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Exact spherically symmetric classical solutions for the *f*-*g* theory of gravity

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We find a class of exact spherically symmetric solutions to the coupled classical field equations of f-g theory. The f and g metrics each induce a cosmological constant in the field equations of the other, and are both of Schwarzschild–plus–de Sitter type.

I. INTRODUCTION

The f-g theory was originally introduced[1,2] as a gravitational analog of the vector-dominance hypothesis for electromagnetism. It was postulated that while leptons would couple directly with gravitation, hadrons would only do so by virtue of a mixing between the gravitational field and a massive spin-2 meson which would couple universally to all hadronic matter. In the simplest version of the theory only one such meson is considered and is denoted by f. The mixing term is chosen in such a way that in the weak-field limit it is also responsible for the f-meson mass, having the Fierz-Pauli form required for the mass term of a quantized, ghost-free spin-2 field.

Recently the significance in a quantum context of *classical* solutions of field theories has been established, following investigations into the properties of solitons, pseudoparticles, and related objects. An important class in a (3+1)-dimensional theory is that composed of static, spherically symmetric solutions, and their occurrence in f-g theory is the subject of this paper. (Solutions with other symmetries are discussed in Refs. 3–7.) The possible relevance to strong-interaction physics is twofold. Firstly,[8,9] the existence of horizons in f spacetime might be associated with Hawking radiation,[10] as for ordinary gravity, perhaps connected in some way with the temperature concept in hadron physics.[11] Secondly, it has been suggested[12] that interpretation of the classical f-field solutions as potentials may provide an interesting mechanism for quark confinement.

The coupled f-g equations are highly nonlinear and it is a nontrivial task to obtain genuine solutions. Various attempts[13,14] were made in the past to find such spherically symmetric solutions but concrete progress was only made recently by Salam and Strathdee,[15] who found an explicit solution in the approximation that the g metric is that of Minkowski space.

Although this might seem a physically reasonable approximation, many important questions cannot be satisfactorily resolved within this framework. For example, the role played by coordinate singularities is difficult to discuss in a situation where the g metric is not completely known.

The plan of this paper is as follows. In Sec. II we find a class of exact spherically symmetric solutions of the coupled f and g field equations. Part of this derivation, the computation of the Ricci curvature components, is relegated to the Appendix. In Sec. III we discuss the nature of these solutions, noting that there is a subset for which the g metric is simply that of Minkowski space, and for which the Salam-Strathdee solution becomes exact.

We will use the Landau-Lifshitz (1971) "timelike convention" in which the signature of the metric is −2 and the curvature components are defined in terms of the affine connection by

$$R^{\alpha}{}_{\mu\beta\nu}=\Gamma^{\alpha}{}_{\mu\nu,\beta}-\Gamma^{\alpha}{}_{\mu\beta,\nu}+\Gamma^{\alpha}{}_{\sigma\beta}\Gamma^{\sigma}{}_{\mu\nu}-\Gamma^{\alpha}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\mu\beta}\,, \quad (1.1)$$

$$R_{\mu\nu}=R^{\alpha}{}_{\mu\alpha\nu}. \quad (1.2)$$

In this convention the usual Einstein equations are

$$G_{\mu\nu}-\Lambda g_{\mu\nu}=R_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R-\Lambda g_{\mu\nu}=+\kappa_g{}^2T_{\mu\nu}\,, \quad (1.3)$$

where $\kappa_g{}^2=8\pi G$ (setting $c=\hbar=1$). The metric of Minkowski space will be written in spherical polar coordinates as

$$\eta_{\mu\nu}=\text{diag}(1,-1,-r^2,-r^2\sin^2\theta). \quad (1.4)$$

Quantities pertinent to the f and g metrics will be labeled with superscripts f and g, respectively.

$$\text{Det}\|f_{\mu\nu}\|\equiv f \quad\text{and}\quad \text{Det}\|g_{\mu\nu}\|\equiv g \quad\text{as usual.}$$

II. EXACT SOLUTIONS

The Lagrangian to be considered is

$$L=\int \mathfrak{L}\, d^4x\,, \quad (2.1)$$

where

$$\mathfrak{L}=\mathfrak{L}_g+\mathfrak{L}_f+\mathfrak{L}_{fg}=\frac{-1}{\kappa_g{}^2}\sqrt{-g}\,R^g-\frac{1}{\kappa_f{}^2}\sqrt{-f}\,R^f+\mathfrak{L}_{fg}\,, \quad (2.2)$$

that is, Einstein Lagrangians for the g and f fields and a generally covariant mixing term chosen to give a mass to the quantized physical f-meson field. κ_f is the strong analog of the gravitational coupling constant κ_g.

