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Rotating stratified ellipsoids of revolution and their effects on the dragging of inertial frames

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We determine here the off-diagonal metric components (the Lense-Thirring terms) resulting from the rotation of an oblate ellipsoid of revolution stratified into similar concentric elliptic layers of equal density. To find the unique function H involved in these off-diagonal components, we use a set of characteristic properties which may be compared with the well-known characteristic properties of the Newtonian potentials. We get H in terms of usual functions when the spinning ellipsoid is homogeneous. Applying this result to the Galaxy, we show that the dragging of inertial frames induced by the galactic rotation is too small to be presently detected. On the other hand, we perform the multipole expansion of H outside the stratified ellipsoid. Assuming then the terrestrial spheroid to be layered into similar ellipsoids following Roche's density law, we calculate the preponderant relativistic multipole term arising in the development of the function H of the earth. The value of this term agrees with what we have previously obtained from Bullen's model A.

I. INTRODUCTION

In an earlier paper,[1] denoted I in the following, we have determined the multipole structure of the off-diagonal metric components g_{0i}—the so-called Lense-Thirring[2] terms—arising from the rotation of an axisymmetric massive body. Our aim was to discuss the influence of the Earth's nonsphericity on the precession of the spin of a relativistic gyroscope, so we were mainly concerned with spinning bodies made of nearly spherical layers of equal density. Nevertheless, strongly nonspherically-symmetric configurations are relevant in astrophysics and geophysics. For this reason we study here the field of oblate spheroids[3] stratified into similar concentric layers of uniform density. Thus we are led to generalize the previous results of Clark[4] on the slightly flattened homogeneous ellipsoids of revolution.

As has been shown in I the off-diagonal potentials g_{0i} involve a unique function H, invariant under the rotations about the axis of symmetry. The method used here consists of rigorously determining H from a set of characteristic properties formulated in Sec. II of the present work for volume and surface distributions of matter. These properties may be compared with Dirichlet's characteristic properties of the Newtonian potentials.[5]

In Sec. III we use this method to find the function H of an infinitely thin homogeneous shell bounded by two similar concentric ellipsoids (i.e., an infinitely thin elliptic homoeoid). Then in Sec. IV we get by simple integration the field of a solid spheroid made of similar concentric layers of equal density. Moreover, we perform the multipole expansion of H, and we derive an explicit relation between the multipole relativistic terms involved in this development and the Newtonian 2^n-pole moments.

The last section is devoted to special cases in which the matter is distributed inside the spheroid (i) with a uniform density and (ii) following Roche's density law. Assuming the latter distribution within the terrestrial spheroid we estimate the preponderant relativistic multipole term in the function H of the Earth, and we compare its value with that previously obtained in paper I from Bullen's model A.

II. CHARACTERISTIC PROPERTIES OF H

Let us consider an isolated axisymmetric body slowly spinning about its axis of symmetry with a uniform angular velocity $\vec{\omega}$. The assumptions and notations are the same as in Secs. II and III of paper I. The gravitational field is assumed to be weak, axisymmetric, stationary, and related to a harmonic quasi-Galilean coordinate system x^0, $x^i=(x, y, z)$ $(i=1,2,3)$ compatible with the symmetries of the space-time. So the components of the metric may be written as

$$g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}\,,\quad \eta_{\mu\nu}=\mathrm{diag}(1,-1,-1,-1),\quad |h_{\mu\nu}|\ll 1\,,$$

and since the potentials are time-independent the corresponding linearized Einstein equations are

$$\nabla^2 h_{\mu\nu}=\begin{cases}0, & \text{outside of the matter,}\\ 2\kappa(T_{\mu\nu}-\tfrac{1}{2}T\eta_{\mu\nu}), & \text{inside of the matter,}\end{cases} \tag{1}$$

where $T_{\mu\nu}$ is the energy-momentum tensor, $T=\eta^{\rho\sigma}T_{\rho\sigma}$ and κ is the Einstein constant:

$$\kappa=8\pi G/c^2\,,$$

where G is the Newtonian gravitational constant and c is the speed of light. As in I we neglect the pressure and the ω^2 terms in $T_{\mu\nu}$.

The angular velocity $\vec{\omega}$ is directed along the z axis. It will be helpful to use the cylindrical coordinate system (ξ, φ, z) defined by

$$x = \xi \cos\varphi, \quad y = \xi \sin\varphi, \quad \xi \geq 0, \quad 0 \leq \varphi \leq 2\pi .$$

In this coordinate system all the matter and field variables (density, Newtonian potential, etc.) depend only upon ξ and z.

A. Rotating volume distributions

We consider a rotating axisymmetric solid body of volume density ρ. Up to the approximation order discussed above the nonvanishing independent components of the energy-momentum tensor are, according to the definition of κ,

$$T_{00} = \rho, \quad T_{01} = \rho(\omega y/c), \quad T_{02} = -\rho(\omega x/c). \tag{2}$$

We suppose the matter to be confined in a finite number of domains $\mathfrak{D}_i$ bounded by closed Liapunov surfaces[6] S_i. In each $\mathfrak{D}_i$ the density ρ is assumed to admit a gradient satisfying a uniform Hölder condition. However, ρ or its first partial derivatives may be discontinuous across the surfaces S_i. Under these assumptions it is well known that the field equations (1) admit one and only one system of solutions such that the metric deviations $h_{\mu\nu}$ (a) are continuously differentiable everywhere, (b) have piecewise continuously differentiable partial derivatives of the second order, and (c) are regular at infinity. It has been shown in I that the nonvanishing $h_{\mu\nu}$ take on the form

$$h_{00} = h_{ii} = -\frac{\kappa}{4\pi} U, \quad h_{01} = -\frac{\kappa\omega}{2\pi c} H y, \quad h_{02} = \frac{\kappa\omega}{2\pi c} H x, \tag{3}$$

where U is the Newtonian potential and H is a function of ξ and z given by

$$H(\xi, z) = \int_{\mathfrak{D}} \rho(\xi', z') \frac{\xi'^2 \sin^2(\varphi - \varphi')}{R^3} d\tau . \tag{4}$$

In the above integral $\mathfrak{D}$ is the union of the domains $\mathfrak{D}_i$, $d\tau$ is the volume element of the Euclidean 3-space at the point (ξ', φ', z'), and $R = |\vec{r} - \vec{r}'|$, $\vec{r}$ and $\vec{r}'$ denoting respectively the position vectors of (ξ, φ, z) and (ξ', φ', z').

