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Generalization of the Lagrange equilateral-triangle solution and the Euler collinear solution to nongravitational forces in the three-body problem

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In the gravitational three-body problem there are special configurations that allow exact solutions. They are the equilateral-triangle and collinear configurations. In this paper I show that exact solutions for these special configurations and others as well exist for forces other than gravitational. The interactions include "scalar" forces or those derived from a world scalar potential as well as Coulomb or "vector" forces. In the formalism used for this classical treatment the aspects emphasized are those which can be carried over to quantum mechanics. An application to an atomic system yields a reasonable estimate of the energy levels for helium-like atoms.

INTRODUCTION

The simplest known solutions to the three-body problem are the Lagrange equilateral triangle and the Euler collinear configurations. These solutions require very special initial conditions. Nevertheless, there is evidence in our solar system for the Lagrange solution. This is the equilateral triangle consisting of the sun, Jupiter, and a cluster of asteroids sharing Jupiter's orbit known as the Trojan asteroids.[1] The aim of this paper is to explore the possibility that such special configurations may exist for types of forces other than gravitational. This paper deals primarily with the classical aspects of the problem. In particular, I demonstrate exact equilateral-triangle and general triangle solutions for forces that can be derived from a world scalar potential. These forces are like gravity except that they are mass independent. For short, I call these "scalar" forces. I also show that these forces as well as Coulomb or "vector" forces allow exact collinear solutions. The existence of these "classical" solutions is demonstrated in this paper. Their possible generalizations to special quantum mechanical configurations are also discussed.

This paper is to be regarded as providing the classical starting point of a quantum-mechanical generalization. To this end I express the characteristics of these classical solutions in terms of the energies of the fictitious particles of relative motion. This, together with separability of the equations of motion, provides criteria on which to restrict quantum solutions rather than the nonapplicable one of a special orbital configuration.

The coordinates used in this paper are the relative coordinates of the three particles. This allows a symmetric reduction of the three-body problem. There have been symmetrical reductions of the three-body problem in the past.[2] As a general rule the only interaction considered in these reductions is the gravitational interaction. A recent example is given in Ref. 3. The symmetric reduction of the equations of motion for an arbitrary potential is given in Sec. I. Section II discusses the Hamiltonian formulation of the three-body problem in terms of these relative variables. Part of the presentation given there is based on unpublished notes on the subject by Arenstorf.[4] As with the treatment of the gravitational potential in Refs. 2 and 3 the equations of motion contain the center-of-mass (c.m.) restriction $\ddot{\vec{R}}=0=\dot{\vec{R}}=\vec{R}$ and yet is symmetrical in the three variables.

In Sec. III, it is shown how the use of the relative coordinates facilitates a derivation of the equilateral-triangle solution for the gravitational forces.[3] An adaption of this technique allows a derivation of the conditions for the collinear solution for the gravitational force. This is presented in Sec. IV. Section V generalizes the results of Sec. III to "scalar" forces other than gravitational and Sec. VI generalizes the adaptive techniques of Sec. IV to these scalar forces as well as the Coulomb force. The summary includes an adaptation of these solutions to a quantum system.

I. EQUATIONS OF MOTION

The Hamiltonian for a system of three mutually interacting point particles is

$$H=T+V=\sum_{i=1}^{3}\frac{\vec{p}_i^{\,2}}{2m_i}+\frac{1}{2}\sum_{\substack{i,j\\ i\neq j}}\phi_{ij}(|\vec{r}_i-\vec{r}_j|). \tag{1}$$

The forces arise from action at a distance and are assumed to be velocity independent. Hamilton's equation leads to

$$m_i \ddot{\vec{r}}_i = -\vec{\nabla}_i\phi_{ij}(|\vec{r}_i-\vec{r}_j|) - \vec{\nabla}_i\phi_{ik}(|\vec{r}_i-\vec{r}_k|)$$
$$= -\frac{\vec{r}_i-\vec{r}_j}{|\vec{r}_i-\vec{r}_j|}\phi'_{ij}(|\vec{r}_i-\vec{r}_j|) - \frac{\vec{r}_i-\vec{r}_k}{|\vec{r}_i-\vec{r}_k|}\phi'_{ik}(|\vec{r}_i-\vec{r}_k|), \quad i,j,k \text{ cyclic}, \tag{2}$$

where ϕ' is the derivative of ϕ with respect to its argument. As a first integral of these equations of motion,

$$M\dot{\vec{R}} = \sum_{i=1}^{3} m_i \dot{\vec{r}}_i = \text{const}, \tag{3}$$

where $M = m_1 + m_2 + m_3$. The constant can be chosen to be zero and the position $\vec{R}$ of the center of mass can be taken as the origin.

