Embedding classical fields in quantum field theories

Stephen Blaha*
Physics Department, Syracuse University, Syracuse, New York 13210
(Received 2 August 1976; revised manuscript received 7 November 1977)

We describe a procedure for quantizing a classical field theory which is the field-theoretic analog of Sudarshan's method for embedding a classical-mechanical system in a quantum-mechanical system. The essence of the difference between our quantization procedure and Fock-space quantization lies in the choice of vacuum states. The key to our choice of vacuum is the procedure we outline for constructing Lagrangians which have gradient terms linear in the field variables from classical Lagrangians which have gradient terms which are quadratic in field variables. We apply this procedure to model electrodynamic field theories, Yang-Mills theories, and a vierbein model of gravity. In the case of electrodynamics models we find a formalism with a close similarity to the coherent-soft-photon-state formalism of QED. In addition, photons propagate to $t = +\infty$ via retarded propagators. We also show how to construct a quantum field for action-at-a-distance electrodynamics. In the Yang-Mills case we show that a previously suggested model for quark confinement necessarily has gluons with principal-value propagation which allows the model to be unitary despite the presence of higher-order-derivative field equations. In the vierbein-gravity model we show that our quantization procedure allows us to treat the classical and quantum parts of the metric field in a unified manner. We find a new perturbation scheme for quantum gravity as a result.

I. INTRODUCTION

The relation between classical and quantum systems has been a subject of continuing interest over the years: First, in the original development of quantum mechanics, second, in the study of the classical limit and infrared divergences of quantum-electrodynamic processes,[1,2] and third, in recent attempts to construct strong-interaction models of quark confinement which are for the most part either classical field theory models in search of quantization[3] or quantized gluon models wherein quark confinement is a consequence of infrared behavior.[4,5]

We will describe a new quantization procedure (called pseudoquantization) for field theory which is the analog of Sudarshan's method for embedding a classical-mechanical system in a quantum-mechanical system. It can be used with advantage to either embed a classical field theory in a quantum field theory in such a way as to maintain the classical character of the embedded fields (while studying the interaction between the classical and quantum sectors on essentially the same footing), or to quantize a class of field theories, members of which have been used as models for gravity and as models for the strong interaction with quark confinement.[7-9]

We shall begin (Sec. II) by pseudoquantizing a classical simple harmonic oscillator. This case is of particular importance because of the analogy between the mode amplitudes of a quantum field and the coordinates of a set of simple harmonic oscillators which we will take advantage of in later sections.

In Sec. III we describe the pseudoquantization procedure for field theory. We apply it to electrodynamic models and show that the propagation of photons to $t=+\infty$ is necessarily retarded in this formalism. Further, we display a close analogy between the present formalism and the coherent-soft-photon-state formalism[10] of QED.

In Sec. IV we apply the pseudoquantization procedure to a classical Yang-Mills field. The resulting field theory (with a slight but important modification) has been used as a model for the strong interactions with quark confinement.[7-9] We also apply the pseudoquantization procedure to a vierbein model of gravity and obtain a new perturbation theory for quantum gravity.

In Sec. V we show that principal-value propagators naturally arise in certains sectors of pseudoquantized theories thus verifying an *ad hoc* procedure devised to unitarize a model of quark confinement.[7-9] We also show how to construct a quantum version of action-at-a-distance electrodynamics.

We shall now briefly outline the procedure for embedding a classical-mechanical system in a quantum system.[6] Consider a classical Hamiltonian system with one degree of freedom, and commuting canonical variables, x_1 and p_1, which have the equations of motion

$$\dot{x}_1 = -i[x_1, \hat{H}], \tag{1}$$

$$\dot{p}_1 = -i[p_1, \hat{H}], \tag{2}$$

where defining

$$\hat{H} = -i\left(\frac{\partial H(x_1,p_1)}{\partial p_1}\frac{\partial}{\partial x_1} - \frac{\partial H(x_1,p_1)}{\partial x_1}\frac{\partial}{\partial p_1}\right) \tag{3}$$

allows us to write Hamilton's equations in commutator form. With Sudarshan[6] we define

$$x_2 = i\frac{\partial}{\partial p_1} \tag{4}$$

and

$$p_2 = -i\frac{\partial}{\partial x_1} \tag{5}$$

so that

$$[x_1, x_2] = [p_1, p_2] = 0\,, \tag{6}$$

$$[x_1, p_2] = [x_2, p_1] = i\,, \tag{7}$$

and $\hat{H}$ can now be taken to be the operator

$$\hat{H} = \frac{\partial H(x_1, p_1)}{\partial p_1} p_2 + \frac{\partial H(x_1, p_1)}{\partial x_1} x_2\,. \tag{8}$$

It is now apparent that we can take the above quantities and equations of motion to describe a quantum mechanical system with two degrees of freedom in the "coordinate" representation where the "coordinates" are (x_1, p_1) and the canonical momenta are $\Pi = (p_2, -x_2)$. As we will see below the linearity of $\hat{H}$ in the momenta is crucial for the maintenance of the classical character of x_1 and p_1, and for the observability of the phase-space trajectory. Since we choose to identify the physical observables with the commutative algebra of the coordinate operators, x_1 and p_1, we are led to impose the superselection condition that the momenta, Π, are unobservable. As a result the Hamiltonian and other generators of canonical transformations, which are all linear in the momenta, are also unobservable. However, in each case there is an associated dynamical quantity which is observable.

The required unobservability of the momenta restricts the form of the interaction between a classical-made-quantum system and an inherently quantum system to

$$H_{\text{int}} = \Phi_1 x_2 + \Phi_2 p_2 + X\,, \tag{9}$$

where Φ_1, Φ_2, and X are functions of x_1, p_1, and the quantum system variables. The commutation relations of these functions are also constrained[6] by the superselection rule and the commutativity of the classical variables, x_1 and p_1, and their time derivatives. In the next section we will study the simple harmonic oscillator in order to exemplify the quantum-mechanical case described above and also for direct use in the field-theoretic generalizations of subsequent sections.

II. SIMPLE HARMONIC OSCILLATOR

In this section we discuss the embedding of a classical simple harmonic oscillator in a quantum system. We shall see that the space of states for the indefinite-metric classical-made-quantum system is far larger than the set of states of a classical harmonic oscillator. However, there is a subset of coherent states which may be placed in one-to-one correspondence with the classical harmonic-oscillator states. The classical-made-quantum oscillator is necessarily an indefinite-metric quantum theory for the simple physical reason that the classical bound states cannot have quantized energy levels. Indefinite-metric quantum theories normally have severe problems of physical interpretation. The present work raises the possibility of a partial resolution of some of these problems through a reinterpretation of an indefinite-metric quantum system as a system composed of a classical subsystem interacting with an essentially quantum subsystem of positive metric.

The classical simple harmonic oscillator of frequency ω has the Hamiltonian

$$\mathcal{H} = \frac{1}{2m}(p_1{}^2 + m^2\omega^2 x_1{}^2)\,, \tag{10}$$

and the motion is described by

$$x_1 = A\sin(\pi t + \delta)\,, \tag{11}$$

where A and δ are constants. To embed this classical system in a quantum-mechanical system we introduce the variables x_2 and p_2, and, using Eq. (8), obtain the quantum Hamiltonian

$$\hat{H} = \frac{1}{m} p_1 p_2 + m\omega^2 x_1 x_2\,. \tag{12}$$

We eliminate constants by defining (for $i = 1, 2$)

$$x_i = \left(\frac{1}{m\omega}\right)^{1/2} Q_i\,, \tag{13}$$

$$p_i = (m\omega)^{1/2} P_i\,, \tag{14}$$

and

$$\hat{H} = H\omega \tag{15}$$

so that

$$H = P_1 P_2 + Q_1 Q_2\,. \tag{16}$$

The raising and lowering operators are defined by

$$a_j = \frac{1}{\sqrt{2}}(Q_j + iP_j)\,, \tag{17}$$

and

$$a_j^\dagger = \frac{1}{\sqrt{2}}(Q_j - iP_j) \tag{18}$$

for $j = 1, 2$. They have the commutation relations

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0\,, \tag{19}$$

$$[a_i, a_j^\dagger] = 1 - \delta_{ij} \tag{20}$$

for $i,j=1,2$. As a result H is seen to have the form

$$H=\tfrac{1}{2}(a_1a_2^\dagger+a_2a_1^\dagger+a_1^\dagger a_2+a_2^\dagger a_1)\,. \tag{21}$$

The number operators are defined by

$$N_1=a_2a_1^\dagger \tag{22}$$

and

$$N_2=a_2^\dagger a_1 \tag{23}$$

and are not Hermitian. However, their sum is Hermitian and we see that

$$H=N_1+N_2\,. \tag{24}$$

The number operators have the following commutation relations with the raising and lowering operators:

$$N_ia_j=a_j(N_i+\delta_{ij}-1) \tag{25}$$

and

$$N_ia_j^\dagger=a_j^\dagger(N_i-\delta_{ij}+1) \tag{26}$$

for $i,j=1,2$.