Let us now replace the potentials h_{0i} by their expressions (3) and $T_{\mu\nu}$ by (2) in the field equations. It is easy to see that H satisfies the equation

$$D_2 H = \begin{cases} 0, & \text{outside of the matter,} \\ -4\pi\rho, & \text{inside of the matter,} \end{cases} \tag{5}$$

where D_2 is the differential operator defined by

$$D_2 H \equiv \frac{\partial^2 H}{\partial \xi^2} + \frac{3}{\xi}\frac{\partial H}{\partial \xi} + \frac{\partial^2 H}{\partial z^2} \equiv \nabla^2 H + \frac{2}{\xi}\frac{\partial H}{\partial \xi} . \tag{6}$$

In order to determine the potentials h_{0i} of a given rotating volume distribution it is sufficient to find a solution of Eq. (5) possessing the following properties:

(P_1) H is continuously differentiable everywhere.

(P_2) H has piecewise continuously differentiable partial derivatives of the second order.

(P_3) H vanishes at infinity as r^{-3}, and its first-order derivatives as r^{-4}. More precisely, if $r = (x^2 + y^2 + z^2)^{1/2}$ becomes larger and larger then

$$\lim_{r\to\infty} (r^3 H) = \tfrac{1}{2} I , \tag{7}$$

where I is the moment of inertia of the body about its axis of symmetry. This last property is a trivial consequence of (4) since $R^{-1} \sim r^{-1}$ when $r \to \infty$ and

$$\int_{\mathfrak{D}} \rho \xi'^2 \sin^2(\varphi - \varphi') d\tau = \tfrac{1}{2} I.$$

Equation (5) and the properties (P_i) constitute a set of characteristic properties of H. In effect if two distinct solutions of (5) satisfying all the (P_i)'s existed, then two distinct solutions of field equations would exist, which is excluded.

B. Rotating single layers

Let us now consider a spinning single layer distributed over a closed Liapunov surface S with a surface density σ satisfying a Hölder condition.[7] We get the energy-momentum tensor components by replacing ρ with $\sigma\delta_S$ in (2), where δ_S is the Dirac distribution of support S. Then Eqs. (1) must be read as distribution equations of which the only solutions $h_{\mu\nu}$, continuous everywhere, regular at infinity, and admitting continuous partial derivatives of the second order everywhere except on S, are the single-layer potentials

$$h_{00} = h_{ii} = -\frac{\kappa}{4\pi} \int_S \frac{\sigma}{R} dS ,$$

$$\begin{Bmatrix} h_{01} \\ h_{02} \end{Bmatrix} = \frac{\kappa\omega}{2\pi c} \int_S \frac{\sigma}{R} \begin{Bmatrix} -y' \\ x' \end{Bmatrix} dS ,$$

where dS is the surface element at the point (x', y', z') on S.

A reasoning similar to that applied in I to the volume distributions shows that the metric deviations $h_{\mu\nu}$ are still given by Eqs. (3) where U is now the Newtonian potential of the single layer and H is a function given by (4) after replacing $\rho d\tau$ with σdS. It is easy to verify that

$$D_2 H = 0 \tag{8}$$

everywhere except on S, on which the second-order partial derivatives of H are not defined.

The potentials h_{0i} and their first tangential derivatives are continuous across S. Therefore H and its first tangential derivatives are also continuous across S.

In order to study the normal derivative of H, let the normal to S be positively oriented outwards and, for any point $P_0(x_0, y_0, z_0)$ on S, let $(df/dn_-)(P_0)$ and $(df/dn_+)(P_0)$ denote the limits—when they exist—of the normal derivative of a function $f(P)$ as P approaches P_0 along the normal on the negative side and on the positive side, respectively. Following a classical result of potential theory such limits exist for the normal derivative of h_{01}, and the discontinuity of dh_{01}/dn at P_0 is

$$\left[\frac{dh_{01}}{dn}\right]_S(P_0) = \frac{dh_{01}}{dn_+}(P_0) - \frac{dh_{01}}{dn_-}(P_0) = 2\frac{\kappa\omega}{c}\sigma(P_0)y_0 .$$

Replacing h_{01} with $-(\kappa\omega/2\pi c)Hy$, then dividing by $-(\kappa\omega/2\pi c)\,y$ and dropping P_0, we get the value of the discontinuity of dH/dn across S:

$$\left[\frac{dH}{dn}\right]_S = \frac{dH}{dn_+} - \frac{dH}{dn_-} = -4\pi\sigma . \tag{9}$$

It is the same law as for the Newtonian potential U of the single layer. Thus to determine the potentials h_{0i} of a given spinning single layer it is sufficient to find a solution H of Eq. (8) possessing the following properties:

(P_1') H is continuous throughout space.

(P_2') The tangential derivatives of H are continuous across S and the normal derivative satisfies Eq. (9).

(P_3') The partial derivatives of second order of H are continuous everywhere except on S.

(P_4') H is regular at infinity and (7) holds.