We take a one-parameter family of mixing terms whose basic structure was first suggested in Ref. 1 (M is the f-meson mass):

$$\mathcal{L}_{fg}=\frac{-M^2}{4\kappa_f^2}(-g)^u(-f)^v(f^{\alpha\beta}-g^{\alpha\beta})(f^{\sigma\tau}-g^{\sigma\tau})\times(g_{\alpha\sigma}g_{\beta\tau}-g_{\alpha\beta}g_{\sigma\tau})\,, \qquad (2.3)$$

where u and v are arbitrary real numbers such that

$$u+v=\tfrac{1}{2}\,, \qquad (2.4)$$

the latter condition ensuring that $\mathcal{L}_{fg}$ is a tensor density of the correct weight. This mixing term has the correct Fierz-Pauli form in the limit $g^{\mu\nu}=\eta^{\mu\nu}$ and $f^{\mu\nu}=\eta^{\mu\nu}+\kappa_f F^{\mu\nu}$ where $F^{\mu\nu}$ is interpreted as the physical f-meson field and terms of order higher than bilinear in $F^{\mu\nu}$ are neglected.

Salam and Strathdee have considered[15] the case in which $u=\frac{1}{2}$ and $v=0$, with the approximation $g_{\mu\nu}=\eta_{\mu\nu}$. We make no such approximation but extend their methods to solve the coupled equations for the f and g fields.

The value of the parameter u is left unspecified for the sake of generality. It will be seen that it plays an important role in determining the "cosmological" (i.e., large-r) behavior of the solutions.

Upon varying $f^{\mu\nu}$, the action principle $\delta L=0$ gives the f-field equations

$$G^f_{\mu\nu}=R^f_{\mu\nu}-\tfrac{1}{2}f_{\mu\nu}R^f=\kappa_f^2T^f_{\mu\nu}\,, \qquad (2.5)$$

where

$$T^f_{\mu\nu}=\frac{M^2}{4\kappa_f^2}\left(\frac{g}{f}\right)^u[vf_{\mu\nu}(f^{\alpha\beta}-g^{\alpha\beta})(f^{\sigma\tau}-g^{\sigma\tau})\times(g_{\sigma\alpha}g_{\beta\tau}-g_{\alpha\beta}g_{\sigma\tau})-2(f^{\alpha\beta}-g^{\alpha\beta})(g_{\alpha\mu}g_{\beta\nu}-g_{\alpha\beta}g_{\mu\nu})]\,. \qquad (2.6)$$

A useful way of re-expressing (2.5) is

$$R^f_{\mu\nu}=\kappa_f^2(T^f_{\mu\nu}-\tfrac{1}{2}f_{\mu\nu}T^f)\,, \qquad (2.7)$$

$$T^f\equiv f^{\mu\nu}T^f_{\mu\nu}\,. \qquad (2.8)$$

Variation of $g^{\mu\nu}$ yields the Einstein equations

$$G^g_{\mu\nu}=R^g_{\mu\nu}-\tfrac{1}{2}g_{\mu\nu}R^g=\kappa_g^2T^g_{\mu\nu}\,, \qquad (2.9)$$

where

$$T^g_{\mu\nu}=\frac{M^2}{4\kappa_f^2}\left(\frac{f}{g}\right)^v[2(f^{\alpha\beta}-g^{\alpha\beta})(g_{\alpha\mu}g_{\beta\nu}-g_{\alpha\beta}g_{\mu\nu})+(f^{\alpha\beta}-g^{\alpha\beta})(f^{\sigma\tau}-g^{\sigma\tau})(ug_{\mu\nu}g_{\alpha\sigma}g_{\beta\tau}-ug_{\mu\nu}g_{\alpha\beta}g_{\sigma\tau}+2g_{\alpha\mu}g_{\sigma\nu}g_{\beta\tau}-2g_{\alpha\mu}g_{\beta\nu}g_{\sigma\tau})]\,. \qquad (2.10)$$

