be possible that such a state would be stable due to nonlinear effects even though the individual solutions are not. If such a bound state were to exist then it would seem that the two particles would be permanently confined.

V. FUNCTIONAL INTERACTIONS

The field for which $F(\phi)$ is given by Eq. (1.2b) will now be considered. The Lagrangian density in this case will be

$$\mathcal{L} = \tfrac{1}{2}(\phi_t)^2 - \tfrac{1}{2}(\phi_x)^2 + \frac{m^4}{\lambda}\left(A\cos\frac{\sqrt{\lambda}\phi}{m} + B\sin\frac{\sqrt{\lambda}\phi}{m} - 1\right), \tag{5.1}$$

where A and B are constants with $A^2 + B^2 = 1$.

The motivation for considering this particular Lagrangian density is that classical solutions are available[1] and there is the possibility that due to the presence of the sine function in the Lagrangian density the interaction between two of the basic static solutions may be of some interest.

If we let $A = \cos\alpha$ and $B = \sin\alpha$ then the Lagrangian density may be written as

$$\mathcal{L} = \tfrac{1}{2}(\phi_t)^2 - \tfrac{1}{2}(\phi_x)^2 + \frac{m^4}{\lambda}\left[\cos\left(\frac{\sqrt{\lambda}\phi}{m} - \alpha\right) - 1\right]. \tag{5.2}$$

The Hamiltonian density will be

$$\mathcal{H} = \tfrac{1}{2}(\phi_t)^2 + \tfrac{1}{2}(\phi_x)^2 - \frac{m^4}{\lambda}\left[\cos\left(\frac{\sqrt{\lambda}\phi}{m} - \alpha\right) - 1\right] \tag{5.3}$$

and the equation of motion is

$$\phi_{tt} - \phi_{xx} + \frac{m^3}{\sqrt{\lambda}}\sin\left(\frac{\sqrt{\lambda}\phi}{m} - \alpha\right) = 0. \tag{5.4}$$

If we now introduce a change in the dependent variable, let

$$\phi' = \phi - \frac{\alpha m}{\sqrt{\lambda}}, \tag{5.5}$$

then the equation of motion becomes

$$\phi'_{tt} - \phi'_{xx} + \frac{m^3}{\sqrt{\lambda}}\sin\frac{\sqrt{\lambda}\phi'}{m} = 0, \tag{5.6}$$

which is just the familiar sine-Gordon equation.

Because of the existence of the transformation, Eq. (5.5), it is possible to generate solutions to the equation of motion, Eq. (5.4), from the well-known solutions to the sine-Gordon equation. This procedure may be applied to both static and time-dependent solutions. For example, static solutions to the pendulum equation and cosine-Gordon equation, obtained by Hu[1] using direct integration, may be obtained from the kink solutions to the sine-Gordon equation[5] and the previous transformation, Eq. (5.5), with the values of $\alpha = \pi$ and $\alpha = -\pi/2$, respectively.

A. Classical mass

The classical mass of any solution to Eq. (5.4) may be computed from the Hamiltonian density,

$$M = \int_{-\infty}^{\infty}\mathcal{H}(x)dx = \int_{-\infty}^{\infty}dx\left\{\tfrac{1}{2}(\phi_t)^2 + \tfrac{1}{2}(\phi_x)^2 - \frac{m^4}{\lambda}\left[\cos\left(\frac{\sqrt{\lambda}\phi}{m} - \alpha\right) - 1\right]\right\}. \tag{5.7}$$

Using the transformation, Eq. (5.5), then

$$M = \int_{-\infty}^{\infty}dx\left[\tfrac{1}{2}(\phi'_t)^2 + \tfrac{1}{2}(\phi'_x)^2 - \frac{m^4}{\lambda}\left(\cos\frac{\sqrt{\lambda}\phi'}{m} - 1\right)\right], \tag{5.8}$$

which is just the expression for the classical mass of the corresponding solution to the sine-Gordon equation.

B. Quantum mass, static solutions

The procedure used to compute the quantum mass of static solutions to Eq. (5.4) will be the same as for the $\lambda\phi^3$ field. The equation for the normal modes will be

$$\frac{d^2u_i}{dx^2} + \left[\omega_i^2 - m^2\cos\left(\frac{\sqrt{\lambda}\phi}{m} - \alpha\right)\right]u_i = 0, \tag{5.9}$$

but with the transformation, Eq. (5.5), this becomes

$$\frac{d^2u_i}{dx^2} + \left(\omega_i^2 - m^2\cos\frac{\sqrt{\lambda}\phi'}{m}\right)u_i = 0. \tag{5.10}$$

This is just the equation for the normal modes of the sine-Gordon field.[5] Therefore, when the quantum mass of static solutions to Eq. (5.4) is computed it will be found to be the same value as the quantum mass of the corresponding solution to the sine-Gordon equation.