Up to this point we have maintained a symmetry of the dynamics under the exchange of the subscripts, $1\longleftrightarrow 2$. Now we must break that symmetry by choosing a vacuum state which is an eigenstate of Q_1 and P_1 or alternately a_1 and $a_1^\dagger$. The commutativity of Q_1 and P_1 permit this. The observability of Q_1 and P_1 for all time requires it. So we define

$$a_1^\dagger|0\rangle=a_1|0\rangle=0\,. \tag{27}$$

As a result $a_2|0\rangle\neq 0$ and $a_2^\dagger|0\rangle\neq 0$. The eigenstates of the number operators are

$$|n_+,n_-\rangle=(a_2^\dagger)^{n_+}(a_2)^{n_-}|0,0\rangle \tag{28}$$

and satisfy

$$N_1|n_+,n_-\rangle=-n_-|n_+,n_-\rangle\,, \tag{29}$$

$$N_2|n_+,n_-\rangle=n_+|n_+,n_-\rangle\,, \tag{30}$$

so that

$$H|n_+,n_-\rangle=(n_+-n_-)|n_+,n_-\rangle\,. \tag{31}$$

The lack of a lower bound to the energy spectrum is in a sense a problem but a necessary one in that it leads to the possibility of bound states with a continuous energy spectrum—a requirement of a faithful representation of the classical oscillator states. There is a subset of coherent states which can be put in a one-to-one relation with the set of classical oscillator states. The defining property of that subset is that its elements are eigenstates of the operators a_1 and $a_1^\dagger$. If we expand an element of that subset in terms of the number eigenstates

$$|z\rangle=\sum_{n_+,n_-=0}^{\infty}f(z|n_+,n_-)|n_+,n_-\rangle \tag{32}$$

and use

$$a_1^\dagger|n_+,n_-\rangle=-n_-|n_+,n_--1\rangle\,, \tag{33}$$

$$a_1|n_+,n_-\rangle=n_+|n_+-1,n_-\rangle \tag{34}$$

to evaluate the eigenvalue equations

$$a_1|z\rangle=iz^*|z\rangle\,, \tag{35}$$

$$a_1^\dagger|z\rangle=-iz|z\rangle\,, \tag{36}$$

we find

$$f(z|n_+,n_-)=\frac{C(iz^*)^{n_+}(iz)^{n_-}}{n_+!\,n_-!}\,, \tag{37}$$

where C is a constant. As a result

$$|z\rangle=C\exp[i(za_2+z^*a_2^\dagger)]|0,0\rangle\,. \tag{38}$$

We shall call the $|z\rangle$ states coherent states because of their close formal resemblance to the coherent states used in the study of the classical limit of harmonic oscillators, and of quantum electrodynamics[11] (which were eigenstates of the lowering operator but not of the raising operator).

Since $[H,a_1]=-a_1$, and $[H,a_1^\dagger]=a_1^\dagger$, it is clear that the (x_1,p_1) phase-space trajectory is sharp on the set of coherent $|z\rangle$ states. The classical trajectory represented by the state $|z\rangle$ is easily seen to be

$$x_1=\left(\frac{2}{m\omega}\right)^{1/2}R\sin(\omega t+\delta) \tag{39}$$

and

$$p_1=(2m\omega)^{1/2}R\cos(\omega t+\delta)\,, \tag{40}$$

where $z=Re^{i\delta}$. The linearity of H in the "momenta", $\Pi=(p_2,-x_2)$, is crucial for the observability of the phase-space trajectory. In fact, the linearity of all generators of canonical transformations in the momenta is necessary if the canonical transformations are not to take states out of the subset of coherent states.

The superselection rule which follows from the unobservability of the momenta, Π, is best approached by a consideration of the momentum-and coordinate-space representations of the coherent states. In the coordinate-space representation we find that Eqs. (35) and (36) give

$$\left[\left(\frac{m\omega}{2}\right)^{1/2}x_1+i\left(\frac{1}{2m\omega}\right)^{1/2}p_1\right]\langle x_1p_1|z\rangle=iz^*\langle x_1p_1|z\rangle \tag{41}$$

and

$$\left[\left(\frac{m\omega}{2}\right)^{1/2}x_1-i\left(\frac{1}{2m\omega}\right)^{1/2}p_1\right]\langle x_1p_1|z\rangle=-iz\langle x_1p_1|z\rangle\,, \tag{42}$$

so that

$$\langle x_1 p_1 | z\rangle = \sqrt{2}\,\delta\left(x_1 - \left(\frac{2}{m\omega}\right)^{1/2} \mathrm{Im}z\right) \times \delta(p_1 - (2m\omega)^{1/2}\mathrm{Re}z). \tag{43}$$

We have normalized $\langle x_1 p_1 | z\rangle$ so that

$$\langle z' | z\rangle = \int_{-\infty}^{\infty} dx_1 dp_1 \langle z' | x_1 p_1\rangle\langle x_1 p_1 | z\rangle = \delta(\mathrm{Re}z - \mathrm{Re}z')\delta(\mathrm{Im}z - \mathrm{Im}z') . \tag{44}$$

In momentum space Eqs. (35) and (36) lead to the differential equations

$$\left[\left(\frac{m\omega}{2}\right)^{1/2} i\frac{d}{dp_2} + \left(\frac{1}{2m\omega}\right)^{1/2} \frac{d}{dx_2}\right]\langle x_2 p_2 | z\rangle = iz^*\langle x_2 p_2 | z\rangle \tag{45}$$

and

$$\left[\left(\frac{m\omega}{2}\right)^{1/2} i\frac{d}{dp_2} - \left(\frac{1}{2m\omega}\right)^{1/2} \frac{d}{dx_2}\right]\langle x_2 p_2 | z\rangle = -iz\langle x_2 p_2 | z\rangle . \tag{46}$$

They are easily integrated to give

$$\langle x_2 p_2 | z\rangle = \frac{1}{\sqrt{2}\pi}\exp\left[-ip_2\left(\frac{2}{m\omega}\right)^{1/2} \mathrm{Im}z + ix_2(2m\omega)^{1/2}\mathrm{Re}z\right] \tag{47}$$

with the normalization condition

$$\langle z' | z\rangle = \int_{-\infty}^{\infty} dx_2 dp_2 \langle z' | x_2 p_2\rangle\langle x_2 p_2 | z\rangle = \delta(\mathrm{Re}z - \mathrm{Re}z')\delta(\mathrm{Im}z - \mathrm{Im}z') . \tag{48}$$

The transformation function between the two representations is

$$\langle x_1 p_1 | x_2 p_2\rangle = \frac{1}{2\pi}\exp(+ip_2 x_1 - ip_1 x_2) , \tag{49}$$

so that

$$\langle x_1 p_1 | z\rangle = \int_{-\infty}^{\infty} dx_2 dp_2 \langle x_1 p_1 | x_2 p_2\rangle\langle x_2 p_2 | z\rangle . \tag{50}$$

Each coherent state, $|z\rangle$, is a superselection sector in itself. There is no measurable dynamical variable $F = F(a_1, a_1^\dagger)$ which connects different states:

$$\langle z' | F(a_1, a_1^\dagger) | z\rangle = F(iz^*, -iz)\delta^2(z - z') . \tag{51}$$

This reflects the lack of a superposition principle in classical mechanics.

The operator formalism for coherent states is incomplete in that we have not defined an inner product. To remedy this deficiency we define the vacuum dual to $|0,0\rangle$ to satisfy

$$\langle 0,0 | a_2 = \langle 0,0 | a_2^\dagger = 0 \tag{52}$$

with $\langle 0,0 | 0,0\rangle = 1$. The dual state corresponding to the physical state, z, we define to be

$$\langle z | = \langle 0,0 | \delta(ia_1 + z^*)\delta(ia_1^\dagger - z) \equiv \langle 0,0 | \int_{-\infty}^{\infty} \frac{d\alpha\, d\beta}{(2\pi)^2} \exp[i\alpha(\mathrm{Im}z - 2^{-1/2}Q_1) + i\beta(\mathrm{Re}z - 2^{-1/2}P_1)] \tag{53}$$

so that Eqs. (48) and (51) follow if we choose $C = 1$.

Sometimes the dynamical state of a classical system is incompletely known and one only has a set of probabilities that the system is at a particular phase-space point at $t = 0$. If we let $P(z)$ be the probability that the system is at a phase-space point corresponding to z (as defined above), then using the properties

$$P(z) \geq 0, \quad \int d^2z\, P(z) = 1 \tag{54}$$

one sees that a density operator

$$\rho\delta^2(0) = \int d^2z\, |z\rangle P(z)\langle z| \tag{55}$$

may be defined which satisfies

$$\mathrm{Tr}\rho = 1 \tag{56}$$

and

$$\langle z' | \rho | z'\rangle \equiv \lim_{z'' \to z'} \langle z'' | \rho | z'\rangle = P(z') . \tag{57}$$

The mean value of an observable $A = A(a_1, a_1^\dagger)$ is given by

$$\langle A\rangle = \mathrm{Tr}\rho A = \int d^2z\, A(iz^*, -iz)P(z) , \tag{58}$$

and one can develop a formalism similar to the density-matrix formalism of quantum mechanics.

We now turn to a closer investigation of the relation of the pseudoquantum mechanics discussed above and true quantum-mechanical systems. We shall be particularly interested in the relation of the coherent states described above and the coherent states of a quantum-mechanical harmonic oscillator—to which they bear such a remarkable resemblance. We shall see that the pseudoquantum oscillator system is equivalent to an indefinite-metric quantum system composed of a harmonic oscillator (thus the connection to the coherent-state quantum oscillator formalism) and an "inverted" oscillator to be described below.