Only the spherically symmetric case will be investigated. Then, without loss of generality, the metrics may be written in the form

$$f_{\mu\nu}dx^\mu dx^\nu=C\,dt^2-2D\,dt\,dr-A\,dr^2-B(d\theta^2+\sin^2\theta\,d\phi^2)\,, \qquad (2.11)$$

$$g_{\mu\nu}dx^\mu dx^\nu=J\,dt^2-K\,dr^2-r^2(d\theta^2+\sin^2\theta\,d\phi^2) \qquad (2.12)$$

with inverses

$$f^{\mu\nu}\partial_\mu\partial_\nu=\frac{A}{\Delta}\partial_t^2-\frac{2D}{\Delta}\partial_t\partial_r-\frac{C}{\Delta}\partial_r^2-\frac{1}{B}(\partial_\theta^2+\sin^{-2}\theta\,\partial_\phi^2)\,, \qquad (2.13)$$

$$g^{\mu\nu}\partial_\mu\partial_\nu=\frac{1}{J}\partial_t^2-\frac{1}{K}\partial_r^2-\frac{1}{r^2}(\partial_\theta^2+\sin^{-2}\theta\,\partial_\phi^2)\,, \qquad (2.14)$$

$$\Delta\equiv AC+D^2>0\,, \qquad (2.15)$$

where A, B, C, D, J, and K are functions of r only. The theory is invariant under simultaneous coordinate transformations for both metrics, and we have used these to optimally simplify the form of $g_{\mu\nu}$.

The components of the Ricci curvature are now computed. The details and intermediate results of the calculation of the components of $R^f_{\mu\nu}$ are given as an appendix. The nonvanishing components are

$$R^f_{tt}=\frac{C}{2\Delta}\left(C''+\frac{B'C'}{B}-\frac{C'\Delta'}{2\Delta}\right),$$

$$R^f_{tr}=R^f_{rt}=\frac{-D}{2\Delta}\left(C''+\frac{B'C'}{B}-\frac{C'\Delta'}{2\Delta}\right), \qquad (2.16)$$

$$R^f_{rr}=\frac{-B''}{B}+\frac{B'^2}{2B^2}-\frac{A}{2\Delta}\left(C''+\frac{B'C'}{B}-\frac{B'\Delta'}{BA}-\frac{C'\Delta'}{2\Delta}\right),$$

$$R^f_{\theta\theta}=\sin^{-2}\theta\,R^f_{\phi\phi}=1-\frac{C}{2\Delta}\left(B''+\frac{B'C'}{C}-\frac{B'\Delta'}{2\Delta}\right).$$

Then the components of $R^g_{\mu\nu}$ are simply obtained by making the replacements

$$C\to J,\quad A\to K,\quad D\to 0,\quad \Delta\to JK,\quad B\to r^2. \tag{2.17}$$

So the nonvanishing components of $R^g_{\mu\nu}$ are

$$R^g_{tt}=\frac{J''}{2K}+\frac{J'}{rK}-\frac{J'^2}{4JK}-\frac{J'K'}{4K^2},$$

$$R^g_{rr}=-\frac{J''}{2J}+\frac{K'}{rK}+\frac{J'K'}{4JK}+\frac{J'^2}{4J^2}, \tag{2.18}$$

$$R^g_{\theta\theta}=\sin^{-2}\theta R^g_{\phi\phi}=1-\frac{1}{K}-\frac{3rJ'}{2JK}-\frac{rK'}{2K^2}.$$

Expressions (2.16) display the simple algebraic identity

$$DR^f_{tt}+CR^f_{tr}=0. \tag{2.19}$$

Hence, from the f-field equations (2.7),

$$DT^f_{tt}+CT^f_{tr}=0, \tag{2.20}$$

which becomes upon substituting the explicit form of the metrics, (2.11)–(2.15),

$$\left(\frac{2r^2}{B}-3\right)D=0. \tag{2.21}$$

It transpires that precisely the same result is obtained from the identity

$$T^g_{tr}=0, \tag{2.22}$$

which is a consequence of $R^g_{tr}=0$ and the Einstein equations (2.9). Thus we can consistently set either

$$B=\tfrac{2}{3}r^2 \tag{2.23}$$

or

$$D=0. \tag{2.24}$$

The resulting solutions will be labeled type I and type II, respectively, following Salam.[9,15]

Unfortunately no explicit type-II solution has yet been found, even in the approximation $g_{\mu\nu}=\eta_{\mu\nu}$. The large-r asymptotic structure has been investigated in detail by Aragone and Chela Flores,[13,14] using this approximation, and has a Yukawa-type behavior for asymptotically flat solutions.