Like any massive rotating body, the spinning single layer induces a dragging of locally inertial frames with respect to the rest frame at infinity. The angular velocity of the rotation of the inertial axes may be written as [see Eq. (13) in I]

$$\vec{\Omega} = \Omega_\xi \frac{\partial}{\partial\xi} + \Omega_z \frac{\partial}{\partial z} ,$$

with (10)

$$\Omega_\xi = -\frac{\kappa\omega}{4\pi}\,\xi\,\frac{\partial H}{\partial z}, \quad \Omega_z = \frac{\kappa\omega}{4\pi}\left(2H + \xi\frac{\partial H}{\partial\xi}\right).$$

It follows from (9) and (10) that the 3-vector $\vec{\Omega}$ has a discontinuity on passing through S in the direction of the outward unit normal $\vec{n}$,

$$[\vec{\Omega}]_S = \kappa\sigma\vec{v}_S \times \vec{n} , \tag{11}$$

where $\vec{v}_S$ denotes the 3-velocity field of the matter constituting the single layer.

III. INFINITELY THIN HOMOGENEOUS ELLIPTIC HOMOEOID

Let us now consider an infinitely thin homogeneous shell bounded by two concentric similar spheroids (E) and (E') represented by the equations

$$\begin{aligned} (E)\ \ & \frac{\xi^2}{a^2} + \frac{z^2}{b^2} = 1 , \\ (E')\ \ & \frac{\xi^2}{a^2} + \frac{z^2}{b^2} = (1+\epsilon)^2 , \end{aligned} \tag{12}$$

where ϵ is a very small dimensionless positive quantity. We always assume $a > b > 0$. Following a terminology introduced by Lord Kelvin and Tait,[8] such a configuration will be called a thin elliptic homoeoid of revolution.

It will be helpful to introduce the oblate spheroidal coordinates (λ, μ, φ) of a point $P(x, y, z)$, defined as follows: λ and μ are the greatest and the smallest roots, respectively, of the equation with respect to τ:

$$\frac{\xi^2}{a^2+\tau} + \frac{z^2}{b^2+\tau} - 1 = 0 . \tag{13}$$

We have $-b^2 \leq \lambda < \infty$ and $-a^2 \leq \mu \leq -b^2$. The λ surfaces are confocal oblate spheroids and the μ surfaces are confocal hyperboloids of one sheet. These two families of surfaces are orthogonal. Evidently the third coordinate φ is the longitude angle.

A straightforward calculation gives the expression for D_2H:

$$\begin{aligned} D_2H \equiv \frac{4}{\lambda-\mu}\Bigg\{&(b^2+\lambda)^{1/2}\frac{\partial}{\partial\lambda}\left((a^2+\lambda)^2(b^2+\lambda)^{1/2}\frac{\partial H}{\partial\lambda}\right) \\ &+[-(b^2+\mu)]^{1/2} \\ &\times\frac{\partial}{\partial\mu}\left[(a^2+\mu)^2[-(b^2+\mu)]^{1/2}\frac{\partial H}{\partial\mu}\right]\Bigg\} . \end{aligned} \tag{14}$$

A. Integral representations of the potentials U and H

Let ρ be the volume density of the homogeneous homoeoid bounded by (E) and (E'). If terms of the second order with respect to ϵ are neglected the homoeoid is equivalent to a single layer of total mass

$$M = 4\pi\rho a^2 b\epsilon$$

distributed over (E) with the surface density

$$\sigma = \frac{M}{4\pi a}\frac{1}{\sqrt{-\mu}} . \tag{15}$$

The corresponding Newtonian potential of such a single layer is well known[9]:

$$U=\frac{M}{2}\int_{\lambda}^{\infty}\frac{d\tau}{(a^2+\tau)(b^2+\tau)^{1/2}}\quad\text{outside }(E)\qquad(0\leq\lambda<\infty).\qquad(16)$$

$$U=\frac{M}{2}\int_{0}^{\infty}\frac{d\tau}{(a^2+\tau)(b^2+\tau)^{1/2}}\quad\text{inside }(E)\qquad(-b^2\leq\lambda<0).\qquad(17)$$

A similar reasoning leads to a solution of Eq. (8) depending only upon λ and possessing all the properties (P_i') (see Appendix A):

$$H=\frac{Ma^2}{2}\int_{\lambda}^{\infty}\frac{d\tau}{(a^2+\tau)^2(b^2+\tau)^{1/2}}\quad\text{outside }(E),\qquad(18)$$

$$H=\frac{Ma^2}{2}\int_{0}^{\infty}\frac{d\tau}{(a^2+\tau)^2(b^2+\tau)^{1/2}}\quad\text{inside }(E).\qquad(19)$$

Making the substitution $\nu=(b^2+\tau)^{1/2}$ we can integrate the above expressions. We get outside the ellipsoid (E)

$$U=\frac{M}{(a^2-b^2)^{1/2}}\arctan\left(\frac{a^2-b^2}{b^2+\lambda}\right)^{1/2},\qquad(20)$$

$$H=\frac{Ma^2}{2(a^2-b^2)}\left[\frac{1}{(a^2-b^2)^{1/2}}\arctan\left(\frac{a^2-b^2}{b^2+\lambda}\right)^{1/2}-\frac{(b^2+\lambda)^{1/2}}{a^2+\lambda}\right].\qquad(21)$$

Now introducing the eccentricity of the ellipsoid (E)

$$e=\frac{(a^2-b^2)^{1/2}}{a},\quad 0\leq e\leq 1,$$

and noting that

$$\arctan\left(\frac{a^2-b^2}{b^2+\lambda}\right)^{1/2}=\arcsin\left(\frac{a^2-b^2}{a^2+\lambda}\right)^{1/2},$$

we may write U and H inside the surface (E):

$$U=\frac{M}{a}\frac{\arcsin e}{e},\qquad(22)$$

$$\begin{aligned}H&=\frac{M}{2a}\frac{\arcsin e-e(1-e^2)^{1/2}}{e^3}\\&=\frac{M}{a}\left[\frac{1}{3}+\frac{1}{2}\times\frac{e^2}{5}+\cdots+\frac{1\times3\cdots(2n-1)}{2\times4\cdots(2n)}\frac{e^{2n}}{2n+3}+\cdots\right].\end{aligned}\qquad(23)$$

If one puts $e=0$, the above formulas give the expressions of U and H inside an infinitely thin homogeneous spherical layer of mass M.