Let us define the following rotated raising and lowering operators in terms of the operators defined in Eqs. (17) and (18):

$$b_1 = a_1\cos\theta + a_2\sin\theta , \tag{59}$$

$$b_2 = -a_1\sin\theta + a_2\cos\theta . \tag{60}$$

Their commutation relations are

$$[b_1, b_1^\dagger] = \sin(2\theta)\,, \tag{61}$$

$$[b_2, b_2^\dagger] = -\sin(2\theta)\,, \tag{62}$$

$$[b_2, b_1^\dagger] = [b_1, b_2^\dagger] = \cos(2\theta) \tag{63}$$

with all other commutators equal to zero. The Hamiltonian of Eq. (21) becomes

$$H = \tfrac{1}{2}(\{b_1, b_1^\dagger\} - \{b_2, b_2^\dagger\})\sin(2\theta) + \tfrac{1}{2}(\{a_1, a_2^\dagger\} + \{a_2, a_1^\dagger\})\cos(2\theta)\,, \tag{64}$$

where $\{u, v\} = uv + vu$.

Now θ is an arbitrary angle and it is obvious that choosing $\theta = 0$ gives the commutation relations and Hamiltonian studied above. However, the choice $\theta = \pi/4$ results in a new form for H and the commutation relations, which can be interpreted as a harmonic oscillator (the b_1 and $b_1^\dagger$ sector) and an "inverted" harmonic oscillator (the b_2 and $b_2^\dagger$ sector) where the commutator and b_2 terms in the Hamiltonian have the wrong sign. The commutativity of the oscillator raising and lowering operators with the inverted oscillator raising and lowering operators leads to a simple factorization of the coherent states which lays bare the basic of the close similarity of form for our coherent states and the coherent states of a quantum oscillator[10]:

$$|z\rangle = \frac{1}{\sqrt{2\pi}} \exp\left[\frac{i}{\sqrt{2}}(zb_1 + z^* b_1^\dagger)\right] \times \exp\left[\frac{i}{\sqrt{2}}(zb_2 + z^* b_2^\dagger)\right] |0, 0\rangle\,, \tag{65}$$

while the coherent state of Ref. 11 has the form

$$|\alpha\rangle = \exp(\alpha b^\dagger - \alpha^* b)|0\rangle\,, \tag{66}$$

where α is a complex numer and $[b, b^\dagger] = 1$. It should be remembered that our choice of vacuum state such that $a_1|0,0\rangle = a_1^\dagger|0,0\rangle = 0$ obviates a simple direct relationship.

Since we have uncovered an interesting relation between a classical-made-quantum system and a "quantum" system of indefinite metric the possibility of reinterpreting indefinite-metric quantum systems as systems containing classical subsystems naturally arises.

III. EMBEDDING OF CLASSICAL FIELDS

In this section we shall discuss the embedding of a classical field theory in a quantum field theory. We shall study the embedding in detail for a scalar field and then describe the features of a classical-made-quantum electrodynamics which we shall call pseudoquantum electrodynamics for the sake of brevity.

Consider a classical field, $\phi_1(x)$, with canonically conjugate momentum, $\pi_1(x)$, and Hamiltonian equations of motion

$$\frac{d}{dt}\phi_1(x) = \frac{\delta \hat{H}}{\delta \pi_1(x)}\,, \tag{67}$$

$$\frac{d}{dt}\pi_1(x) = \frac{-\delta H}{\delta \phi_1(x)}\,, \tag{68}$$

where $\hat{H}$ is the Hamiltonian. We wish to define a "quantum" Hamiltonian, H, which allows us to rewrite Eqs. (67) and (68) in commutator form:

$$\frac{d}{dt}\phi_1(x) = i[H, \phi_1(x)]\,, \tag{69}$$

$$\frac{d}{dt}\pi_1(x) = i[H, \pi_1(x)]\,. \tag{70}$$

Equations (69) and (70) are satisfied if

$$H = \int d^3x \left[\frac{\delta H}{\delta \pi_1(x)} \frac{1}{i} \frac{\delta}{\delta \phi_1(x)} - \frac{\delta H}{\delta \phi_1(x)} \frac{1}{i} \frac{\delta}{\delta \pi_1(x)}\right]. \tag{71}$$

We now formally define

$$\phi_2(x) = i\frac{\delta}{\delta \pi_1(x)} \tag{72}$$

and

$$\pi_2(x) = -i\frac{\delta}{\delta \phi_1(x)}\,, \tag{73}$$

so that

$$H = \int d^3x \left[\frac{\delta \hat{H}}{\delta \pi_1(x)} \pi_2(x) + \frac{\delta \hat{H}}{\delta \phi_1(x)} \phi_2(x)\right]. \tag{74}$$

The fields satisfy the equal-time commutation relations

$$[\phi_i(x), \pi_j(y)] = i(1 - \delta_{ij})\delta^3(\vec{x} - \vec{y})\,, \tag{75}$$

$$[\phi_i(x), \phi_j(y)] = 0\,, \tag{76}$$

$$[\pi_i(x), \pi_j(y)] = 0\,, \tag{77}$$

where δ_{ij} is the Kronecker δ.

We note that the linearity of H in ϕ_2 and π_2 is necessary to maintain the classical character of ϕ_1 and π_1. This is best seen by an examination of Eqs. (69) and (70) and the corresponding Hamiltonian equations for ϕ_2 and π_2. (Other generators of canonical transformations are also linear in π_2 and ϕ_2.)

$\phi_2(x)$ and $\pi_2(x)$ will not be observables on the set of physical states, so that $\phi_1(x)$ and $\pi_1(x)$ will both be sharp on the set of physical states and satisfy superselection rules.

If we wish to couple the classical field to a truly quantum system and maintain the classical nature of the field then certain restrictions exist on the form of the total Hamiltonian H_{tot} and on the commutation relations of the various terms occurring in it. First, the coupling must satisfy the requirement that H_{tot} is linear in $\phi_2(x)$ and $\pi_2(x)$. If we denote the quantum fields by ψ and write the general form of the Hamiltonian as

$$H_{tot}=H+H_Q(\psi)+H_{int}\,, \tag{78}$$

where H is given by Eq. (74), $H_Q(\psi)$ depends only on the quantum fields, ψ, and

$$H_{int}=\int d^3x[\tilde{A}(\phi_1,\pi_1,\psi)\phi_2(x)+\tilde{B}(\phi_1,\pi_1,\psi)\pi_2(x)+\tilde{C}(\phi_1,\pi_1,\psi)]\,, \tag{79}$$

then we can rearrange the Hamiltonian so that

$$H_{tot}=\int d^3x[A(\phi_1,\pi_1,\psi)\phi_2(x)+B(\phi_1,\pi_1,\psi)\pi_2(x)+C(\phi_1,\pi_1,\psi)]\,, \tag{80}$$

where

$$A=\frac{\delta\hat{H}}{\delta\phi_1(x)}+\tilde{A}\,, \tag{81}$$

$$B=\frac{\delta\hat{H}}{\delta\pi_1(x)}+\tilde{B}\,, \tag{82}$$

and

$$C=\tilde{C}+\mathcal{H}_Q \tag{83}$$

with $H_Q=\int d^3x\,\mathcal{H}_Q$. An examination of the equations of motion of $\phi_1(x)$, $\pi_1(x)$, and ψ,

$$\frac{d}{dt}\phi_1=B(\phi_1,\pi_1,\psi)\,, \tag{84}$$

$$\frac{d}{dt}\pi_1=A(\phi_1,\pi_1,\psi)\,, \tag{85}$$

$$\frac{d}{dt}\psi=i[H_{tot},\psi]\,, \tag{86}$$

and the second time derivatives of ϕ_1 and π_1, such as

$$\frac{d^2}{dt^2}\phi_1(x)=i[H,B]=\int d^3y\left(-A\frac{\delta B}{\delta\pi_1(y)}+B\frac{\delta B}{\delta\phi_1(y)}+i\phi_2(y)[A,B]+i\pi_2(y)[B(y),B(x)]+i[C,B]\right), \tag{87}$$

leads us to require the equal-time commutation relations

$$[A(x),A(y)]=[A(x),B(y)]=[B(x),B(y)]=0\,, \tag{88}$$

where $A(x)=A(\phi_1(x),\pi_1(x),\psi(x))$, etc., so that $\phi_1(x)$ and $\pi_1(x)$ are independent of ϕ_2 and π_2 and hence observable for all time. An examination of higher time derivatives of ϕ_1 and π_1 lead to further restrictions on the equal-time commutation relations of A, B, and C. Examples are

$$[A,[C,B]]=0\,, \tag{89}$$

$$[B,[C,B]]=0\,, \tag{90}$$

$$[A,[C,[C,[C,B]]]]=0\,, \tag{91}$$

etc. A sufficient condition for satisfying all relations of this class consists of having equal-time commutation relations with the form

$$[A,C]=F_1(A,B,\phi_1,\pi_1) \tag{92}$$

and

$$[B,C]=F_2(A,B,\phi_1,\pi_1)\,. \tag{93}$$

Finally, we note that another obvious requirement [cf. Eqs. (84) and (85)] for the observability of ϕ_1 and π_1 is that A and B depend only on an (equal-time) commutative subset of the quantum field variables, ψ.

The above restrictions on the equal-time commutation relations have a direct interpretation in terms of Feynman diagrams for quantum corrections to the classical field behavior. For example, consider the interaction of the classical field sector with a scalar quantum field, ψ, expressed in the interaction

$$H_{int}=g\phi_2(x)\psi^2(x). \tag{94}$$

If $H_Q(\psi)$ is the conventional free Klein-Gordon Hamiltonian, then we find that Eq. (92) is not satisfied so that the Green's function for the classical ϕ_1 field receives quantum corrections from vacuum polarization loops of ψ particles and thus loses its classical character.