In this paper we will only consider the type-I solutions. It is already clear from (2.23) that there will be no weak-field region, since $f^{\mu\nu}-g^{\mu\nu}\simeq 0$ requires $B\simeq r^2$, *not* $\tfrac{2}{3}r^2$.

At this point it is convenient to display the nonvanishing components of $T^f_{\mu\nu}$ and $T^g_{\mu\nu}$, using the explicit form of the metrics, (2.11)–(2.15), and setting $B=\tfrac{2}{3}r^2$:

$$C^{-1}T^f_{tt}=-D^{-1}T^f_{tr}=-A^{-1}T^f_{rr}=\frac{M^2}{4\kappa_f^2}\left(\frac{9JK}{4\Delta}\right)^u\left[\frac{3v}{2}+\frac{2JK}{\Delta}(1-v)\right], \tag{2.25}$$

$$T^f_{\theta\theta}=\sin^{-2}\theta\, T^f_{\phi\phi}=\frac{M^2r^2}{4\kappa_f^2}\left(\frac{9JK}{4\Delta}\right)^u\left[\frac{4vJK}{3\Delta}-v+3-\frac{2}{\Delta}(JA+KC)\right], \tag{2.26}$$

$$J^{-1}T^g_{tt}=-K^{-1}T^g_{rr}=\frac{M^2}{4\kappa_f^2}\left(\frac{4\Delta}{9JK}\right)^v\left[\frac{3u}{4}-(1+u)\frac{JK}{\Delta}\right], \tag{2.27}$$

$$T^g_{\theta\theta}=\sin^{-2}\theta\, T^g_{\phi\phi}=\frac{M^2r^2}{4\kappa_f^2}\left(\frac{4\Delta}{9JK}\right)^v\left[\frac{uJK}{\Delta}+\frac{3}{2\Delta}(JA+KC)-\frac{3u}{4}-\frac{9}{4}\right]. \tag{2.28}$$

Using the field equations, the simple relations $AT^f_{tt}+CT^f_{rr}=0$ and $KT^g_{tt}+JT^g_{rr}=0$ become

$$AR^f_{tt}+CR^f_{rr}=0 \tag{2.29}$$

and

$$KR^g_{tt}+JR^g_{rr}=0. \tag{2.30}$$

Now (2.16), (2.18), and (2.23) are used to substitute for R^f_{tt}, etc., and following a fair amount of algebra it is found that

$$\Delta'=(JK)'=0, \tag{2.31}$$

i.e., Δ and JK are constants of integration. It is convenient to choose $JK=1$ by a suitable rescaling of the time parameter t, this being the last remaining degree of freedom in our choice of coordinates.

Using these results, the general solution of the remaining f-field equations is found:

$$JA+J^{-1}C=\tfrac{3}{2}\Delta+\tfrac{2}{3}, \tag{2.32}$$

$$C=\tfrac{3}{2}\Delta\left(1-\frac{2\mu_f}{r}-\frac{2\lambda r^2}{9}\right), \tag{2.33}$$

where λ is a constant given by

$$\lambda=\frac{M^2}{4}\left(\frac{9}{4\Delta}\right)^u\left[\frac{3v}{2}+\frac{2}{\Delta}(1-v)\right] \tag{2.34}$$

and μ_f is an integration constant.

The constant λ seems very much like a cosmological constant and in fact, substituting (2.32) back into $T^f_{\mu\nu}$, one finds

$$\kappa_f^2T^f_{\mu\nu}=\lambda f_{\mu\nu}, \tag{2.35}$$

whence

$$G^f_{\mu\nu} - \lambda f_{\mu\nu} = 0\,. \tag{2.36}$$

A very similar result is found for $T^g_{\mu\nu}$. Substituting $JK=1$ and (2.32) into (2.27) and (2.28) yields

$$\kappa_g{}^2 T^g_{\mu\nu} = \Lambda g_{\mu\nu}\,, \tag{2.37}$$

where

$$\Lambda = \frac{M^2\kappa_g{}^2}{4\kappa_f{}^2}\left(\frac{4\Delta}{9}\right)^v\left[\frac{3u}{2} - \frac{2}{\Delta}(1+u)\right]. \tag{2.38}$$

So we see that the f-field configuration induces a cosmological constant in the g-field equations and vice versa.