B. Dragging of locally inertial frames

The partial derivatives of $H(\lambda)$ with respect to ξ and z are given by

$$\frac{\partial H}{\partial\xi}=2\xi\frac{b^2+\lambda}{\lambda-\mu}\frac{dH}{d\lambda},\quad\frac{\partial H}{\partial z}=2z\frac{a^2+\lambda}{\lambda-\mu}\frac{dH}{d\lambda}.$$

Replacing $H(\lambda)$ in the above expressions by the right member of (18) and substituting the obtained values in (10), we get the components of the angular velocity $\vec{\Omega}$ outside the homoeoid,

$$\Omega_\xi=\frac{2GMa^2\omega}{c^2}\frac{\xi z}{(\lambda-\mu)(a^2+\lambda)(b^2+\lambda)^{1/2}},\qquad\lambda>0\qquad(24)$$

$$\Omega_z=\frac{2GM\omega}{c^2e^2}\left(\frac{1}{ae}\arctan\frac{ae}{(b^2+\lambda)^{1/2}}-\frac{(b^2+\lambda)^{1/2}}{\lambda-\mu}\right),\qquad\lambda>0.\qquad(25)$$

H is constant inside the homoeoid. It follows from Eqs. (10) that the vector field $\vec{\Omega}$ is uniform and parallel to $\vec{\omega}$ in this region:

$$\vec{\Omega}=\frac{2GM}{c^2a}\frac{\arcsin e-e(1-e^2)^{1/2}}{e^3}\vec{\omega}.\qquad(26)$$

The linear character of the present theory implies the uniformity of the field $\vec{\Omega}$ in the free space enclosed by a thick homogeneous spinning shell bounded by two similar concentric ellipsoids of revolution about the z axis. It is an interesting generalization of the classical Thirring result[10] on the dragging of inertial frames inside a thick homogeneous spherical shell spinning about one of its diameters.

C. Multipole expansions of H and $\vec{\Omega}$

In paper I we have performed the multipole expansion of H in a region of free space far enough from the rotating body for a volume distribution of matter. A similar reasoning applied to a single layer leads to the same expansion of H provided that $\rho d\tau$ is replaced by σdS in the expressions of the relativistic multipole coefficients K_n given by Eqs. (19) and (20) in I.

Let us again turn our attention to the homoeoid. On the part of the z axis outside the homoeoid and such that $z>(a^2-b^2)^{1/2}$ the value of H may be developed as a power series in the reciprocal of z:

$$\begin{aligned}H&=\frac{M}{2e^2}\left(\frac{1}{ae}\arctan\frac{ae}{z}-\frac{z}{z^2+a^2e^2}\right)\\&=\frac{I}{2z^3}\left[1+3\sum_{n=1}^{\infty}\frac{(-1)^n(n+1)e^{2n}}{2n+3}\left(\frac{a}{z}\right)^{2n}\right],\end{aligned}$$

where I is the moment of inertia about the z axis,

$$I=\tfrac{2}{3}Ma^2.$$

Let (r,θ,φ) be the spherical coordinates relative to the center of the homoeoid. The relation

(44) established in Appendix A of I enables us to derive immediately the expansion of H at the point (r, θ, φ):

$$H = \frac{I}{2r^3}\left[1 - 3\sum_{n=1}^{\infty} \frac{(-1)^{n+1}e^{2n}}{(2n+1)(2n+3)}\left(\frac{a}{r}\right)^{2n} \times P'_{2n+1}(\cos\theta)\right], \quad (27)$$

where P'_{2n+1} denotes the first derivative of the Legendre polynomial P_{2n+1}. This development is to be compared with the multipole expansion of the Newtonian potential outside the homoeoid:

$$U = \frac{M}{r}\left[1 - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}e^{2n}}{2n+1}\left(\frac{a}{r}\right)^{2n} P_{2n}(\cos\theta)\right]. \quad (28)$$

Now we derive the expansions of the components of $\vec{\Omega}$ from Eqs. (21) and (22) in I:

$$\Omega_\xi = \frac{3GI\omega}{c^2r^3}\sin\theta\left[\cos\theta - \sum_{n=1}^{\infty}(-1)^{n+1}\frac{e^{2n}}{2n+3}\left(\frac{a}{r}\right)^{2n} \times P'_{2(n+1)}(\cos\theta)\right], \quad (29)$$

$$\Omega_z = \frac{GI\omega}{c^2r^3}\left[3\cos^2\theta - 1 - 6\sum_{n=1}^{\infty}(-1)^{n+1}\frac{(n+1)e^{2n}}{2n+3}\left(\frac{a}{r}\right)^{2n} \times P_{2(n+1)}(\cos\theta)\right]. \quad (30)$$

All these series are convergent if $r > (a^2 - b^2)^{1/2}$. This condition is always satisfied in the region $\lambda \geq 0$ when $b > a/\sqrt{2}$. Therefore, the expansions (27), (28), (29), and (30) represent H, U, and $\vec{\Omega}$, respectively, in the whole space outside of the homoeoid and on the surface (E) itself provided the oblateness $\alpha = (a - b)/a$ is less than $(\sqrt{2} - 1)/\sqrt{2} = 0.29289\ldots$.