The constraint $\vec{R}=0$ can be used to eliminate one of the coordinates, but this leads to unsymmetrical equations of motion. This constraint, or in general the c.m. condition $\ddot{\vec{R}}=0$, can be imposed in a symmetrical way be introducing[2,3]

$$\vec{u}_i = \vec{r}_j - \vec{r}_k, \quad i,j,k \text{ cyclic} . \tag{4}$$

This can be written as

$$\begin{bmatrix}\vec{u}_1\\ \vec{u}_2\\ \vec{u}_3\end{bmatrix} = \begin{bmatrix}0 & 0 & -1\\ -1 & 0 & 1\\ 1 & -1 & 0\end{bmatrix}\begin{bmatrix}\vec{r}_1\\ \vec{r}_2\\ \vec{r}_3\end{bmatrix}. \tag{5}$$

The matrix relating the two sets of variables is singular. This is because the variables $\vec{u}_i$ are not independent:

$$\vec{u}_1 + \vec{u}_2 + \vec{u}_3 = 0. \tag{6}$$

(The $\vec{r}$ variables are not independent either but this is not by definition. It follows from the equation of motion.)

Now (5) cannot be inverted directly because the matrix is singular. However, one has the identity

$$\vec{r}_i = \frac{m_j\vec{u}_k - m_R\vec{u}_j}{M} + \vec{R} . \tag{7}$$

This identity serves as an inverse of (5). With $\vec{R}=0$, the "inverse" of (5) is therefore

$$\begin{bmatrix}\vec{r}_1\\ \vec{r}_2\\ \vec{r}_3\end{bmatrix} = \frac{1}{M}\begin{bmatrix}0 & -m_3 & m_2\\ m_3 & 0 & -m_1\\ -m_2 & m_1 & 0\end{bmatrix}\begin{bmatrix}\vec{u}_1\\ \vec{u}_2\\ \vec{u}_3\end{bmatrix}. \tag{8}$$

This matrix also has a zero determinant. Combining (5) and (8) leads to

$$M\begin{bmatrix}\vec{u}_1\\ \vec{u}_2\\ \vec{u}_3\end{bmatrix} = \begin{bmatrix}m_2+m_3 & -m_1 & -m_1\\ -m_2 & m_3+m_1 & -m_2\\ -m_3 & -m_3 & m_1+m_2\end{bmatrix}\begin{bmatrix}\vec{u}_1\\ \vec{u}_2\\ \vec{u}_3\end{bmatrix}. \tag{9}$$

The matrix is of course not a unit matrix. This equation is simply a restatement of (6).

The equations of motion (2) can be rewritten using the new variables by way of (8):

$$\ddot{\vec{u}}_i + \vec{\nabla}_i \frac{\phi_{jk}(u_i)M}{m_j m_k} = m_i\vec{Z}, \quad i,j,k \text{ in cyclic order}, \tag{10}$$

where $\vec{\nabla}_i = \vec{\nabla}_{u_i}$ and

$$\vec{Z} = \frac{\vec{\nabla}_1\phi_{23}(u_1)}{m_2 m_3} + \frac{\vec{\nabla}_2\phi_{31}(u_2)}{m_3 m_1} + \frac{\vec{\nabla}_3\phi_{12}(u_3)}{m_1 m_2}. \tag{11}$$

These equations of motion contain the c.m. restriction $\ddot{\vec{R}}=0$ and are symmetrical in the three variables. They are also independent of the nature of the potential.

II. SYMMETRIC REDUCTION OF THE THREE-BODY PROBLEM IN THE HAMILTONIAN FORMULATION

If the Hamiltonian (1) is written in terms of the variables $\vec{u}_i$ and $\dot{\vec{u}}_i$ it has the form

$$H = T + V = \frac{1}{2}\,\frac{m_1 m_2 m_3}{M}\sum_{i=1}^{3}\frac{\dot{\vec{u}}_i^{\,2}}{m_i} + V. \tag{12}$$

This form contains the c.m. restriction $\dot{\vec{R}}=0$ [viz., (8)] as well as the restriction $\sum_{i=1}^{3}\dot{\vec{u}}_i=0$. Although the variables $\vec{u}_i$ are not independent, they can be treated as independent if a Lagrange multiplier of the form $\vec{\lambda}\cdot(\vec{u}_1+\vec{u}_2+\vec{u}_3)$ is added to this Hamiltonian (12). The equations of motion (10) then follow from Hamilton's equation[3] in terms of the canonically conjugate variables $\vec{u}_i$ and $\vec{\mathcal{P}}_i = m_j m_k \dot{\vec{u}}_i/M$.

As an alternative to the Lagrange multiplier approach of incorporating the constraint $\dot{\vec{u}}_1+\dot{\vec{u}}_2+\dot{\vec{u}}_3=0$, one can construct a Hamiltonian that has the following properties:

(i) It is symmetrical in the relative variables $[\vec{u}_1, \vec{u}_2, \vec{u}_3]$.

(ii) It yields the constraint $\dot{\vec{u}}_1+\dot{\vec{u}}_2+\dot{\vec{u}}_3=0$ as a consequence of the equations of motion

(iii) It is the Hamiltonian in the c.m. frame.

An outline of how the Hamiltonian can be obtained from the Hamiltonian (1) is given in the Appendix.