The g-field equations

$$G^g_{\mu\nu} - \Lambda g_{\mu\nu} = 0 \tag{2.39}$$

in the chosen coordinate system have the standard general spherically symmetric solution

$$J = 1 - \frac{2\mu_g}{r} - \frac{\Lambda r^2}{3}\,, \tag{2.40}$$

where μ_g is another integration constant. Summarizing,

$$g_{\mu\nu}dx^\mu dx^\nu = \left(1 - \frac{2\mu_g}{r} - \frac{\Lambda r^2}{3}\right)dt^2 - \left(1 - \frac{2\mu_g}{r} - \frac{\Lambda r^2}{3}\right)^{-1}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \tag{2.41}$$

$$f_{\mu\nu}dx^\mu dx^\nu = \frac{3\Delta}{2}\left(1 - \frac{2\mu_f}{r} - \frac{2\lambda r^2}{9}\right)dt^2 - 2D\,dt\,dr - A\,dr^2 - \tfrac{2}{3}r^2(d\theta^2 + \sin^2\theta\, d\phi^2)\,, \tag{2.42}$$

$$D^2 = \Delta\left[1 - \left(1 + \frac{9\Delta}{4}\right)X + \frac{9\Delta}{4}X^2\right], \tag{2.43}$$

$$A = \left(1 - \frac{2\mu_g}{r} - \frac{\Lambda r^2}{3}\right)^{-1}\left[\frac{2}{3} + \frac{3\Delta}{2}(1-X)\right], \tag{2.44}$$

where

$$X = \left(1 - \frac{2\mu_f}{r} - \frac{2\lambda r^2}{9}\right)\left(1 - \frac{2\mu_g}{r} - \frac{\Lambda r^2}{3}\right)^{-1} \tag{2.45}$$

and Λ, λ are given by, respectively, Eqs. (2.38) and (2.34). This is the general type-I solution.

III. DISCUSSION

The f- and g-field equations are in a sense decoupled, each set becoming the Einstein equations with (related) cosmological terms. The coordinate system was chosen to simplify the structure of the g metric by expressing it in a conventional diagonal form. Since the general covariance of the theory refers to *simultaneous* coordinate transformations of both metrics the functional form of the f metric is already determined by that of $g_{\mu\nu}$. It is perhaps not surprising that $f_{\mu\nu}$ appears in a form which, were it the only metric in the theory, would be regarded as being associated with a rather unconventional choice of coordinates. In our case this "locking together" of the two metrics in the chosen coordinate system is manifested in Eqs. (2.23) and (2.32) and in the relation between the two cosmological constants.

Both metrics describe a Schwarzschild (or anti-Schwarzschild) plus de Sitter (or anti–de Sitter) spacetime. For a given value of u the cosmological constants are not independent, but are related to each other through their dependence on the constant of integration Δ. On the other hand, the Schwarzschild masses μ_f and μ_g are completely independent. This suggests that it is the cosmological structure that is the most important aspect of the solution, particularly if interpreting it as being "solitonic."

An interesting special case results from the choice

$$\mu_g = 0,\quad \Delta = \frac{4}{3}\left(1 + \frac{1}{u}\right) \text{which implies } \Lambda = 0\,. \tag{3.1}$$

Then $g_{\mu\nu} = \eta_{\mu\nu}$ and so g spacetime is simply Minkowski spacetime. (This shows that the Salam-Strathdee solution happens to be exact for $\Delta = 4$.) The f cosmological constant is fixed by (3.1), taking the value

$$\lambda\,|_{\Lambda=0} = \frac{3M^2}{16(u+1)}\left[\frac{27u}{16(u+1)}\right]^u. \tag{3.2}$$

It is interesting that the object described by this special case has no mass in the gravitational sense, since $T^g_{\mu\nu}$ vanishes everywhere. This feature is shared by the Yang-Mills pseudoparticles of Belavin *et al.*,[16] whose gravitational energy-momentum tensor also vanishes. Similarly the "ghost neutrino" solutions of Davies and Ray[17] propagate without gravitational mass in a plane-symmetric spacetime. Massless fermions in a spatially flat Robertson-Walker universe also possess nontrivial zero-energy solutions.