We now define a Lagrangian appropriate to our pseudoquantum field theory and then verify the reasonableness of our definition, and the pseudoquantization procedure described above, by studying the equivalent path-integral formulation. The Lagrangian corresponding to the pseudoquantum Hamiltonian, H, is

$$L=\int d^3x(\pi_1\mathring{\phi}_2+\pi_2\mathring{\phi}_1)-H\,, \tag{95}$$

where $L=L(\phi_1,\mathring{\phi}_1,\phi_2,\mathring{\phi}_2)$ and

$$\pi_1=\frac{\delta L}{\delta\mathring{\phi}_2}\,, \tag{96}$$

$$\pi_2 = \frac{\delta L}{\delta \dot{\phi}_1} . \tag{97}$$

The vacuum-vacuum transition amplitude for the field theory corresponding to the H_{tot} of Eq. (78) will be shown to be

$$W = \int \prod_x d\phi_1(x) d\phi_2(x) d\pi_1(x) d\pi_2(x) d\psi(x) \exp(iS) , \tag{98}$$

where $S = \int dt\, L_{tot}$ up to external source terms. We begin by considering the vacuum-vacuum transition amplitude corresponding to H_Q,

$$W_Q = \int \prod_x d\psi(x) \exp(iS_Q) , \tag{99}$$

where ϕ_1 has the character of an external source.

We can now introduce the classical behavior of the ϕ_1 field through functional δ functions

$$\int \prod_x d\psi(x) d\phi_1(x) d\pi_1(x) \delta(B(\phi_1, \pi_1, \psi) - \dot{\phi}_1) \times \delta(A(\phi_1, \pi_1, \psi) + \dot{\pi}_1) e^{iS_Q} , \tag{100}$$

which can be put in the form

$$\int \prod_x d\phi_1(x) d\pi_1(x) d\phi_2(x) d\pi_2(x) \times \exp\left\{ i \int d^4x [(\dot{\phi}_1 - B)\pi_2 - (\dot{\pi}_1 + A)\phi_2] + iS_Q \right\} . \tag{101}$$

After performing a partial integration on the $\dot{\pi}_1\phi_2$ term and discarding a surface term we see that the definition of L in Eq. (95) is correct and that the vacuum-vacuum transition amplitude is indeed given by Eq. (98).

The restrictions on the commutation relations of the various terms in the H_{tot} [expressed in Eqs. (88)–(93)] translate into the requirement that the "quantum completion"[11] of the ϕ_2 field does not take place, i.e., that all N-point functions of the ϕ_2 field are zero:

$$\frac{\delta^n W}{\delta J_2(x_1)\delta J_2(x_2) \cdots \delta J_2(x_n)} = 0 , \tag{102}$$

where J_2 is an external source coupled to ϕ_2.

We now discuss the embedding of a free classical Klein-Gordon field in a quantum field theory. The Lagrangian density is

$$\mathcal{L} = \frac{\partial \phi_1}{\partial x^\mu} \frac{\partial \phi_2}{\partial x_\mu} - m^2 \phi_1 \phi_2 , \tag{103}$$

from which one obtains the Euler-Lagrange equations (for $i = 1, 2$)

$$(\Box + m^2)\phi_i(x) = 0. \tag{104}$$

The canonical momenta are (note that π_2 is conjugate to ϕ_1, etc.)

$$\Pi_i = \dot{\phi}_i \tag{105}$$

for $i = 1, 2$ with the equal-time commutation relations given by Eqs. (75)–(77). We expand the fields in Fourier integrals:

$$\phi_1(\vec{x}, t) = \int d^3k [a_1(k) f_k(x) + a_1^\dagger f_k^*(x)] \tag{106}$$

and

$$\phi_2(\vec{x}, t) = \int d^3k [a_2(k) f_k(x) + a_2^\dagger(k) f_k^*(x)] , \tag{107}$$

where

$$f_k(x) = (2\pi)^{-3/2}(2\omega_k)^{-1/2} e^{-ik\cdot x} \tag{108}$$

with $\omega_k = (\vec{k}^2 + m^2)^{1/2}$. The Fourier component operators satisfy the commutation relations

$$[a_i(k), a_j^\dagger(k')] = (1 - \delta_{ij})\delta^3(\vec{k} - \vec{k}') \tag{109}$$

and

$$[a_i(k), a_j(k')] = [a_i^\dagger(k), a_j^\dagger(k')] = 0 \tag{110}$$

for $i, j = 1, 2$.

In terms of the Fourier coefficients

$$H \equiv \int d^3x (\dot{\phi}_1 \dot{\phi}_2 + \vec{\nabla}\phi_1 \cdot \vec{\nabla}\phi_2 + m^2 \phi_1 \phi_2) \tag{111}$$

becomes

$$H = \int d^3k\, \omega_k [\{a_1(k), a_2^\dagger(k)\} + \{a_2(k), a_1^\dagger(k)\}] . \tag{112}$$

The analogy between the mode amplitudes of the fields and the raising and lowering operators of the simple harmonic oscillator has been previously remarked. We can therefore use the considerations of Sec. II to establish the spectrum of physical states. The defining properties of a physical state are that $\phi_1(x)$ and $\pi_1(x)$ are sharp on it for all time:

$$\phi_1(x) |\Phi, \Pi\rangle = \Phi(x) |\Phi, \Pi\rangle \tag{113}$$

and

$$\pi_1(x) |\Phi, \Pi\rangle = \Pi(x) |\Phi, \Pi\rangle , \tag{114}$$

where $\Phi(x)$ and $\Pi(x)$ are c-number functions of x:

$$\Phi(x) = \int d^3k [\alpha(k) f_k(x) + \alpha^*(k) f_k^*(x)] \tag{115}$$

and

$$\Pi(x) = -i \int d^3k\, \omega_k [\alpha(k) f_k(x) - \alpha^*(k) f_k^*(x)] \tag{116}$$

with $\alpha(k)$ a c-number function of k.

As a result we are led to define a set of physical states, $|\alpha\rangle$, which are in one-to-one correspondence with the classical solutions of the Klein-Gordon equation and satisfy

$$a_1(k)|\alpha\rangle = \alpha(k)|\alpha\rangle, \tag{117}$$

$$a_1^\dagger(k)|\alpha\rangle = \alpha^*(k)|\alpha\rangle. \tag{118}$$

In analogy with the states of the simple harmonic oscillator (Sec. II) we further define

$$|\alpha\rangle = C\exp\left\{\int d^3k'[\alpha(k')a_2^\dagger(k') - \alpha^*(k')a_2(k')]\right\}|0\rangle, \tag{119}$$

where the vacuum state, $|0\rangle$, satisfies

$$a_1(k)|0\rangle = a_1^\dagger(k)|0\rangle = 0. \tag{120}$$

The physical states, $|\alpha\rangle$, lie in a space which is the infinite tensor product of single-mode spaces. While ϕ_1 and π_1 are sharp for all time on the subset of physical states, we see that ϕ_2 and π_2 are not and, in fact, when applied to a physical state map it into an unphysical state. The superselection rules are embodied in

$$\langle\alpha'|\mathcal{O}|\alpha\rangle = \mathcal{O}_\alpha\delta^2(\alpha-\alpha'), \tag{121}$$

where $\mathcal{O}$ is the operator corresponding to any observable, $\mathcal{O}_\alpha$ is its eigenvalue for the state $|\alpha\rangle$, and $\delta^2(\alpha-\alpha')$ is a functional δ function in the real and imaginary parts of $\alpha-\alpha'$. The functional δ functions have their origin in the definition of the dual set of physical states. We define the dual vacuum state $\langle 0|$ by

$$\langle 0|a_2(k) = 0 \tag{122a}$$

and

$$\langle 0|a_2^\dagger(k) = 0 \tag{122b}$$

for all k with $\langle 0|0\rangle = 1$. The dual state corresponding to $\alpha(k)$ we define by

$$\begin{aligned}\langle\alpha| &= \langle 0|\prod_k \delta(\alpha(k)-a_1(k))\delta(\alpha^*(k)-a_1^\dagger(k))\\ &\equiv \langle 0|\delta(\alpha-a_1)\delta(\alpha^*-a_1^\dagger),\end{aligned} \tag{123}$$

so that

$$\langle\alpha'|\alpha\rangle = \delta^2(\alpha'-\alpha) \tag{124}$$

if $C=1$.

We have now established a procedure for embedding a classical field in a quantum field theory. Given a Lagrangian, L, for a classical field theory describing a field $\phi_1(x)$, the Lagrangian density for the pseudoquantum field theory, $\mathcal{L}_{PQ}$ is

$$\mathcal{L}_{PQ}(\phi_1,\dot\phi_1,\phi_2,\dot\phi_2) = \frac{\delta L}{\delta\phi_1(x)}\phi_2(x) + \frac{\delta L}{\delta\dot\phi_1(x)}\pi_2(x) \tag{125}$$

up to a divergence with

$$\pi_2(x) = \frac{\delta}{\delta\dot\phi_1(x)}\int d^3x\,\mathcal{L}_{PQ}. \tag{126}$$