There it is shown that

$$H=\frac{(\vec{P}_1-\vec{P}_2)^2}{2m_3}+\frac{(\vec{P}_2-\vec{P}_3)^2}{2m_1}+\frac{(\vec{P}_3-\vec{P}_1)^2}{2m_2}+V(u_1,u_2,u_3). \tag{13}$$

The variables $(\vec{P}_i, \vec{u}_1)$ are independent canonically conjugate variables. That is, they satisfy the fundamental Poisson-bracket relations[5]

$$\{\vec{P}_i, \vec{P}_j\}=0=\{\vec{u}_i,\vec{u}_j\}, \quad \{(\vec{u}_i)_k, (\vec{P}_j)_l\}=\delta_{ij}\,\delta_{kl}. \tag{14}$$

Regarding these variables as independent canonical variables is not, as it first appears, contradicted by the relation $\vec{u}_1+\vec{u}_2+\vec{u}_3=0$. The Hamiltonian (13) is derived without reference to this constraint. As mentioned above, however, it does imply the constraint $\vec{u}_1+\vec{u}_2+\vec{u}_3=\vec{c}_1$, a constant vector. To see this consider Hamilton's equation in terms of these canonical variables. From the first set of equations

$$\vec{\nabla}_{\vec{P}} H=\dot{\vec{u}}_i \tag{15}$$

follows the vector equations

$$\begin{bmatrix}\dot{\vec{u}}_1\\ \dot{\vec{u}}_2\\ \dot{\vec{u}}_3\end{bmatrix}=\begin{bmatrix}\frac{1}{m_3} & 0 & -\frac{1}{m_2}\\ -\frac{1}{m_3} & \frac{1}{m_1} & 0\\ 0 & -\frac{1}{m_1} & \frac{1}{m_2}\end{bmatrix}\begin{bmatrix}\vec{P}_1-\vec{P}_2\\ \vec{P}_2-\vec{P}_3\\ \vec{P}_3-\vec{P}_1\end{bmatrix}. \tag{16}$$

From this follows $\dot{\vec{u}}_1+\dot{\vec{u}}_2+\dot{\vec{u}}_3=0$. *Hence the constraint $\vec{u}_1+\vec{u}_2+\vec{u}_3=\vec{c}_1$ is a first integral of the equation of motion. It does not need to be imposed as an extra condition on (13).* This is an analogous to the situation for the original Hamiltonian (1) with respect to the c.m. constraint. It is not imposed on the Hamiltonian (1). It follows as a first integral of the equation of motion. The initial conditions of the three-body problem lead to $\vec{c}_1=0$. Hereafter the condition $\vec{u}_1+\vec{u}_2+\vec{u}_3=\vec{c}_1=0$ will be referred to as the triangle constraint.

The second set of Hamilton's equations

$$\vec{\nabla}_i H=-\dot{\vec{P}}_i \tag{17}$$

when combined with (16) leads to the equations of motion (10).

III. LAGRANGE'S EQUILATERAL-TRIANGLE SOLUTION FOR GRAVITATIONAL INTERACTIONS

For gravitational forces, the potentials are

$$\phi_{ij}(u_k)=-\frac{\kappa\, m_i m_j}{|\vec{u}_k|}, \quad i,j,k \text{ cyclic}. \tag{18}$$

The equations of motion (10) are

$$\ddot{\vec{u}}_i+\frac{M\kappa\vec{u}_i}{|\vec{u}_i|^3}=m_i\vec{Z}, \quad i=1,2,3 \tag{19}$$

with

$$\vec{Z}=\kappa\left(\frac{\vec{u}_1}{|\vec{u}_1|^3}+\frac{\vec{u}_2}{|\vec{u}_2|^3}+\frac{\vec{u}_3}{|\vec{u}_3|^2}\right). \tag{20}$$

Now suppose $|\vec{u}_1|=|\vec{u}_2|=|\vec{u}_3|\equiv|\vec{u}|$ (hereafter called the equilateral condition). Then

$$\vec{Z}=\frac{\kappa}{|\vec{u}|^3}(\vec{u}_1+\vec{u}_2+\vec{u}_3)=0 \tag{21}$$

and Eq. (19) separates into three two-body problems. The particles revolve about the center of an equilateral triangle. The relative motion that the three sets of two-body equations indicate can either be identical circles, ellipses, or parabolas and hyperbolas for unbounded motion. For bounded motion, the periodic motion that results will be described as a rotating equilateral triangle with either fixed sides or periodically varying sides. These are Lagrange's equilateral-triangle solutions. This condition, $\vec{Z}=0$ is synonymous with the Lagrange solution for the gravitational interactions.