IV. SOLID STRATIFIED SPHEROIDS

We are now in a position to determine by simple integration the function H of a spinning heterogeneous spheroid stratified into similar concentric surfaces of uniform density. The solid ellipsoid may be in effect decomposed into infinitely thin homogeneous homoeoids: The function H of the whole body will be obtained by summing the contributions of each elementary homoeoid. A similar method has been used to find the Newtonian potential of a solid ellipsoid.[11]

A. Integral expressions of H and $\vec{\Omega}$

Let (E) be the external surface of the rotating solid spheroid and (E_u) be the similar oblate spheroid of semiaxes au and bu, where u is a parameter going from 0 to 1. We assume the density ρ to be a continuous function of u on the interval $0 \leq u \leq 1$. The mass of the elementary homoeoid bounded by the surfaces (E_u) and (E_{u+du}) is thus

$$dM_u = 4\pi a^2 b\rho(u)u^2du.$$

If the point $P(x, y, z)$ is outside of the solid spheroid we obtain the contribution of the elementary homoeoid to the value of H at P by replacing in (18) M, a, b with dM_u, au, bu, respectively, and λ with the greatest root λ_u of the equation

$$\frac{\xi^2}{a^2u^2+\tau} + \frac{z^2}{b^2u^2+\tau} - 1 = 0.$$

We have thus

$$dH_u = 2\pi a^4 b\rho(u)u^4\int_{\lambda_u}^{\infty}\frac{d\tau}{(a^2u^2+\tau)^2(b^2u^2+\tau)^{1/2}}\,du. \quad (31)$$

Making the substitution $\tau = u^2s$, putting $\tau_u = \lambda_u/u^2$ and integrating with respect to u over the interval $0 \leq u \leq 1$, we obtain the whole H at (x, y, z):

$$H_e = 2\pi a^4 b\int_0^1 \rho(u)u\int_{\tau_u}^{\infty}\frac{ds}{(a^2+s)^2(b^2+s)^{1/2}}\,du. \quad (32)$$

Putting

$$\psi(u) = 2\int_u^1 \rho(\nu)\nu d\nu, \quad (33)$$

and integrating (32) by parts, the expression for H at an exterior point becomes (see the computational details in Appendix B)

$$H_e = \pi a^4 b\int_\lambda^{\infty}\psi(u)\frac{ds}{(a^2+s)^2(b^2+s)^{1/2}}, \quad (34)$$

where λ is the greatest root of Eq. (13) and u must be replaced by

$$u = \left(\frac{\xi^2}{a^2+s} + \frac{z^2}{b^2+s}\right)^{1/2}. \quad (35)$$

To find H at an interior point let us denote by u_0 the parameter of the spheroid of the family of similar spheroids (E_u) passing through this point. We get the contribution of the matter inside of (E_{u_0}) by integrating (31) on the interval $0 \leq u \leq u_0$ since P is an exterior point for the spheroids $u \leq u_0$. To obtain the whole contribution of the layers characterized by $u > u_0$ we have merely to replace the lower limit by 0 in (31) and to integrate over $u_0 < u \leq 1$. Making again the substitution $\tau = u^2s$, we get

$$H_i=2\pi a^4 b\left[\int_0^{u_0}\rho(u)u\int_{\tau_u}^{\infty}\frac{ds}{(a^2+s)^2(b^2+s)^{1/2}}\,du+\int_{u_0}^{1}\rho(u)u\int_0^{\infty}\frac{ds}{(a^2+s)^2(b^2+s)^{1/2}}\,du\right]. \quad (36)$$

Integrating by parts (see Appendix B) we find H at an interior point

$$H_i=\pi a^4 b\int_0^{\infty}\psi(u)\frac{ds}{(a^2+s)^2(b^2+s)^{1/2}}, \quad (37)$$

where u is again given by (35).

A similar reasoning leads to the expressions of the Newtonian potential outside and inside of the spheroid:

$$U_e=\pi a^2 b\int_{\lambda}^{\infty}\psi(u)\frac{ds}{(a^2+s)(b^2+s)^{1/2}},$$

$$U_i=\pi a^2 b\int_0^{\infty}\psi(u)\frac{ds}{(a^2+s)(b^2+s)^{1/2}}.$$

In order to determine the components of $\vec{\Omega}$ let us now carry out the calculation of the derivatives of H. At an exterior point a formal application of the Leibniz rule gives

$$\frac{\partial H_e}{\partial\xi}=\pi a^4 b\int_{\lambda}^{\infty}\frac{\partial\psi(u)}{\partial\xi}\frac{ds}{(a^2+s)^2(b^2+s)^{1/2}}-\pi a^4 b\frac{\partial\lambda}{\partial\xi}\frac{\psi(u(\lambda))}{(a^2+\lambda)^2(b^2+\lambda)^{1/2}}. \quad (38)$$

We have $u(\lambda)=1$ because the point $P(x,y,z)$ lies on the coordinate λ surface. But $\psi(1)=0$. So the second term of the right member vanishes. On the other hand, the first term is a uniformly convergent integral in the region $\lambda\geq 0$. Indeed, we have

$$\frac{\partial\psi(u)}{\partial\xi}=-\frac{2\xi}{a^2+s}\rho(u).$$

But $s\geq\lambda\geq 0$ and the definition of λ implies that $\xi/(a^2+s)^{1/2}\leq 1$. So, $\xi/(a^2+s)\leq 1/a$. Then $|\partial\psi(u)/\partial\xi|$ is uniformly bounded since $\rho(u)$ is bounded. Therefore, the integral in the right member of (38) is uniformly convergent as the integral giving H_e. As a consequence the application of the Leibniz rule is justified. A similar reasoning holds for $\partial H_e/\partial z$, and we have finally

$$\frac{\partial H_e}{\partial\xi}=-2\pi a^4 b\xi\int_{\lambda}^{\infty}\frac{\rho(u)}{(a^2+s)^3(b^2+s)^{1/2}}\,ds,$$

$$\frac{\partial H_e}{\partial z}=-2\pi a^4 bz\int_{\lambda}^{\infty}\frac{\rho(u)}{(a^2+s)^2(b^2+s)^{3/2}}\,ds.$$

Then inserting these expressions in (10), we get the components of $\vec{\Omega}$:

$$\Omega_\xi=\tfrac{1}{2}\kappa\omega a^4 b\xi z\int_{\lambda}^{\infty}\frac{\rho(u)}{(a^2+s)^2(b^2+s)^{3/2}}\,ds, \quad \lambda\geq 0 \quad (39)$$

$$\Omega_z=\frac{\kappa\omega}{2\pi}\left[H_e-\pi a^4 b\xi^2\int_{\lambda}^{\infty}\frac{\rho(u)}{(a^2+s)^3(b^2+s)^{1/2}}\,ds\right], \quad \lambda\geq 0. \quad (40)$$

To find the components of $\vec{\Omega}$ at an interior point we have only to replace H_e with H_i and λ with 0 in the above expressions.