In the case of a classical electromagnetic field interacting with a quantum electron field, one pseudoquantum model, which describes some electromagnetic processes, has the Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}F^1_{\mu\nu}F^2_{\mu\nu} + \bar\psi(i\nabla\!\!\!/ - eA\!\!\!/_1 - m_0)\psi, \tag{127}$$

where $A^1_\mu(x)$ is the classical electromagnetic field, ψ is the electron field, $A^2_\mu(x)$ is the unobservable auxiliary field, and $F^i_{\mu\nu} = \partial_\nu A^i_\mu - \partial_\nu A^i_\mu$ for $i=1,2$. Although our interpretation of the free electromagnetic part of the Lagrangian, $-\frac{1}{2}F^1_{\mu\nu}F^2_{\mu\nu}$, is new, the actual form of this term appeared some time ago in a generalization of electrodynamics by Mie,[12] and was recently used in an Abelian prototype model for quark confinement.[8] The equations of motion are

$$\partial^\mu F^1_{\mu\nu} = 0, \tag{128}$$

$$\partial^\mu F^2_{\mu\nu} + eJ_\nu = 0, \tag{129}$$

and

$$(i\nabla\!\!\!/ - eA\!\!\!/^1 - m)\psi = 0. \tag{130}$$

The canonical momentum which is conjugate to A^1_μ is

$$\Pi^2_\mu = F^2_{0\mu} \tag{131}$$

and that conjugate to A^2_μ is

$$\Pi^1_\mu = F^1_{0\mu}. \tag{132}$$

We take A^1_μ and Π^1_μ to be classical fields which are observable for all time. A^2_μ and Π^2_μ are not observable. Note that $\mathcal{L}$ is invariant under the independent gauge transformations

$$A^1_\mu \to A^1_\mu + \partial_\mu\Lambda^1(x) \tag{133}$$

and

$$A^2_\mu \to A^2_\mu + \partial_\mu\Lambda^2(x). \tag{134}$$

Since $\Pi^1_0 = \Pi^2_0 = 0$, it is apparent that A^1_0 and A^2_0 are c numbers. If we chose the Coulomb gauge for A^1_μ,

$$\vec\nabla\cdot\vec{A}^1 = 0, \tag{135}$$

and for A^2_μ,

$$\vec\nabla\cdot\vec{A}^2 = 0, \tag{136}$$

then we can establish the equal-time commutation relations

$$[\Pi_i^a(\vec{x},t),A_j^b(\vec{y},t)]=i(1-\delta_{ab})\times\int\frac{d^3k}{(2\pi)^3}e^{i\vec{k}\cdot(\vec{x}-\vec{y})}\left(\delta_{ij}-\frac{k_ik_j}{|\vec{k}|^2}\right)=i(1-\delta_{ab})\delta_{ij}^{tr}(\vec{x}-\vec{y}) \quad (137)$$

for $a,b=1,2$ and $i,j=1,2,3$.

This pseudoquantum field theory describes the dynamics of quantum electron fields interacting with a free, classical electromagnetic field. A typical perturbation theory matrix element would have the form

$$\langle \mathcal{A}',0\,|\,T(\bar{\psi}(x)J^{\mu_1}(x_1)A^1_{\mu_1}(x_1)J^{\mu_2}(x_2)A^1_{\mu_2}(x_2)\cdots J^{\mu_n}(x_n)A^1_{\mu_n}(x_n)\psi(y))\,|\,\mathcal{A},0\rangle, \quad (138)$$

where $|\mathcal{A},0\rangle$ is the tensor product of an electron vacuum state and an electromagnetic state corresponding to the classical field $\mathcal{A}_\mu(z)$. Because $A^1_\mu(x)$ is sharp on this state, the matrix element becomes

$$\langle 0\,|\,T(\bar{\psi}(x)J^{\mu_1}(x_1)\cdots J^{\mu_n}(x_n)\psi(y))\,|\,0\rangle\mathcal{A}_{\mu_1}(x_1)\mathcal{A}_{\mu_2}(x_2)\cdots\mathcal{A}_{\mu_n}(x_n) \quad (139)$$

modulo a functional δ function in $\mathcal{A}'-\mathcal{A}$. Thus this model is equivalent to a quantized electron field interacting with an external electromagnetic field.

Another possibility for a model electrodynamics is realized by letting the interaction term in Eq. (127) above be replaced with

$$L_{\text{int}}=-e\bar{\psi}A\!\!\!/_2\psi. \quad (140)$$

Because the equivalent of the equal-time commutation relation, Eq. (92), is not true in this model, the A^1_μ field loses its purely classical character due to quantum corrections. However, this model may be of value for the study of the modification of the A^1_μ field resulting from the emission of many soft photons by a current.

Since vacuum polarization effects modify the electromagnetic field in this case we define in-field eigenstates (in the transverse gauge) by

$$\vec{A}^1_{\text{in}}|\mathcal{A}\rangle_{\text{in}}=\vec{\mathcal{A}}_{\text{in}}|\mathcal{A}\rangle_{\text{in}}, \quad (141)$$

where

$$|\mathcal{A}\rangle_{\text{in}}=\exp\left[\int d^3k\sum_{\lambda=1}^{2}(\alpha(k,\lambda)a_2^\dagger(k,\lambda)-\alpha^*(k,\lambda)a_2(k,\lambda))\right]|0\rangle \quad (142)$$

and

$$\vec{\mathcal{A}}_{\text{in}}=\int d^3k\sum_{\lambda=1}^{2}\vec{\epsilon}(k,\lambda)[\alpha(k,\lambda)f_k(x)+\alpha^*(k,\lambda)f_k^*(x)] \quad (143)$$

with

$$\vec{A}^i_{\text{in}}=\int d^3k\sum_{\lambda=1}^{2}\vec{\epsilon}(k,\lambda)[a_i(k,\lambda)f_k(x)+a_i^\dagger(k,\lambda)f_k^*(x)] \quad (144)$$

for $i=1,2$. The vacuum state is defined by

$$a_1(k,\lambda)\,|\,0\rangle=a_1^\dagger(k,\lambda)\,|\,0\rangle=0$$

for all k,λ. The interacting field, $\vec{A}^1$, is apparently not sharp on $|\mathcal{A}\rangle_{\text{in}}$ but is sharp on

$$|\mathcal{A}\rangle=U^{-1}(t,-\infty)\,|\mathcal{A}\rangle_{\text{in}}, \quad (145)$$

where

$$U(t,-\infty)=T\left(\exp\left[-i\int_\infty^t d^4x\,H_{\text{int}}(A^2_{\text{in}},\psi_{\text{in}})\right]\right) \quad (146)$$

because

$$\vec{A}^1(\vec{x},t)=U^{-1}(t,-\infty)\vec{A}^1_{\text{in}}(\vec{x},t)U(t,-\infty). \quad (147)$$

With these preliminaries completed, the study of physical processes within the framework of these models is now possible, although we shall not pursue it in this report.

Before turning to a discussion of non-Abelian gauge field theories, it is worth noting that the choice of vacuum state we have made necessitates a redefinition of normal-ordering. By normal-ordering a Lagrangian term we shall mean that the observable fields (to which we have consistently appended the superscript or subscript one) are to be placed to the right, and unobservable fields, labeled by two, are to be placed to the left. Thus Wick's theorem (with our definition of normal-ordering) becomes in the case of two fields

$$\begin{aligned}T(\phi_{1\,\text{in}}(x_1)\phi_{2\,\text{in}}(x_2))&=:\phi_{1\,\text{in}}(x_1)\phi_{2\,\text{in}}(x_2):+\langle 0\,|\,T(\phi_{1\,\text{in}}(x_1)\phi_{2\,\text{in}}(x_2))\,|\,0\rangle\\&=\phi_{2\,\text{in}}(x_2)\phi_{1\,\text{in}}(x_1)+\theta(x_{10}-x_{20})[\phi_{1\,\text{in}}(x_1),\phi_{2\,\text{in}}(x_2)].\end{aligned} \quad (148)$$

Note that the Green's function

$$G(x_1,x_2)=\langle 0\,|\,T(\phi_{1\,\text{in}}(x_1)\phi_{2\,\text{in}}(x_2))\,|\,0\rangle \quad (149)$$

is necessarily retarded. From this we can conclude that the models of electrodynamics, which we have considered, naturally embody the observed retarded nature of classical electrodynamics. Another way of stating this result is: If classical electrodynamics is to have a pseudoquantum formulation, its Green's functions are necessarily retarded. The origin of the asymmetry is the definition of the vacuums (which is equivalent to a specification of boundary conditions). Just as in classical electrodynamics retarded propagation is implemented by a choice of boundary conditions which do not require a commitment to any specific cosmological model.

Finally we would like to note that the Lagrangian obtained from adding L_{int} of Eq. (140) to the Lagrangian of Eq. (127) is equivalent to the usual Lagrangian of electrodynamics plus a term describing a massless Abelian gauge field with the wrong sign. (This is seen by defining new fields equal to the sum and difference of A^1_μ and A^2_μ.) This field theory may be quantized following the procedure we have outlined. A^1_μ loses its classical character due to quantum corrections.

IV. NON-ABELIAN GAUGE THEORIES

In this section we shall describe the procedure for embedding a classical non-Abelian Yang-Mills field in a quantum field theory. Then we will discuss a vierbein formulation of quantum gravity which could have been interpreted as a pseudoquantum field theory for a classical metric field if it were not for one term in the Lagrangian which makes it a truly quantum field theory. Nevertheless we suggest a new canonical quantization procedure based on our pseudoquantum approach.