Equation (19), or more generally Eq. (10), follows by applying Hamilton's equation to the Hamiltonian (13). The triangle constraint follows from the first of Hamilton's equations. The separability of these equations (19) is imposed by requiring the equilateral condition. These same separable equations can be obtained by applying Hamilton's equation to the following separable Hamiltonian in which the $\vec{u}_i$ are regarded as independent:

$$H=\frac{\vec{P}_1^{\,2}}{2\mu_1}+\frac{\vec{P}_2^{\,2}}{2\mu_2}+\frac{\vec{P}_3^{\,2}}{2\mu_3}-\frac{\kappa m_1 m_2}{|\vec{u}_3|}-\frac{\kappa m_2 m_3}{|\vec{u}_1|}-\frac{\kappa m_3 m_1}{|\vec{u}_2|}, \tag{22}$$

where $\mu_i=m_j m_k/M$ $(i,j,k$ cyclic). Of course, the triangle constraint does not follow from (22) as it does from (13); it must be imposed. Whereas the separability of the equations of motion (19) that follow from (13) is imposed by requiring the equilateral condition and combining this with the derived triangle constraint, the separability of the equations of motion that follow from (22) is automatic. The triangle constraint as well as the equilateral condition is imposed on the solutions. Of course, these are the only conditions under which (22) yields the correct equation of motion.

The Hamiltonian (22) cannot be derived from (13) except by integrating (19) for $i=1,2,3$ under the $\vec{Z}=0$ condition and then adding the results. In that sense of the word it is a rewriting of the Hamiltonian (13). One can gain further insight into the relation between (13) and (22) by considering the derivation of (13) given in the Appendix. In that derivation, the triangle constraint is not imposed and as a consequence the resultant canonical momenta $\vec{P}_i$ are independent variables. If in the derivation, however, the constraint is imposed, then one finds $\vec{P}_i = m_j m_k \vec{u}_i / M$ [(A4) in the Appendix]. These are not independent canonical variables in view of the triangle constraint. If one nevertheless treats them as independent then the resultant Hamiltonian will be (22). As should be expected, the equations of motion (19) do not follow from (22), unless the triangle constraint is reinstated by way of a Lagrange multiplier. The exception to this is the Lagrange solution $\vec{Z}=0$.

The utility of separable Hamiltonians such as (22) lies in their possible applications for approximate solutions to the quantum three-body problem. Separable Hamiltonians would lead to separable Schrödinger equations. There remains the problem of how to impose the $\vec{Z}=0$ solution on the resultant eigenvalue spectrum. For example, another way to view the $\vec{Z}=0$ restriction is to look at the fictitious individual energies. Let $E=E_1+E_2+E_3$ be the total energy of the three-body bound systems. The individual energies must be restricted so that classically the three individual periods are the same.

The period of the ith particle is given by

$$T_i = \pi\kappa m_j m_k \left(\frac{\mu_i}{2E_i^{\,3}}\right)^{1/2} . \tag{23}$$

Hence equality of the periods requires

$$E_1 : E_2 : E_3 = m_2 m_3 : m_3 m_1 : m_1 m_2 . \tag{24}$$

Such energy considerations are of importance in discussions of the quantum analogs of the Lagrange solution for other types of interactions.

IV. EULER'S COLLINEAR SOLUTION FOR GRAVITATIONAL INTERACTIONS

The Euler collinear solution exists if one can find a λ and ρ such that $\vec{u}_1 = -\lambda\vec{u}_3$, $\vec{u}_2 = -\rho\vec{u}_3$ where $\lambda+\rho=1$. The $\vec{Z}=0$ condition characteristic of the Lagrange equilateral triangle is not allowed for the Euler solutions:

$$\vec{Z} = \frac{\vec{u}_3}{|\vec{u}_3|^3}\left(-\frac{1}{\lambda^2}-\frac{1}{\rho^2}+1\right) \neq 0 . \tag{25}$$

The reason is that

$$\eta(\lambda) \equiv \frac{1}{\lambda^2} + \frac{1}{(1-\lambda)^2} > 1, \quad 0<\lambda<1 \tag{26}$$

has a minimum value of 8 at $\lambda=\frac{1}{2}$.

To obtain the Euler solution, the equations of motion (10) are modified by adding $m_i\kappa(\eta-1)\vec{u}_3/|\vec{u}_3|^3$ to each side. This changes $\vec{Z}$ into

$$\vec{Z}' = \kappa\left(\frac{\vec{u}_1}{|\vec{u}_1|^3} + \frac{\vec{u}_2}{|\vec{u}_2|^3} + \frac{\eta\vec{u}_3}{|\vec{u}_3|^3}\right) \tag{27}$$

and (10) into

$$\ddot{\vec{u}}_i + \kappa\left[\frac{M\vec{u}_i}{|\vec{u}_i|^3} + \frac{m_i(\eta-1)\vec{u}_3}{|\vec{u}_3|^3}\right] = m_i\vec{Z}' , \quad i=1,2,3 . \tag{28}$$

The value of η that gives rise to the Euler solution ($\vec{u}_1=-\lambda\vec{u}_3$, $\vec{u}_2=-\rho\vec{u}_3$, $\lambda+\rho=1$) is defined in (26). The variable λ satisfies a fifth-order equation first noted by Euler. As this equation will appear in a later section for other types of interactions, a derivation of it is given here. Use $\vec{u}_1=-\lambda\vec{u}_3$ and (28) for $i=1,3$ ($\vec{Z}'=0$). This leads to

$$\eta = 1 + \frac{M}{m_1+m_3\lambda}\left(\frac{1}{\lambda^2}-\lambda\right) . \tag{29}$$

Using $\vec{u}_2=-\rho\vec{u}_3$ and (28) for $i=2,3$ ($\vec{Z}'=0$) leads to

$$\eta = 1 + \frac{M}{m_2+m_3\rho}\left(\frac{1}{\rho^2}-\rho\right) . \tag{30}$$

Combining (29) and (30) gives

$$\left(\lambda+\frac{m_1}{m_3}\right)\left(\frac{1}{\rho^2}+\frac{m_2}{m_3}\right) - \left(\rho+\frac{m_2}{m_3}\right)\left(\frac{1}{\lambda^2}+\frac{m_1}{m_3}\right) = 0 , \tag{31}$$

an equation of the fifth degree in λ.