It is noteworthy that the function D, defined by Eq. (2.43), will in general become imaginary for some range(s) of values of r. This indicates the presence of coordinate singularities removable by a suitable coordinate transformation. In the flat g spacetime case such coordinate transformations will produce a g metric representing Minkowski space, but in a peculiar coordinate system, which we prefer to avoid. One particularly simple way of ensuring that D is real everywhere (i.e., in all of the coordinate chart being used) is to choose $\Delta=\frac{4}{9}$ whence, from (2.43),

$$D=\pm\Delta^{1/2}(1-X)\,. \tag{3.3}$$

Of course we can only have $\Delta=\frac{4}{9}$ and flat g spacetime if we postulate $u=-\frac{3}{2}$ [cf. Eq. (3.1)]. Then, from (3.2),

$$\lambda\big|_{\Lambda=0,\,u=-3/2}=\pm\frac{8M^2}{243}\,. \tag{3.4}$$

Such a fixing of the parameter λ may be of relevance to the physical interpretation of solutions of the Klein-Gordon equation in f spacetime.[8] It should be emphasized, however, that $\Delta=\frac{4}{9}$ is not the only way of making D real everywhere. Other more complicated possibilities exist, involving restrictions on the ranges of values of the parameters μ_f, μ_g, and Δ.

We conclude with a few remarks on one of the more unexpected features of our solution, namely the result that $g_{\mu\nu}$ is in general not asymptotically flat. Since κ_g/κ_f is very small one is tempted to assume that the difference between g spacetime and Minkowski spacetime can be treated as a small perturbation. But it can now be seen that while being locally acceptable (setting $\mu_g=0$) such an approximation will tend to mask the important global structure described by $g_{\mu\nu}$.

Consideration of the related global structures for the two metrics leads naturally to the following question. How can the manifold on which $f_{\mu\nu}$ and $g_{\mu\nu}$ are defined by analytically extended? One seeks an analog of the well-known extensions found for the standard solutions in general relativity. This and other questions remain to be answered, and novel problems are likely to arise when dealing with two metrics defined on the same manifold.

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APPENDIX: CURVATURE COMPUTATION

The Ricci curvature components may be conveniently computed using the method of curvature 2-forms,[18] requiring the choice of a local orthonormal frame of 1-forms. This technique is an efficient one and is, in addition, a natural choice if considering the introduction of half-integral spin.

A suitable orthonormal frame $\omega^{\hat\mu}$ (where $\hat\mu$ labels the member of the tetrad of 1-forms) is

$$\omega^{\hat t}=\sqrt{C}\,dt-\frac{D}{\sqrt{C}}dr,\quad \omega^{\hat r}=\left(\frac{\Delta}{C}\right)^{1/2}dr,\qquad \omega^{\hat\theta}=\sqrt{B}\,d\theta,\quad \omega^{\hat\phi}=\sqrt{B}\sin\theta\,d\phi\,, \tag{A1}$$

so that

$$f_{\mu\nu}dx^\mu dx^\nu=(\omega^{\hat t})^2-(\omega^{\hat r})^2-(\omega^{\hat\theta})^2-(\omega^{\hat\phi})^2\,, \tag{A2}$$

i.e.,

$$f_{\hat\mu\hat\nu}=\eta_{\hat\mu\hat\nu}=\mathrm{diag}(1,-1,-1,-1)\,. \tag{A3}$$

The connection 1-forms $\omega^{\hat\mu}{}_{\hat\nu}$ are deduced from the structure equations

$$d\omega^{\hat\mu}=-\omega^{\hat\mu}{}_{\hat\nu}\wedge\omega^{\hat\nu} \tag{A4}$$

and

$$\omega^{\hat\mu\hat\nu}=-\omega^{\hat\nu\hat\mu}\,, \tag{A5}$$

the latter resulting from $df_{\hat\mu\hat\nu}=d\eta_{\hat\mu\hat\nu}=0$. The nonvanishing connection 1-forms are

$$\begin{aligned}
\omega^{\hat t}{}_{\hat r}&=\omega^{\hat r}{}_{\hat t}=\frac{C'}{2\sqrt{C\Delta}}\omega^{\hat t}\,,\\
-\omega^{\hat r}{}_{\hat\theta}&=\omega^{\hat\theta}{}_{\hat r}=\frac{B'}{2B}\left(\frac{C}{\Delta}\right)^{1/2}\omega^{\hat\theta}\,,\\
-\omega^{\hat r}{}_{\hat\phi}&=\omega^{\hat\phi}{}_{\hat r}=\frac{B'}{2B}\left(\frac{C}{\Delta}\right)^{1/2}\omega^{\hat\phi}\,,\\
-\omega^{\hat\theta}{}_{\hat\phi}&=\omega^{\hat\phi}{}_{\hat\theta}=\frac{\cot\theta}{\sqrt{B}}\omega^{\hat\phi}\,.
\end{aligned} \tag{A6}$$