C. Bound states

It is not surprising that the static solutions to Eq. (5.4) have the same mass as the corresponding sine-Gordon kinks, but how do they interact? Owing to the presence of the sine function in the Lagrangian density it is possible that static solutions to Eq. (5.4) may not interact in the same manner as the sine-Gordon kinks. If this is so then the energy spectrum of any solution to Eq. (5.4) corresponding to a kink-antikink bound state would not be the same as for the sine-Gordon case.

A solution to Eq. (5.4) representing a kink-antikink bound state may be obtained by a transformation from the doublet solution of the sine-Gordon equation.[5] The WKB quantization condition in this case will be[9]

$$\int_{-\tau/2}^{+\tau/2} dt \int_{-\infty}^{\infty} dx\pi(x,t)\phi_t(x,t) = 2n\pi, \qquad (5.11)$$

where τ is the period of the doublet solution. For this field $\pi = \phi_t$, and so the quantization condition becomes

$$\int_{-\tau/2}^{\tau/2} dt \int_{-\infty}^{\infty} (\phi_t)^2 = 2n\pi, \qquad (5.12)$$

which will yield the appropriate bound-state spectrum. Again apply the transformation, Eq. (5.5), and the quantization condition becomes

$$\int_{-\tau/2}^{\tau/2} dt \int_{-\infty}^{\infty} dx(\phi_t')^2 = 2n\pi. \qquad (5.13)$$

The period of the solution will remain unchanged by the above transformation and so the WKB quantization condition will yield the same bound-state spectrum as obtained for the sine-Gordon doublet. From these arguments it seems that the behavior of a field whose Lagrangian density contains a linear combination of sine and cosine functions will be basically no different from that of the sine-Gordon field.

Other forms of functional interactions have been examined by Hu[1] at the classical level. However, the solutions to the field equations for these other cases seem generally to be unbounded and therefore have not been considered here.

VI. CONCLUSIONS

The purpose of this work has been to examine the behavior of two particular fields, each of which contain odd-power self-interactions. The idea has been that solutions of the quantum field theory which are constructed from exact solutions to the classical field theory might be considered as model particles.

In the case of the $\lambda\phi^3$ field it would seem that owing to the apparent instabilities in both the vacuum and heavy-particle sectors that this field would not be suitable for constructing model particles. However, the existence of a separately conserved quantity could prevent the radiative collapse of the heavy-particle sector. It is interesting that this field seems to provide a constant, attractive force between two of the heavy particles at large separations.

For the second case it appears that the behavior of a field whose Lagrangian density contains an arbitrary linear combination of sine and cosine functions is essentially no different from that of the sine-Gordon field.

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[1]B. Hu, Nuovo Cimento 38, 441 (1977).
[2]R. F. Dashen, B. Hasslacher, and A. Neveu, Phys. Rev. D 10, 4114 (1974).
[3]G. Baym, Phys. Rev. 117, 886 (1960).
[4]R. Feynman and A. Hibbs, *Quantum Mechanics and Path Integrals* (McGraw-Hill, New York, 1965).
[5]R. Rajaraman, Phys. Rep. 21C, 227 (1975).
[6]P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill, New York, 1953).
[7]W. Thirring, Helv. Phys. Acta 26, 33 (1953).
[8]J. Rubenstein, J. Math. Phys. 11, 258 (1970).
[9]R. Jackiw, Rev. Mod. Phys. 49, 681 (1977).

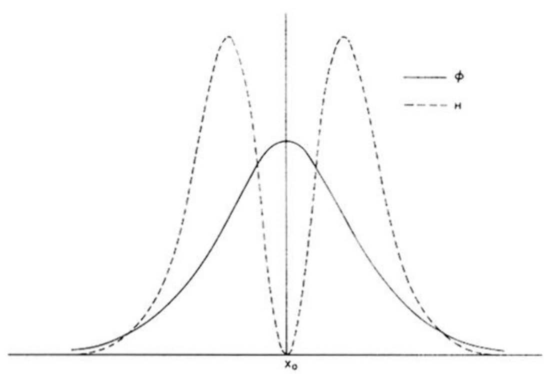

FIG. 1. The field ϕ and Hamiltonian density $\mathcal{H}$ corresponding to the static solution for the $\lambda\phi^3$ field. x_o denotes the location of the center of the extended object.

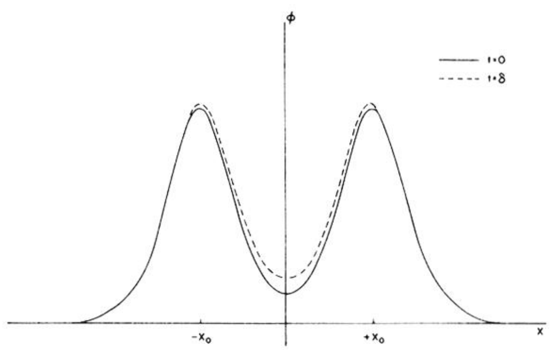

FIG. 2. The field ϕ, as a function of position x, at two times, $t=0$ and $t=\delta$, for the case of two extended objects initially located at $x=+x_o$ and $x=+x_o$.