Consider a classical Yang-Mills field, $A^1_\mu = \underline{A}^1_\mu \cdot \underline{T}$, where the jth component of $\underline{T}$ is a matrix representing a generator of a non-Abelian group G in the defining representation with commutation relations

$$[T_j, T_k] = it_{jkl}T_l. \tag{150}$$

We can define a pseudoquantum field theory, wherein the classical character of A^1_μ is maintained, which has the Lagrangian density

$$\begin{aligned}\mathcal{L} = {} & \tfrac{1}{2}\underline{F}^1_{\mu\nu}\cdot\underline{F}^{2\mu\nu} - \tfrac{1}{2}\underline{F}^{2\mu\nu}\cdot(\partial_\mu\underline{A}^1_\nu - \partial_\nu\underline{A}^1_\mu + g\underline{A}^1_\mu\times\underline{A}^1_\nu)\\ & - \tfrac{1}{2}\underline{F}^{1\mu\nu}\cdot(\partial_\mu\underline{A}^2_\nu - \partial_\nu\underline{A}^2_\mu + g\underline{A}^1_\mu\times\underline{A}^2_\nu - g\underline{A}^1_\nu\times\underline{A}^2_\mu)\\ & + \bar\psi(i\not\nabla + g\not A^1 - m)\psi,\end{aligned} \tag{151}$$

where ψ is a fermion field. The theory is invariant under the local gauge transformation, $S\in G$,

$$\psi' = S^{-1}\psi, \tag{152}$$

$$A^{1\prime}_\mu = S^{-1}A^1_\mu S + \frac{i}{g}S^{-1}\partial_\mu S\,, \tag{153}$$

$$F^{1\prime}_{\mu\nu} = S^{-1}F^1_{\mu\nu}S, \tag{154}$$

$$A^{2\prime}_\mu = S^{-1}A^2_\mu S, \tag{155}$$

$$F^{2\prime}_{\mu\nu} = S^{-1}F^2_{\mu\nu}S. \tag{156}$$

Except for one important term this Lagrangian with its attendant gauge invariance properties has been suggested as a possible model for the quark-confining strong interaction.[8] Since the omitted term has a masslike character $\Lambda^2\underline{A}^2_\mu\cdot\underline{A}^{2\mu}$, where Λ has the dimensions of a mass, it is clear that the strong-interaction model's ultraviolet behavior approaches that of the present pseudoquantum theory if the same quantization procedure is followed in both cases. We shall discuss this question further in the next section and show that the *ad hoc* procedure followed in Ref. 8 leads to the same result as the quantization procedure developed in this report.

The Euler-Lagrange equations of motion which are obtained from $\mathcal{L}$ in the canonical manner are

$$\underline{F}^1_{\mu\nu} = \partial_\mu\underline{A}^1_\nu - \partial_\nu\underline{A}^1_\mu + g\underline{A}^1_\mu\times\underline{A}^1_\nu, \tag{157}$$

$$\underline{F}^2_{\mu\nu} = \partial_\mu\underline{A}^2_\nu - \partial_\nu\underline{A}^2_\mu + g\underline{A}^1_\mu\times\underline{A}^2_\nu - g\underline{A}^1_\nu\times\underline{A}^2_\mu, \tag{158}$$

$$(\partial_\mu + g\underline{A}^1_\mu\times)\underline{F}^{1\mu\nu} = 0, \tag{159}$$

$$(\partial_\mu + g\underline{A}^1_\mu\times)\underline{F}^{2\mu\nu} + g\underline{A}^2_\mu\times\underline{F}^{1\mu\nu} + g\underline{J}^\nu = 0, \tag{160}$$

$$(i\not\nabla + g\not A^1 - m)\psi = 0, \tag{161}$$

with the conservation law

$$(\partial_\nu + g\underline{A}^1_\nu\times)\underline{J}^\nu = 0. \tag{162}$$

The canonical momentum which is conjugate to $\underline{A}^1_j$ is

$$\underline{\Pi}^2_j = \underline{F}^2_{0j} \tag{163}$$

and the canonical momentum conjugate to $\underline{A}^2_j$ is

$$\underline{\Pi}^1_j = \underline{F}^1_{0j} \tag{164}$$

for $j = 1, 2, 3$. The canonical momentum corresponding to the fields $\underline{A}^i_0$ is zero for $i = 1, 2$. The existence of equations of constraint among the Euler-Lagrange equations implies that not all field components are independent, so that we must isolate the independent components prior to defining the canonical equal-time commutation relations.

Following Ref. 8 we choose to work in the Coulomb gauge, $\nabla_i\underline{A}^1_i = 0$, and define the field variables

$$\underline{A}^2_i = \underline{A}^{2T}_i + \underline{A}^{2L}_i, \tag{165}$$

$$\underline{\Pi}^a_i = \underline{\Pi}^{aT}_i + \underline{\Pi}^{aL}_i, \tag{166}$$

where

$$\nabla_i\cdot\underline{A}^{2T}_i = \nabla_i\cdot\underline{\Pi}^{aT}_i = 0 \tag{167}$$

and $a = 1, 2$. Then the nonzero equal-time commutation relations are

$$[\Pi^{aT}_{ip}(x), A^{bT}_{jq}(y)] = i\delta_{pq}(1-\delta_{ab})\delta^{tr}_{ij}(\vec{x}-\vec{y}), \tag{168}$$

where p and q are internal-symmetry indices, $a, b = 1, 2$, and $i, j = 1, 2, 3$.

While the classical character of $\underline{A}_\mu^1$ can be maintained with our choice of $\mathcal{L}$, this theory has features due to its non-Abelian nature which make it less trivial and therefore more interesting than the corresponding Abelian theory discussed in the last section. If we follow a procedure similar to that in the Abelian case [Eq. (127)] and introduce a set of states appropriate to the quadratic part of the Lagrangian, then the cubic and quartic Yang-Mills terms in the interaction part of the Lagrangian will act to transform $\underline{A}_{\text{in}\,\mu}^1$ eigenstates into eigenstates of the interacting field $\underline{A}_\mu^1$. This is, of course, necessary for the classical Yang-Mills equations of motion to be satisfied. Our formalism, thus, offers a perturbative method for calculating solutions of the classical Yang-Mills equations. In addition, it gives an interesting interpretation to the short-distance behavior of the quark-confining field theory of Ref. 8. At short distances the gluon field $\underline{A}_\mu^1$ effectively decouples from the quark sector and becomes, in effect, a free field. This type of short-distance behavior is certainly not at odds with the seemingly simple behavior observed in hadron processes at high energy. Therefore, it is possible that pseudoquantum field theory may be relevant to the short-distance behavior of hadron interaction. Certainly, it is interesting that elementary fermions fall into two similar groups: those which appear to be individually observable (leptons) and those which are not individually observable (quarks).

We now turn to a consideration of a vierbein model of gravity which has certain close similarities to the pseudoquantum field theories we have been studying. In Weyl's formulation[13] of the Einstein-Cartan theory of gravity a vierbein field, $l^{\mu a}(x)$, is introduced which is the "square root" of the metric tensor

$$g^{\mu\nu}=\eta_{ab}l^{\mu a}l^{\nu b}, \tag{169}$$

where η_{ab} is the constant metric tensor of special relativity, where Roman indices transform as vectors under the SL(2, C) group of local Lorentz transformations, and where Greek indices transform as vectors under general coordinate transformations. It is useful to introduce the constant Dirac matrices, γ_a and $4S_{ab}=i[\gamma_a,\gamma_b]$. Under an SL(2, C) transformation,

$$S=\exp[iC^{ab}(x)S_{ab}], \tag{170}$$

a spinor, $\psi(x)$, becomes

$$\psi'=S\psi. \tag{171}$$

The local nature of the transformation requires the introduction of a gauge field

$$B_\mu^{ab}=-B_\mu^{ba} \tag{172}$$

which transforms inhomogeneously,

$$B_\mu\to SB_\mu S^{-1}-\frac{i}{g}S\partial_\mu S^{-1}, \tag{173}$$

so that a Lorentz transformation gauge-covariant derivative can be defined

$$\nabla_\mu\psi=(\partial_\mu+igB_\mu)\psi, \tag{174}$$

where $B_\mu=B_\mu^{ab}S_{ab}$ and $g=12\pi G$ where G is Newton's constant. Under a gauge transformation we have

$$l^\mu=l^{\mu a}\gamma_a\to Sl^\mu S^{-1}, \tag{175}$$

so that the gauge-covariant derivative of l^μ is defined to be

$$\nabla_\nu l^\mu=(\partial_\nu+igB_\nu\times)\,l^\mu, \tag{176}$$

where $B_\nu\times l^\mu=[B_\nu,l^\mu]$. The commutator

$$igB_{\mu\nu}=[\partial_\mu+igB_\mu,\partial_\nu+igB_\nu] \tag{177}$$

transforms homogeneously under a gauge transformation

$$B_{\mu\nu}\to SB_{\mu\nu}S^{-1}, \tag{178}$$

and as a second-rank tensor under general coordinate transformations. With these field quantities we are able to construct a Lagrangian $\mathcal{L}_{\text{Weyl}}$ which reduces to the Einstein Lagrangian for gravity when no matter is present,[13]

$$\mathcal{L}=\mathcal{L}_{\text{Weyl}}+\mathcal{L}_{\text{matter}}, \tag{179}$$

where

$$\mathcal{L}_{\text{Weyl}}=\frac{i}{8l}\,\text{Tr}\,l^\mu l^\nu B_{\mu\nu} \tag{180}$$

and where, for example, we might let

$$l\,\mathcal{L}_{\text{matter}}=\bar{\psi}(il^\mu\nabla_\mu+m)\psi \tag{181}$$

with $l=\det(l^{\mu a})$.