B. Multipole expansions of H and $\vec{\Omega}$

To find the multipole expansion of dH_u at a point far enough from the origin, let us replace in (27) a with au and I with the moment of inertia about the z axis of the homoeoid bounded by (E_u) and (E_{u+du}):

$$dI_u=\frac{8\pi}{3}a^4 b\rho(u)u^4 du.$$

Considered as a function of u, dH_u is then represented by a power series in u, convergent in the interval $0\leq u\leq 1$ provided $r>(a^2-b^2)^{1/2}$. Now, integrating over the interval $0\leq u\leq 1$ and permuting the summation symbols, we get the following uniformly convergent expansion for H in the region of the free space such that $r>(a^2-b^2)^{1/2}$:

$$H=\frac{I}{2r^3}\left[1-\sum_{n=1}^{\infty}K_{2n}\left(\frac{a}{r}\right)^{2n}P'_{2n+1}(\cos\theta)\right], \quad (41)$$

where I is the moment of inertia of the solid ellipsoid about the z axis

$$I=\frac{8\pi}{3}a^4 b\int_0^1\rho(u)u^4 du, \quad (42)$$

and the K's are given by

$$K_{2n}=(-1)^{n+1}\frac{3e^{2n}}{(2n+1)(2n+3)}\frac{\int_0^1\rho(u)u^{2(n+2)}du}{\int_0^1\rho(u)u^4 du}. \quad (43)$$

A similar reasoning leads to the expansion of the Newtonian potential

$$U=\frac{M}{r}\left[1-\sum_{n=1}^{\infty}J_{2n}\left(\frac{a}{r}\right)^{2n}P_{2n}(\cos\theta)\right],$$

where M is the total mass and the J's are given by

$$J_{2n}=(-1)^{n+1}\frac{e^{2n}}{2n+1}\frac{\int_0^1\rho(u)u^{2(n+1)}du}{\int_0^1\rho(u)u^2 du}. \quad (44)$$

Comparing (43) and (44) we find an algebraic relation between the relativistic coefficients K_{2n} and the Newtonian multipole moments J_{2n}:

$$K_{2n} = -\frac{2}{2n+1}\frac{Ma^2}{Ie^2}J_{2(n+1)} . \quad (45)$$

It is easy to obtain a bound of the J's and of the K's when the density is a monotonically decreasing function of u. Under this assumption an integration by parts shows in effect that

$$\int_0^1 \rho(u)u^{2n+p}du \leq \frac{p+1}{2n+p+1}\int_0^1 \rho(u)u^p du$$

for any $n \geq 0$ and $p \geq 0$. Putting then $p=2$ and $p=4$ successively, we deduce immediately from these inequalities

$$|J_{2n}| \leq \frac{3e^{2n}}{(2n+1)(2n+3)} ,$$
$$|K_{2n}| \leq \frac{15e^{2n}}{(2n+1)(2n+3)(2n+5)} . \quad (46)$$

The equalities hold when ρ does not depend upon u.

Following the formula (23) of paper I, the vector field $\vec{\Omega}$ may be expanded as

$$\vec{\Omega} = \frac{GI\omega}{c^2r^2}\left\{3\cos\theta\vec{u} - \vec{k} - \sum_{n=1}^{\infty}(2n+1)K_{2n}\left(\frac{a}{r}\right)^{2n}[P'_{2(n+1)}(\cos\theta)\vec{u} - P'_{2n+1}(\cos\theta)\vec{k}]\right\},$$

where $\vec{u}$ and $\vec{k}$ denote the unit vectors in the $\vec{r}$ and in the z directions, respectively.

V. PARTICULAR CASES

We shall now turn our attention to some special density distributions within the rotating spheroid.

A. Homogeneous spheroid

When the density ρ is uniform, we have

$$\psi(u(s)) = \rho\left(1 - \frac{\xi^2}{a^2+s} - \frac{z^2}{b^2+s}\right) .$$

Then, making the substitution $w = [(a^2-b^2)/(a^2+s)]^{1/2}$ we can integrate the expressions for H and $\vec{\Omega}$. Thus we find

(a) outside the spheroid:

$$H_e = \frac{\pi\rho a^4 b}{(a^2-b^2)^{3/2}}\left\{\left(1 - \frac{3}{4}\frac{\xi^2-4z^2}{a^2-b^2}\right)\arcsin\left(\frac{a^2-b^2}{a^2+\lambda}\right)^{1/2} - (a^2-b^2)^{1/2}\left[1 - \frac{3}{4}\frac{\xi^2-4z^2}{a^2-b^2} - \frac{1}{2}\left(\frac{\xi^2}{a^2+\lambda} - \frac{4z^2}{b^2+\lambda}\right)\right]\frac{(b^2+\lambda)^{1/2}}{a^2+\lambda}\right\}, \quad (47)$$

$$\Omega_\xi = \frac{45}{2}\frac{GI\omega}{c^2(a^2-b^2)^{5/2}}\xi z\left[\left(1 - \frac{1}{3}\frac{a^2-b^2}{a^2+\lambda}\right)\left(\frac{a^2-b^2}{b^2+\lambda}\right)^{1/2} - \arcsin\left(\frac{a^2-b^2}{a^2+\lambda}\right)^{1/2}\right],$$

$$\Omega_z = \frac{15}{2}\frac{GI\omega}{c^2(a^2-b^2)^{3/2}}\left\{\left(1 - \frac{3}{2}\frac{\xi^2-2z^2}{a^2-b^2}\right)\arcsin\left(\frac{a^2-b^2}{a^2+\lambda}\right)^{1/2} - (a^2-b^2)^{1/2}\left[1 - \frac{3}{2}\frac{\xi^2-2z^2}{a^2-b^2} - \left(\frac{\xi^2}{a^2+\lambda} - \frac{2z^2}{b^2+\lambda}\right)\right]\frac{(b^2+\lambda)^{1/2}}{a^2+\lambda}\right\}, \quad (48)$$

where I is the moment of inertia about the z axis: $I = (8\pi/15)\rho a^4 b$;