As an example, consider the case $m_1=m_2=m$. An obvious solution to (30) is $\lambda=\rho=\frac{1}{2}$. The common value of η predicted by (26), (29), and (30) is 8.

In the general case, the equations of motion can be rewritten as

$$\begin{aligned}
&\ddot{\vec{u}}_1 + \kappa M\left[1+\frac{m_1}{m_1+m_3\lambda}(1-\lambda^3)\right]\frac{\vec{u}_1}{|\vec{u}_1|^3} = 0 ,\\
&\ddot{\vec{u}}_2 + \kappa M\left[1+\frac{m_2}{m_2+m_3\rho}(1-\rho^3)\right]\frac{\vec{u}_2}{|\vec{u}_2|^3} = 0 ,\\
&\ddot{\vec{u}}_3 + \frac{\kappa M}{2}\left[2+\frac{m_3}{m_1+m_3\lambda}\left(\frac{1}{\lambda^2}-\lambda\right) + \frac{m_3}{m_2+m_3\rho}\left(\frac{1}{\rho^2}-\rho\right)\right]\frac{\vec{u}_3}{|\vec{u}_3|^3} = 0 ,
\end{aligned} \tag{32}$$

where $\rho=1-\lambda$ and λ is the solution to (31) that lies between 0 and 1. As λ varies from 0 to 1, (31) varies monotonically from $+\infty$ to $-\infty$. Hence, regardless of the masses, there is only one λ that satisfies (31) for $0<\lambda<1$.

Of particular interest are those aspects of the Euler solution that can be compared with more general interactions and their quantum analogs.

One such aspect is the form of separable Hamiltonian analogous to (22). It is[6]

$$\frac{\vec{P}_1^{\,2}}{2\mu_1}+\frac{\vec{P}_2^{\,2}}{2\mu_2}+\frac{\vec{P}_3^{\,2}}{2\mu_3}-\frac{\kappa_1 m_2 m_3}{|\vec{u}_1|}-\frac{\kappa_2 m_3 m_1}{|\vec{u}_2|}-\frac{\kappa_3 m_1 m_1}{|\vec{u}_3|}. \tag{33}$$

The constants, κ_1, κ_2, and κ_3 differ from κ by the respective factors in brackets in the three equations of (32). The condition on the individual energies E_1, E_2, E_3 for the Euler solution analogous to (24) for the Lagrange solution is

$$E_1:E_2:E_3=(\kappa_1)^{1/3}m_2m_3:(\kappa_2)^{1/3}m_3m_1:(\kappa_3)^{1/3}m_1m_2. \tag{34}$$

V. GENERALIZATION OF THE LAGRANGE SOLUTION TO OTHER INTERACTIONS

Are special solutions similar to the Lagrange equilateral-triangle solution possible for other than gravitational potentials? In the general case

$$\vec{Z}=\frac{\vec{u}_1}{|\vec{u}_1|}\frac{\phi'_{23}(u_1)}{m_2m_3}+\frac{\vec{u}_2}{|\vec{u}_2|}\frac{\phi'_{31}(u_2)}{m_3m_1}+\frac{\vec{u}_3}{|\vec{u}_3|}\frac{\phi'_{12}(u_3)}{m_1m_2}. \tag{35}$$

The first type of potentials gives rise to purely mutually attractive forces among the bodies, called "scalar" forces. Examples include forces derivable from a world scalar potential such as nuclear forces and model forces derived from a Lorentz scalar Coulomb potential. The second category of potentials includes the real Coulomb potential and other "vector" forces which allows only two of the three interparticle potentials to be attractive.

Consider the first type of force. Assume that

$$\phi'_{ij}(u_k)=\alpha_{ij}\phi'(u_k),\quad i,j,k \text{ cyclic}. \tag{36}$$

Consider the condition under which a generalization of the Lagrange equilateral-triangle condition is satisfied. If $|\vec{u}_1|=|\vec{u}_2|=|\vec{u}_3|\equiv|\vec{u}|$, then

$$\vec{Z}=\left(\frac{\vec{u}_1\alpha_{23}}{m_2m_3}+\frac{\vec{u}_2\alpha_{31}}{m_3m_1}+\frac{\vec{u}_3\alpha_{12}}{m_1m_2}\right)\frac{\phi'(u)}{|\vec{u}|}. \tag{37}$$