We observe that the terms containing derivatives in $\mathcal{L}_{\text{Weyl}}$ are linear in the field B_μ—a suggestive feature in view of our previous discussion. However, the quadratic term in B_μ eliminates the possibility of regarding $\mathcal{L}_{\text{Weyl}}$ as a pseudoquantum field theory for a classical field $l^{\mu a}$. But, regardless of this consideration, the fact that $l^{\mu a}$ is necessarily classical in part leads us to consider quantizing vierbein gravity in a manner which is based on the pseudoquantization procedure described above. Remembering that a successful perturbation theory requires the perturbation to be around known solutions we introduce a quadratic Lagrangian term via

$$\mathcal{L}=\mathcal{L}_0+(\mathcal{L}-\mathcal{L}_0)=\mathcal{L}_0+\mathcal{L}_{\text{int}}, \tag{182}$$

where

$$\mathcal{L}_0=-\tfrac{1}{4}i\,\text{Tr}(B'_{\mu a}l^\mu\gamma^a+ig\,[B_a,B_b]\,\gamma^a\gamma^b) \tag{183}$$

and

$$B'_{\mu a}=\partial_\mu B_a-\partial_a B_\mu\,. \tag{184}$$

Our plan is to follow the pseudoquantization procedure for the "free" part of the Lagrangian $\mathcal{L}_0$. Therefore we will (i) choose a particular coordinate system (harmonic coordinates) and a particular gauge, the "Lorentz" gauge, $\partial^\mu B_\mu=0$, (ii) establish equal-time commutation relations, (iii) define a set of eigenstates of $l^{\mu a}$, and (iv) proceed to calculate quantum corrections in perturbation theory.

The equations of motion for the "free" Lagrangian $\mathcal{L}_0$ are

$$\partial_\mu B^{ab}_b-\partial_b B^{ab}_\mu=0 \tag{185}$$

and

$$\partial_\mu(l^{\mu a}\eta^{\nu b}-l^{\nu a}\eta^{\mu b})+2g\,(\eta^{\nu a}B^{cb}_c-\eta^{\nu b}B^{ca}_c-\eta^{ac}B^{\nu b}_c+\eta^{bc}B^{\nu a}_c)=0\,. \tag{186}$$

We work in the gravitational equivalent of the Lorentz gauge of electrodynamics,

$$\partial^\mu B^{ab}_\mu=0\,, \tag{187}$$

and choose harmonic coordinates

$$\partial_\mu l^{\mu a}=\tfrac{1}{2}\,\partial^a\eta_{\sigma\tau}l^{\sigma\tau}\,. \tag{188}$$

The Green's function associated with Eq. (185) is

$$G_{\alpha ef,\rho\sigma}(x,y)=-\tfrac{1}{2}\int\frac{d^4k}{k^2}\,e^{-ik\cdot(x-y)}g_{\alpha ef,\rho\sigma}(k)\,, \tag{189}$$

where

$$g_{\alpha ef,\rho\sigma}(k)=k_e\left(\eta_{\alpha\rho}\eta_{f\sigma}+\eta_{\alpha\sigma}\eta_{f\rho}-\eta_{\alpha f}\eta_{\rho\sigma}-\frac{k_\alpha k_\rho\eta_{f\sigma}+k_\alpha k_\sigma\eta_{f\rho}}{k^2}\right)-k_f\left(\eta_{\alpha\rho}\eta_{e\sigma}+\eta_{\alpha\sigma}\eta_{e\rho}-\eta_{\alpha e}\eta_{\rho\sigma}-\frac{k_\alpha k_\rho\eta_{e\sigma}+k_\alpha k_\sigma\eta_{e\rho}}{k^2}\right). \tag{190}$$

In order to relate the above Green's function to a time-ordered product of the quantum fields it is first necessary to introduce a set of coherent states, $|L\rangle$, which are eigenstates of $l^{\mu a}$:

$$l^{\mu a}(x)\,|L\rangle=L^{\mu a}(x)\,|L\rangle\,, \tag{191}$$

where $L^{\mu a}(x)$ is a c-number function of x. In particular, we define $|\eta\rangle$ to satisfy

$$l^{\mu a}|\eta\rangle=\eta^{\mu a}|\eta\rangle\,, \tag{192}$$

where $\eta^{\mu a}$ is the constant Lorentz metric tensor of special relativity. Given a state $|L\rangle$ we define the field

$$l^{\mu a}_L=l^{\mu a}-L^{\mu a}. \tag{193}$$

This field corresponds to the quantum part of $l^{\mu a}$ and when applied to the purely classical state $|L\rangle$ has the eigenvalue zero.

We now make the identification

$$iG_{\alpha ef,\rho\sigma}(x,y)=\langle L\,|\,T(B_{\alpha ef}(x),l_{L\rho\sigma}(y))\,|\,L\rangle\,. \tag{194}$$

If we desire to calculate quantum corrections to $l_{\rho\sigma}=\eta_{\rho\sigma}$ we choose $|L\rangle=|\eta\rangle$. (It should be noted that $G_{\alpha ef,\rho\sigma}$ is independent of the choice of $|L\rangle$ as we have defined it.) Because $l_{L\rho\sigma}(y)$ is sharp on $|L\rangle$ we find that the right side of Eq. (194) becomes

$$iG_{\alpha ef,\rho\sigma}(x,y)=\theta(y_0-x_0)[l_{\rho\sigma}(y),B_{\alpha ef}(x)] \tag{195}$$

up to a functional δ function. From the form of $\mathcal{L}_0$ we see that the commutator is not zero. It is fully determined by an equal-time commutation relation of $l_{\rho\sigma}$ and $B_{\alpha ef}$ (which by the way is the only nonzero equal-time commutator if the canonical procedure is followed), the equations of motion, and the requirement that it be zero at spacelike distances. The "retarded" form of $G_{\alpha ef,\rho\sigma}$ fixes the integration contour around poles in Eq. (192). The other nonzero Green's function in the free Lagrangian model specified by $\mathcal{L}_0$ is

$$iH^{\mu\nu,\rho\sigma}(x,y)=\langle L\,|\,T(l^{\mu\nu}_L(x),l^{\rho\sigma}_L(y))\,|\,L\rangle\,. \tag{196}$$

It is nonzero owing to the presence of the $[B_\mu,B_\nu]$ term in $\mathcal{L}_0$. We shall show in the next section that it is a principal-value propagator rather than a Feynman propagator. In coordinate space this results in $H^{\mu\nu,\rho\sigma}$ being the sum of the advanced and retarded propagators. As a result our model is equivalent to an action-at-a-distance theory in some sectors.

The classical part of $l_{\mu a}$ is the solution of the classical linearized field equations with appropriate matter sources. The linearized field equations are derived from a Lagrangian consisting of $\mathcal{L}_0$ plus matter terms. (Note that the form of $\mathcal{L}_0$ is obtained by substituting $l_{\mu a}=\eta_{\mu a}+h_{\mu a}$ in $\mathcal{L}_{\text{Weyl}}$, expanding, and keeping quadratic terms.) Thus the class of possible background metrics is restricted.

A simplification occurs in perturbation theory when the classical part of $l_{\mu\alpha}$ is $\eta_{\mu\alpha}$. In this case $(\mathcal{L}_{\text{Weyl}}-\mathcal{L}_0)\,|\eta\rangle=0$ when $\mathcal{L}_0$ and $\mathcal{L}_{\text{Weyl}}$ are expressed in terms of asymptotic fields.

V. PRINCIPAL-VALUE PROPAGATORS AND ACTION AT A DISTANCE

In this section we shall show that certain propagators, in field theories where the pseudoquantization procedure has been followed, are principal-value propagators (i.e., the sum of the advanced and retarded Green's functions in coordinate space) rather than Feynman propagators. We also describe a quantum field theory for action-at-a-distance electrodynamics which completes the program initiated by Schwarzschild, Tetrode, and Fokker.[14]

To illustrate the origin of the principal-value propagator we return to the scalar field model of Eq. (103) which described a classical field, $\phi_1(x)$. We introduce an interaction term

$$L_{\text{int}} = -\int d^3z\, \tfrac{1}{2}\lambda^2 [\phi_2(z)]^2 \tag{197}$$

(where λ is a constant), which destroys the purely classical nature of ϕ_1. Suppose we consider the Green's function

$$i\tilde{G}(x,y) = \langle 0 | T(\phi_1(x)\phi_1(y)) | 0\rangle , \tag{198}$$

which would be zero if L_{int} were not present. In terms of in-fields we have

$$i\tilde{G}(x,y) = \left\langle 0 \left| T\left(\phi_{1\text{in}}(x)\phi_{1\text{in}}(y)\exp\left(i\int dt\, L_{\text{int}}\right)\right)\right| 0\right\rangle, \tag{199}$$

where the vacuum states, $|0\rangle$ and $\langle 0|$, are defined as in Eqs. (120) and (122). From the definition of the vacuum we find (dropping "in" labels)

$$i\tilde{G}(x,y) = \frac{-i\lambda^2}{2}\int d^4z \langle 0 | T(\phi_1(x)\phi_1(y)\phi_2{}^2(z)) | 0\rangle , \tag{200}$$

which becomes

$$i\tilde{G}(x,y) = \frac{-i\lambda^2}{2}\epsilon(x_0-y_0)\frac{\partial}{\partial m^2}\Delta(x-y) \tag{201}$$

with

$$\Delta(x-y) = -i\int \frac{d^4k}{(2\pi)^3}\delta(k^2-m^2)\epsilon(k_0)e^{-ik\cdot(x-y)} . \tag{202}$$

Using

$$\tfrac{1}{2}\epsilon(x_0-y_0)\Delta(x-y) = \int \frac{d^4k}{(2\pi)^4}\,\mathrm{P}\frac{1}{k^2-m^2} \times e^{-ik\cdot(x-y)} , \tag{203}$$

we see that

$$\tilde{G}(x,y) = -\lambda^2\int\frac{d^4k}{(2\pi)^4}\,\mathrm{P}\frac{1}{(k^2-m^2)^2}e^{-ik\cdot(x-y)} , \tag{204}$$

where

$$\mathrm{P}\frac{1}{(k^2-m^2)^2} \equiv \frac{1}{2}\left[\frac{1}{(k^2-m^2+i\epsilon)^2}+\frac{1}{(k^2-m^2-i\epsilon)^2}\right]. \tag{205}$$