(b) inside the spheroid:

$$H_i = \frac{\pi\rho ab}{e^3}\left\{\arcsin e - e(1-e^2)^{1/2} - \frac{3}{4e}\left[\frac{\arcsin e}{e} - \left(1 + \frac{2e^2}{3}\right)(1-e^2)^{1/2}\right]\left(\frac{\xi}{a}\right)^2 + \frac{3}{e}\left(\frac{\arcsin e}{e} - \frac{3-e^2}{3(1-e^2)^{1/2}}\right)\left(\frac{z}{a}\right)^2\right\}, \quad (49)$$

$$\Omega_\xi = 9\frac{GM}{c^2ae^4}\omega\left[1 - \frac{e^2}{3} - \frac{(1-e^2)^{1/2}}{e}\arcsin e\right]\frac{\xi z}{ab} ,$$

$$\Omega_z = 3\frac{GM}{c^2ae^2}\omega\left[\left(\frac{\arcsin e}{e} - (1-e^2)^{1/2}\right)\left(1 - \frac{3}{2}\frac{\xi^2-2z^2}{a^2e^2}\right) + (1-e^2)^{1/2}\left(\frac{\xi^2}{a^2} - \frac{2z^2}{b^2}\right)\right] , \quad (50)$$

where M is the mass, related to I by the equation $I=\frac{2}{5}Ma^2$.

Suppose now the spheroid to be very flattened ($b \ll a$). At a point on the equatorial plane within the spheroid, Ω_ξ vanishes and Ω_z becomes

$$\Omega_z \simeq 3\,\frac{GM}{c^2a}\,\omega\left[\frac{\pi}{2}-2\frac{b}{a}-\left(\frac{3\pi}{4}-\frac{4b}{a}\right)\left(\frac{\xi}{a}\right)^2\right], \quad (51)$$

if we neglect the terms of the second order with respect to b/a.

This formula enables us to estimate the dragging of inertial frames induced by the rotation of the Galaxy. With $a=1.5\times10^4$ pc, $b=2.5\times10^3$ pc, $M=12\times10^{44}$ g, and $\omega=3\times10^{-8}$ rad/year, we find at the place of the Sun in the Galaxy ($\xi\simeq\frac{2}{3}a$)[12]

$$\Omega\simeq 0.3\times10^{-8} \text{ sec of arc/year}.$$

Such a precession is at least 10^5 times smaller than what can be actually measured. So the effect of the rotation of the Galaxy is negligible in the relativistic gyroscope experiment presently conducted by Everitt, Fairbank, and their co-workers.[13]

B. Roche's density law

Consider a density distribution within the spheroid given by Roche's law

$$\rho(u)=\rho_c(1-ku^2),$$

where ρ_c is the density at the center and k a constant such that $0\leqslant k<1$. The formulas (43) and (44) give immediately the multipole coefficients

$$J_{2n}=(-1)^{n+1}\frac{3e^{2n}}{(2n+1)(2n+3)}\left(1-\frac{2n+3}{2n+5}k\right)\bigg/\left(1-\tfrac{3}{5}k\right), \quad (52)$$

$$K_{2n}=(-1)^{n+1}\frac{15e^{2n}}{(2n+1)(2n+3)(2n+5)}\times\left(1-\frac{2n+5}{2n+7}k\right)\bigg/\left(1-\tfrac{5}{7}k\right). \quad (53)$$

Let us apply these results to the Earth. If we take 5.517 g cm^{-3} as the mean density of the Earth and 2.84 g cm^{-3} as the mean density of the crust, we have $k=0.702$. With a flattening $\alpha=(a-b)/a$ 1/298.25, we find

$$J_2=1.153\times10^{-3},\quad K_2=0.871\times10^{-3}.$$

The other J's and K's are not significant as far as the precession of the spin of a gyroscope is concerned [see I]. The value found for J_2 differs only by 6.53 per cent from the real value of this quantity [$J_2=(1082.64\pm0.01)\times10^{-6}$][14] and the above estimate of the relativistic coefficient K_2 agrees with the result obtained in I from Bullen's realistic model of the Earth.

VI. CONCLUSION

We have found a set of characteristic properties of the function H involved in the off-diagonal metric components due to the rotation of an isolated axisymmetric body. These properties have enabled us to determine under a simple integral form the field and the dragging of inertial frames induced by a spinning solid spheroid stratified into similar concentric surfaces of uniform density.

The integrals can be expressed in terms of usual functions when the spheroid is homogeneous. Applied to the Galaxy, our formulas show that the rotation of the Milky Way gives a contribution to the spin precession of a gyroscope about 10^5 times smaller than the expected experimental error.

We have determined the relativistic multipole coefficients K_n involved in the expansion of H in series of spherical functions. These coefficients are related to the Newtonian multipole terms J_n by a quite simple algebraic equation. This correspondence between the K's and the J's is one of the most interesting features of the models stratified into similar ellipsoids.

Such an internal layering may be assumed for the earth. Using Roche's density law, we have found a numerical estimate of the preponderant term K_2 which agrees with the value that we have previously obtained from the classical Bullen's model A.

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APPENDIX A: FUNCTION H OF AN INFINITELY THIN HOMOGENEOUS ELLIPTIC HOMOEOID

Let us see if there is a solution H of Eq. (8) depending only upon λ and possessing all the properties (P_i'). If this is the case, the expression (14) shows that $H(\lambda)$ must satisfy the equation

$$\frac{d}{d\lambda}\left[(a^2+\lambda)^2(b^2+\lambda)^{1/2}\frac{dH}{d\lambda}\right]=0\,.$$

The general solution of this equation is

$$H(\lambda)=AK(\lambda)+B,$$

$$K(\lambda)=\int_\lambda^\infty\frac{d\tau}{(a^2+\tau)^2(b^2+\tau)^{1/2}}\,,$$

where A and B are arbitrary constants.