The curvature 2-forms are computed from the defining formula

$$\mathcal{R}^{\hat\mu}{}_{\hat\nu}=d\omega^{\hat\mu}{}_{\hat\nu}+\omega^{\hat\mu}{}_{\hat\alpha}\wedge\omega^{\hat\alpha}{}_{\hat\nu}\,. \tag{A7}$$

In this way, one obtains

$$\begin{aligned}
\mathcal{R}^{\hat t}{}_{\hat r}&=\mathcal{R}^{\hat r}{}_{\hat t}=\left(\frac{C'\Delta'}{4\Delta^2}-\frac{C''}{2\Delta}\right)\omega^{\hat t}\wedge\omega^{\hat r}\,,\\
\mathcal{R}^{\hat t}{}_{\hat\theta}&=\mathcal{R}^{\hat\theta}{}_{\hat t}=-\frac{B'C'}{4B\Delta}\omega^{\hat t}\wedge\omega^{\hat\theta}\,,\\
\mathcal{R}^{\hat t}{}_{\hat\phi}&=\mathcal{R}^{\hat t}{}_{\hat\phi}=-\frac{B'C'}{4B\Delta}\omega^{\hat t}\wedge\omega^{\hat\phi}\,,\\
\mathcal{R}^{\hat r}{}_{\hat\theta}&=-\mathcal{R}^{\hat\theta}{}_{\hat r}=\left(-\frac{B''C}{2B\Delta}+\frac{B'^2C}{4B^2\Delta}-\frac{B'C'}{4B\Delta}+\frac{B'\Delta'C}{4B\Delta^2}\right)\omega^{\hat r}\wedge\omega^{\hat\theta}\,,\\
\mathcal{R}^{\hat r}{}_{\hat\phi}&=-\mathcal{R}^{\hat\phi}{}_{\hat r}=\left(-\frac{B''C}{2B\Delta}+\frac{B'^2C}{4B^2\Delta}-\frac{B'C'}{4B\Delta}+\frac{B'\Delta'C}{4B\Delta^2}\right)\omega^{\hat r}\wedge\omega^{\hat\phi}\,,\\
\mathcal{R}^{\hat\theta}{}_{\hat\phi}&=-\mathcal{R}^{\hat\phi}{}_{\hat\theta}=\left(\frac{1}{B}-\frac{B'^2C}{4B^2\Delta}\right)\omega^{\hat\theta}\wedge\omega^{\hat\phi}\,.
\end{aligned} \tag{A8}$$

The nonvanishing Riemann tensor components can then be read off from the relation

$$\mathcal{R}^{\hat\mu}{}_{\hat\nu} = R^{\hat\mu}{}_{\hat\nu\hat\alpha\hat\beta}\,\omega^{\hat\alpha}\wedge\omega^{\hat\beta} \text{ (summation over } \alpha<\beta \text{ only).} \tag{A9}$$

They are as follows, omitting those which are obtainable from the given components using the symmetry properties of the Riemann tensor:

$$R^{\hat t}{}_{\hat r\hat t\hat r} = \frac{C'\Delta'}{4\Delta^2} - \frac{C''}{2\Delta}, \quad R^{\hat t}{}_{\hat\theta\hat t\hat\theta} = R^{\hat t}{}_{\hat\phi\hat t\hat\phi} = -\frac{B'C'}{4B\Delta},$$

$$R^{\hat r}{}_{\hat\theta\hat r\hat\theta} = R^{\hat r}{}_{\hat\phi\hat r\hat\phi} = -\frac{B''C}{2B\Delta} + \frac{B'^2C}{4B^2\Delta} - \frac{B'C'}{4B\Delta} + \frac{B'\Delta'C}{4B\Delta^2}, \tag{A10}$$

$$R^{\hat\theta}{}_{\hat\phi\hat\theta\hat\phi} = \frac{1}{B} - \frac{B'^2C}{4B^2\Delta}.$$

Contracting, using (1.2), to form the Ricci tensor and converting back to the coordinate frame produces the results given in (2.16).

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