An equilateral-triangle solution ($\vec{Z}=0$) is possible if

(1) $\alpha_{12}m_3=\alpha_{23}m_1=\alpha_{31}m_2$ and

(2) the individual equations of motion

$$\ddot{\vec{u}}_i+\frac{\vec{u}_i}{|\vec{u}_i|}\frac{M\alpha_{jk}}{m_jm_k}\phi'(\vec{u}_i)=0,\quad i,j,k \text{ cyclic}$$

can simultaneously retain $|\vec{u}_1|=|\vec{u}_2|=|\vec{u}_3|$ and $\vec{u}_1+\vec{u}_2+\vec{u}_3=0$. For $\phi=-1/u$ and $\alpha_{12}=\alpha_{23}=\alpha_{31}\equiv\alpha$, only the equal-mass solution exists. The appropriate Hamiltonian from which the above equation of motion can be derived is[6]

$$\frac{\vec{P}_1^{\,2}}{2\mu_1}+\frac{\vec{P}_2^{\,2}}{2\mu_2}+\frac{\vec{P}_3^{\,2}}{2\mu_3}-\frac{\alpha}{|\vec{u}_1|}-\frac{\alpha}{|\vec{u}_2|}-\frac{\alpha}{|\vec{u}_3|}, \tag{38}$$

where

$$\mu_1=\mu_2=\mu_3=m/3, \tag{39}$$

and the condition on the energies for the equilateral-triangle solution is

$$E_1:E_2:E_3=1:1:1. \tag{40}$$

For more general potentials such as

$$\phi=-\frac{e^{-\beta u}}{u},\quad \phi=-1/u+\beta/u^2, \tag{41}$$

the same condition on the individual energies would be expected because of the symmetry involved.

Next consider the case of a general triangular solution $\vec{u}_1+\vec{u}_2+\vec{u}_3=0$, $|\vec{u}_1|=\lambda|\vec{u}_3|$, $|\vec{u}_2|=\rho|\vec{u}_3|$. One has $\vec{Z}=0$ under these conditions if

$$\frac{\phi'(u\lambda)}{\lambda\phi'(u)}\frac{\alpha_{23}}{\alpha_{12}}\frac{m_1}{m_3}=\frac{\phi'(u\rho)}{\rho\phi'(u)}\frac{\alpha_{31}}{\alpha_{12}}\frac{m_2}{m_3}=1. \tag{42}$$

To this must be added the triangle inequality $\lambda+\rho\geq 1$. For homogeneous potentials such as $\phi=-1/u$ this is an easy algebraic equation. For example, if $\alpha_{12}=\alpha_{23}=\alpha_{13}$,

$$\lambda=\left(\frac{m_1}{m_3}\right)^{1/3},\quad \rho=\left(\frac{m_2}{m_3}\right)^{1/3}, \tag{43}$$

and we must have

$$\left(\frac{m_1}{m_3}\right)^{1/3}+\left(\frac{m_2}{m_3}\right)^{1/3}\geq 1. \tag{44}$$

For nonhomogeneous potentials such as those in (41) the equations are more complicated. For example, if $\phi=-1/u+\beta/u^2$ and $\alpha_{12}=\alpha_{23}=\alpha_{31}\equiv\alpha$, then

$$(u-2\beta)=\frac{1}{\lambda^4}(\lambda u-2\beta)\frac{m_1}{m_3}=\frac{1}{\rho^4}(\rho u-2\beta)\frac{m_2}{m_3}. \tag{45}$$

Combining these two equations yields

$$u=\frac{2\beta}{\lambda\rho m_3}\left(\frac{m_2\lambda^4-m_1\rho^4}{\lambda^3-\rho^3}\right). \tag{46}$$

Now since $u>0$, this implies that either

$$\lambda>\rho \text{ and } \left(\frac{\lambda}{\rho}\right)^4>\frac{m_1}{m_2} \tag{47a}$$

or

$$\rho>\lambda \text{ and } \left(\frac{\rho}{\lambda}\right)^4>\frac{m_2}{m_1}. \tag{47b}$$

Thus, if $m_1/m_2>1$ then (47a) holds, and if m_2m_1

>1 then (47b) holds. Beyond this, however, (46) implies that the actual scale or size of the triangle is not an arbitrary quantity. The scale is set by the parameter β. This contrasts with the homogeneous potentials which do not have any length scale and therefore do not lead to any restriction on the size of the triangle when $\vec{Z}=0$.

If (46) is substituted back into (45), the two simultaneous equations for $m_1/m_3 \equiv k_1$ and $m_2/m_3 \equiv k_2$ do not depend on β. The two equations are

$$\begin{aligned} &\rho^4 k_1{}^2 + (\rho\lambda^3 - k_2\lambda^4)k_1 + (k_2\lambda^4 - \lambda^4\rho - \lambda\rho^4)\lambda^3 = 0\,,\\ &\lambda^4 k_2{}^2 + (\lambda\rho^3 - k_1\rho^4)k_2 + (k_1\rho^4 - \lambda\rho^4 - \rho\lambda^4)\rho^3 = 0\,, \end{aligned} \tag{48}$$

and their solutions determine the mass ratios (k_1, k_2) as a function of the triangle shape (λ, ρ).