The form of $\tilde{G}$ is consistent with the equations of motion:

$$(\Box + m^2)\phi_1 + \lambda^2\phi_2 = 0 , \tag{206}$$

$$(\Box + m^2)\phi_2 = \delta^4(x-y) . \tag{207}$$

The appearance of the principal-value dipole propagator rather than the Feynman dipole propagator in Eq. (204) is useful because it eliminates certain unitarity problems associated with indefinite-metric fields. However, depending on the model under consideration, it could lead to difficulties with causality. To illustrate the manner in which unitarity problems are resolved, consider the interaction of the ϕ_1 dipole field with a scalar quantum field ψ with

$$L'_{\text{int}} = g\phi_1(x)[\psi(x)]^2 . \tag{208}$$

Suppose we consider the subset of in and out states containing arbitrary numbers of ψ particles but no ϕ_1 or ϕ_2 particles. These states have positive metric. If one could systematically exclude indefinite-metric ϕ_1 and ϕ_2 particles from physical states one would avoid negative probabilities and other problems. But the sum over states in a unitarity sum would normally include states with ϕ_1 particles if the ϕ_1 field had Feynman propagators. In the case of principal-value propagators, no intermediate states with ϕ_1 particles occur, since the pole term is not present. The interaction mediated by the ϕ_1 field is a form of action at a distance and ϕ_1 is properly described by the phrase adjunct field, coined by Feynman and Wheeler.[14] A more detailed discussion of the unitarity question is given in Refs. 7 and 8. In those articles a dipole gluon model for quark confinement was proposed which introduced principal-value propagators in an *ad hoc* manner to resolve unitarity problems. It was pointed out that causality problems did not necessarily exist in those models because the non-Abelian dipole gluons were confined for the same reason as the quarks so that—at the worst—there would be unobservable causality violations at distances of the order of hadron dimensions.

The pseudoquantization procedure may be used to construct a quantum field-theoretic version of action-at-a-distance electrodynamics. Consider the Lagrangian

$$\mathcal{L} = -\tfrac{1}{2}F^{\mu\nu}(\partial_\nu A_\mu - \partial_\mu A_\nu) + \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\not{\partial} - e\not{A} - m_0)\psi . \tag{209}$$

We define the momentum

$$\Pi_\mu = \frac{\delta \mathcal{L}}{\delta \dot{A}^\mu} = F_{0\mu}. \tag{210}$$

Going to the transverse gauge as in Sec. IV, we define the equal-time commutation relation

$$[\Pi_i(\vec{x}, t), A_j(\vec{y}, t)] = i\delta_{ij}^{tr}(\vec{x} - \vec{y}). \tag{211}$$

Suppose we neglect interaction terms in $\mathcal{L}$ for the moment and choose $F_{\mu\nu}$ to be an observable classical field (as it is up to quantum corrections which we neglect) and A_μ to be unobservable (as it is because it is not gauge invariant). Then we follow our pseudoquantization procedure for

$$\mathcal{L}_0 = -\tfrac{1}{2} F^{\mu\nu}(\partial_\nu A_\mu - \partial_\mu A_\nu) + \tfrac{1}{4} F^{\mu\nu} F_{\mu\nu}. \tag{212}$$

In particular, we define a vacuum such that

$$F_{\mu\nu}|0\rangle = 0, \quad A_\mu|0\rangle \neq 0, \tag{213}$$

while

$$\langle 0|A_\mu = 0, \quad \langle 0|F_{\mu\nu} \neq 0. \tag{214}$$

Then

$$iG_{\mu\nu}(x, y) = \langle 0|T(A_\mu(x)A_\nu(y))|0\rangle \tag{215}$$

would be zero were it not for $F_{\mu\nu}F^{\mu\nu}$ in $\mathcal{L}_0$. In terms of appropriate in-fields it becomes

$$2iG_{\mu\nu}(x, y) = \int d^4z\,(\theta(x_0 - y_0)\theta(y_0 - z_0) + \theta(y_0 - x_0)\theta(x_0 - z_0)) \times [A_{\mu\,\text{in}}(x), F_{\alpha\beta\,\text{in}}(z)][A_{\mu\,\text{in}}(y), F^{\alpha\beta}_{\text{in}}(z)]. \tag{216}$$

Note that we are treating $F_{\mu\nu}F^{\mu\nu}$ in $\mathcal{L}_0$ as an interaction term. The structure of $G_{\mu\nu}(x, y)$ is the same as that of Eq. (200) so we can conclude that

$$G_{\mu\nu}(x, y) = -g_{\mu\nu} \int \frac{d^4k}{(2\pi)^4}\, \mathrm{P}\,\frac{1}{k^2}\, e^{-ik\cdot(x-y)} \tag{217}$$

in the Feynman gauge. Thus the action-at-a-distance interaction follows from the pseudoquantization of electrodynamics. The classical character of $F_{\mu\nu}$ is lost owing to quantum corrections resulting from the presence of $J_\mu A^\mu$ in the Lagrangian.

The example we have just studied has a certain parallel in the vierbein model of gravitation studied in the last section. The forms of the Lagrangian and commutation relations are similar. As a result it is clear that

$$D^{\mu\nu,\lambda\sigma}(x, y) \equiv \left\langle L \left| T\left(l^{\mu\nu}_{L\,\text{in}}(x) l^{\lambda\sigma}_{L\,\text{in}}(y) \int d^4z\, \tilde{\mathcal{L}}_{\text{int}}(z)\right) \right| L \right\rangle \tag{218}$$

with

$$\tilde{\mathcal{L}}_{\text{int}} = \tfrac{1}{4}\, g\,\mathrm{Tr}\,[B_{\mu\,\text{in}}, B_{\nu\,\text{in}}]\gamma^\mu\gamma^\nu \tag{219}$$

is a principal-value propagator. Therefore we have constructed an action-at-a-distance version of quantum gravity. Our motivation was to take account of the classical part of $l^{\mu a}$ in a way which did not divorce it from the quantum part to which it is intimately related.

VI. CONCLUSION

We have seen that an alternative to Fock-space quantization exists for a class of field theories which have Lagrangian gradient terms which are linear in field variables. A method was also proposed for constructing Lagrangians of that type from classical Lagrangians with gradient terms which are quadratic in field variables. To some extent this process has a parallel in the passage from Klein-Gordon field Lagrangians which are quadratic in derivatives to Dirac field Lagrangians which are linear in derivatives.

The quantization procedure we have outlined is canonical so far as the fields are concerned. We do, however, make a choice of vacuum states which differs from the usual choice. As a result we have found free propagators which were either retarded, or half-advanced and half-retarded. The choice of vacuum state does not in itself preclude the appearance of Feynman propagators. If one has a good reason to modify the canonical commutation relations then it is possible to obtain Feynman propagators.[15] The procedure we have outlined has, therefore, a greater generality than the particular class of models studied in the present work. It can enable one to embed a classical field theory in a quantum field theory in such a way as to maintain its classical character. It can also be applied to study classical field theories which obtain quantum corrections. Finally it can be applied in order to obtain a fully second-quantized field theory (cf. Ref. 15).

ACKNOWLEDGMENT

This work was supported in part by the U.S. Energy Research and Development Administration.

*Present address: Physics Department, Williams College, Williamstown, Mass. 01267.

[1]D. R. Yennie, S. C. Frautschi, and H. Suura, Ann. Phys. (N.Y.) 13, 379 (1961).
[2]R. J. Glauber, Phys. Rev. 131, 2766 (1963).
[3]W. A. Bardeen, M. S. Chanowitz, S. D. Drell, M. Weinstein, and T.-M. Yan, Phys. Rev. D 11, 1094 (1975).

[4]J. M. Cornwall and G. Tiktopoulos, Phys. Rev. D 13, 3370 (1976).
[5]S. Blaha, Phys. Lett. 56B, 373 (1975).
[6]E. C. G. Sudarshan, Center for Particle Theory report Univ. of Texas—Austin, 1976 (unpublished).
[7]S. Blaha, Phys. Rev. D 10, 4268 (1974).
[8]S. Blaha, Phys. Rev. D 11, 2921 (1975).
[9]S. Blaha, Lett. Nuovo Cimento 18, 60 (1977).
[10]Cf. Ref. 2; T. W. B. Kibble, J. Math. Phys. 9, 315 (1968); Phys. Rev. 173, 1527 (1968); 174, 1882 (1968); 175, 1624 (1968);
[11]A. Salam, lecture at Center for Theoretical Studies, Miami, Florida, 1973 (unpublished).
[12]G. Mie, Ann. Phys. (Leipzig) 37, 511 (1912); 39, 1 (1912); 40, 1 (1913); H. Weyl, *Space, Time, Matter* (Dover, N.Y. 1952).
[13]H. Weyl, Z. Phys. 56, 330 (1929); T. W. B. Kibble, J. Math. Phys. 2, 212 (1961); J. Schwinger, Phys. Rev. 130, 1253 (1963); C. J. Isham, A. Salam, and J. Strathdee, Lett. Nuovo Cimento 5, 969 (1972); F. W. Hehl, P. von der Heyde, G. D. Kerlick, and J. Nester, Rev. Mod. Phys. 48, 393 (1976); and references therein.
[14]K. Schwarzschild, Göttinger Nachrichten 128, 132 (1903); H. Tetrode, Z. Phys. 10, 317 (1922); A. D. Fokker, *ibid.* 58, 386 (1929); J. Wheeler and R. P. Feynman, Rev. Mod. Phys. 17, 157 (1945); 21, 425 (1949).
[15]S. Blaha (unpublished).