The values of A and B in the exterior region $\lambda\geqslant0$ are determined by (P_4'). Since the coordinates λ and μ are the roots of Eq. (13) we have

$$\lambda+\mu=\xi^2+z^2-(a^2+b^2).$$

As μ is bounded, λ becomes infinite as $r^2=\xi^2$

$+z^2$. Now, integrating the inequalities

$$(a^2+\tau)^{-5/2}<(a^2+\tau)^{-2}(b^2+\tau)^{-1/2}<(b^2+\tau)^{-5/2}$$

over $[\lambda,\infty)$, we find

$$\tfrac{2}{3}(a^2+\lambda)^{-3/2}<K(\lambda)<\tfrac{2}{3}(b^2+\lambda)^{-3/2},$$

so that

$$\lim_{r\to\infty}[r^3K(\lambda)]=\lim_{\lambda\to\infty}[\lambda^{3/2}K(\lambda)]=\tfrac{2}{3}\,.$$

But (P_4') requires that

$$\lim_{r\to\infty}(r^3H)=\lim_{\lambda\to\infty}[\lambda^{3/2}H(\lambda)]=\tfrac{1}{2}I\,,$$

with $I=\frac{2}{3}Ma^2$. Therefore, we must put $A=\frac{3}{4}I=\frac{1}{2}Ma^2$ and $B=0$ in the exterior region. So (18) is proved.

Now let us denote by A^- and B^- the values of constants in the interior region ($-b^2\leqslant\lambda<0$). The discontinuity of the normal derivative of H across (E) is given by

$$\left[\frac{dH}{dn}\right]_{(E)}=\left[\frac{dH}{d\lambda}\right]_{(E)}\frac{\partial\lambda}{\partial n}\bigg|_{\lambda=0}\,.$$

But

$$\frac{\partial\lambda}{\partial n}\bigg|_{\lambda=0}=2ab/\sqrt{-\mu}$$

and

$$\left[\frac{dH}{d\lambda}\right]_{(E)}=(\tfrac{1}{2}Ma^2-A^-)\frac{dK}{d\lambda}\bigg|_{\lambda=0}\,.$$

Therefore

$$\left[\frac{dH}{dn}\right]_{(E)}=-\frac{M}{a\sqrt{-\mu}}+\frac{2A^-}{a^3\sqrt{-\mu}}\,.$$

Let us replace σ by its value (15) in equation (9) and compare with the above expression. We find $A^-=0$. Hence $H(\lambda)=B^-$ in the interior region. The value of the constant B^- is determined by the value of H on the boundary since H is continuous across (E). The formula (19) is thus justified.

APPENDIX B: FUNCTION H OF A SOLID STRATIFIED ELLIPSOID

To integrate (32) by parts let us replace $2\rho(u)u$ by $-d\psi/du$, where $\psi(u)$ is defined by (33). We get

$$H_e=-\pi a^4b\left[\left|\psi(u)\int_{\tau_u}^{\infty}\frac{ds}{(a^2+s)^2(b^2+s)^{1/2}}\right|_0^1+\int_0^1\psi(u)\frac{1}{(a^2+\tau_u)^2(b^2+\tau_u)^{1/2}}\frac{d\tau_u}{du}du\right]\,.$$

The first term of the right member vanishes. In effect, $\psi(1)=0$ and the factor of $\psi(0)$ is null since τ_u becomes infinite as u approaches 0.

It is easy to transform the remaining integral. By definition, τ_u is the greatest root of the equation

$$\frac{\xi^2}{a^2+s}+\frac{z^2}{b^2+s}=u^2\,.$$

For a given ξ and z this root is a monotonic function of u: τ_u decreases from ∞ to λ as u increases from 0 to 1, hence τ_u may be used as a new variable of integration in place of u. The expression of H_e becomes

$$H_e=\pi a^4b\int_{\lambda}^{\infty}\psi(u)\frac{d\tau_u}{(a^2+\tau_u)^2(b^2+\tau_u)^{1/2}}\,,$$

where u is now a function of τ_u given by

$$u=\left(\frac{\xi^2}{a^2+\tau_u}+\frac{z^2}{b^2+\tau_u}\right)^{1/2}\,.$$

This expression for H is exactly the same as the formula (34).

Consider now the point (x,y,z) inside the ellipsoid. The integration by parts of (36) gives immediately

$$H_i=-\pi a^4b\left[\left|\psi(u)\int_{\tau_u}^{\infty}\frac{ds}{(a^2+s)^2(b^2+s)^{1/2}}\right|_0^{u_0}+\int_0^{u_0}\frac{\psi(u)}{(a^2+\tau_u)^2(b^2+\tau_u)^{1/2}}\frac{d\tau_u}{du}du+\left|\psi(u)\int_0^{\infty}\frac{ds}{(a^2+s)^2(b^2+s)^{1/2}}\right|_{u_0}^1\right]\,.$$

But $\tau_{u_0}=0$ since (x,y,z) lies on the ellipsoid (E_{u_0}). Therefore

$$\left|\psi(u)\int_{\tau_u}^{\infty}\frac{ds}{(a^2+s)^2(b^2+s)^{1/2}}\right|_0^{u_0}+\left|\psi(u)\int_0^{\infty}\frac{ds}{(a^2+s)^2(b^2+s)^{1/2}}\right|_{u_0}^1=\psi(1)\int_0^{\infty}\frac{ds}{(a^2+s)^2(b^2+s)^{1/2}}=0\,.$$

The remaining integral in the expression of H_i may be transformed by a similar reasoning to that applied in the exterior case. This leads immediately to (37).

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