For electrical or vector forces, the system will not be bounded if all three charges have the same sign. If one has a charge opposite that of the other two, then two of the potentials will be attractive and the other will be repulsive.

For the Coulomb potential

$$\phi_{ij}(u_k) = \frac{e_i e_j}{|\vec{u}_k|}\,, \quad i,j,k \text{ cyclic} \tag{49}$$

and

$$\vec{Z} = -\frac{e_2 e_3}{m_2 m_3}\frac{\vec{u}_1}{|\vec{u}_1|^3} - \frac{e_3 e_1}{m_3 m_1}\frac{\vec{u}_2}{|\vec{u}_2|^3} - \frac{e_1 e_2}{m_1 m_2}\frac{\vec{u}_3}{|\vec{u}_3|^3}\,. \tag{50}$$

Let e_3 be opposite in sign from e_1 and e_2 and define

$$e_3 = -z_1 e_1 = -z_2 e_2\,, \quad m_3 = k_1 m_1 = k_2 m_2\,, \tag{51}$$

where z_1, z_2, k_1, and k_2 are positive constants. Then

$$\vec{Z} = \frac{e_1 e_2}{m_1 m_2}\left(\frac{z_1}{k_1}\frac{\vec{u}_1}{|\vec{u}_1|^3} + \frac{z_2}{k_2}\frac{\vec{u}_2}{|\vec{u}_2|^3} - \frac{\vec{u}_3}{|\vec{u}_3|^3}\right). \tag{52}$$

This Z cannot vanish if $|\vec{u}_1| = |\vec{u}_2| = |\vec{u}_3|$ or for any other triangular configuration. Collinear or Euler configurations are possible as shall be demonstrated in the next section.

VI. GENERALIZATION OF THE EULER SOLUTION TO OTHER INTERACTIONS

Special solutions similar to the Euler collinear solution are possible for interactions other than gravitational. For potentials of the form of (36) (i.e., nuclear forces or scalar potentials such as the scalar Coulomb potential), the Z of (37) vanishes for $\vec{u}_1 = -\lambda\vec{u}_3$, $\vec{u}_2 = -\rho\vec{u}_3$, $\lambda+\rho=1$ if

$$\frac{\alpha_{12}}{m_1 m_2} = \frac{\lambda\alpha_{23}}{m_2 m_3}\frac{\phi'(\lambda u_3)}{\phi'(u_3)} + \frac{\rho\alpha_{31}}{m_3 m_1}\frac{\phi'(\rho u_3)}{\phi'(u_3)}\,. \tag{53}$$

For potentials such as

$$\phi = -1/u\,, \quad \phi = -\beta/u^2\,, \tag{54}$$

λ and ρ would come from a direct solution of an appropriate algebraic equation if the masses and coupling constants are the right size. For example, if $\alpha_{12}=\alpha_{23}=\alpha_{31}=\alpha$ and $m_1 = m_2 = (1/k)m_3$ and $\phi = -1/u$) then (53) is

$$k = \frac{1}{\lambda} + \frac{1}{\rho}\,. \tag{55}$$

This has a solution for $0<\lambda<1$ if $k \geq 2$. If the masses and couplings are not the right size, then one would have to make a modification of $\vec{Z}$ similar to that done in the gravitational Euler solution.

If ϕ is not homogeneous, say $\phi = -e^{-\beta u}/u$, and $\alpha_{12}=\alpha_{31}=\alpha_{23}=\alpha$, $m_1 = m_2 = m_3/k$, then (53) is

$$k = \frac{1}{\lambda}\left(\frac{\lambda\beta u + 1}{\beta u + 1}\right)e^{-\beta u(\lambda-1)} + \frac{1}{\rho}\left(\frac{\rho\beta u + 1}{\rho u + 1}\right)e^{-\beta u(\rho-1)}\,. \tag{56}$$

As with the triangular case, this condition leads to a restriction on the size of the collinear configuration. That is, the magnitude of $\vec{u}$ (for a given k, λ, and $\rho = 1-\lambda$) must satisfy (56). One could regard (56) as defining a surface in a three dimension space (u, k, λ). When (u, k, λ) lie on this surface, a collinear solution is possible for inhomogeneous types of potential.

For Coulomb forces,

$$\vec{Z} = \frac{e_1 e_2}{m_1 m_2}\left(\frac{z_1}{k_1}\frac{\vec{u}_1}{|\vec{u}_1|^3} + \frac{z_2}{k_2}\frac{\vec{u}_2}{|\vec{u}_2|^3} - \frac{\vec{u}_3}{|\vec{u}_3|^3}\right) \tag{57}$$

cannot vanish as $\vec{u}_1$ and $\vec{u}_2$ are in directions opposite to that of $\vec{u}_3$. One can modify $\vec{Z}$ by adding

$$\frac{m_i e_1 e_2}{m_1 m_2}(\eta+1)\frac{\vec{u}_3}{|\vec{u}_3|^3}$$

to both sides of (10) just as was done in the gravitational case of Sec. IV. This yields the following set of equations:

$$\ddot{\vec{u}}_i + \frac{e_1 e_2}{m_1 m_2}\left(\frac{z_i}{k_i}M\frac{\vec{u}_i}{|\vec{u}_i|^3} + m_i(\eta+1)\frac{\vec{u}_3}{|\vec{u}_3|^3}\right) = m_i\vec{Z}'\,, \quad i=1,2 \tag{58}$$

$$\ddot{\vec{u}}_3 + \frac{e_1 e_2}{m_1 m_2}(m_3(\eta+1) - M)\frac{\vec{u}_3}{|\vec{u}_3|^3} = m_3\vec{Z}'\,,$$

where

$$\vec{Z}' = \frac{e_1 e_2}{m_1 m_2}\left(\frac{z_1}{k_1}\frac{\vec{u}_1}{|\vec{u}_1|^3} + \frac{z_2}{k_2}\frac{\vec{u}_2}{|\vec{u}_2|^3} + \eta\frac{\vec{u}_3}{|\vec{u}_3|^3}\right). \tag{59}$$

Since $\vec{u}_1 = -\lambda\vec{u}_3$, $\vec{u}_2 = -\rho\vec{u}_3$, and $\lambda + \rho = 1$,

$$\vec{Z}' = \frac{e_1 e_2}{m_1 m_2}\left(-\frac{z_1}{k_1}\frac{1}{\lambda^2} - \frac{z_2}{k_2}\frac{1}{\rho^2} + \eta\right)\frac{\vec{u}_3}{|\vec{u}_3|^3} = 0 \qquad (60)$$

if

$$\eta = \frac{z_1}{k_1\lambda^2} + \frac{z_2}{k_2\rho^2}\,. \qquad (61)$$

Equation (58) for $i = 1$ has the form

$$\ddot{\vec{u}}_1 + \frac{e_1 e_2}{m_1 m_2}\left(-\frac{z_1}{k_1}\frac{M}{\lambda^2} + m_1(\eta + 1)\right)\frac{\vec{u}_3}{|\vec{u}_3|^3} = 0\,. \qquad (62)$$

Combining this with the last part of (58) implies

$$\lambda(m_3(\eta+1) - M) + m_1(\eta + 1) - \frac{z_1 M}{k_1\lambda^2} = 0\,. \qquad (63)$$

This can be solved for η,

$$\eta = -1 + \frac{1}{\lambda m_3 + m_1} M\left(\frac{z_1}{k_1\lambda^2} + \lambda\right)\,. \qquad (64)$$

Performing a similar analysis for $i = 2$ leads to

$$\eta = -1 + \frac{1}{\rho m_3 + m_2} M\left(\frac{z^2}{k_2\rho^2} + \rho\right)\,. \qquad (65)$$

Equating the two expressions gives

$$\left(\rho + \frac{m_2}{m_3}\right)\left(\frac{z_1}{k_1\lambda^2} - \frac{m_1}{m_3}\right) - \left(\lambda + \frac{m_1}{m_3}\right)\left(\frac{z_2}{k_2\rho^2} - \frac{m_2}{m_3}\right) = 0\,. \qquad (66)$$

This fifth-order equation is similar to that derived earlier for the Euler solution to the gravitational problem.

An example of particular interest is $m_1 = m_2$, $k_1 = k_2 = k$, $z_1 = z_2 = z$. In this case, (66) is satisfied for $\lambda = \frac{1}{2}$. The value of η from either (61) and (64) or (65) is

$$\eta = \frac{8z}{k}\,. \qquad (67)$$

The orbit described by $\vec{u}_3$ is closed and periodic if

$$m_3(\eta + 1) - M > 0 \qquad (68)$$

or

$$z > \tfrac{1}{4}\,. \qquad (69)$$

Assume that conditions allow $\vec{Z}' = 0$. Substituting $\vec{u}_1 = -\frac{1}{2}u_3$ and $\vec{u}_2 = -\frac{1}{2}\vec{u}_3$ the equations of motion (58) are of the form

$$\ddot{\vec{u}}_i + \frac{\kappa_i M}{m_j m_k}\frac{\vec{u}_i}{|\vec{u}_i|^3} = 0\,. \qquad (70)$$

Now with $m_1 = m_2 = m$,

$$\kappa_1 = \kappa_2 = \frac{e^2}{2+k}k(z - \tfrac{1}{4}) \equiv \kappa \qquad (71)$$

and

$$\kappa_3 = \frac{8e^2}{2+k}(z - \tfrac{1}{4}) = \frac{8\kappa}{k}\,. \qquad (72)$$

The separable Hamiltonian that leads directly to (70) if the u_i are treated as independent is[6]

$$\frac{\vec{P}_1^{\,2}}{2\mu_1} + \frac{\vec{P}_2^{\,2}}{2\mu_2} + \frac{\vec{P}_3^{\,2}}{2\mu_3} - \frac{\kappa_1}{|\vec{u}_1|} - \frac{\kappa_2}{|\vec{u}_2|} - \frac{\kappa_3}{|\vec{u}_3|}\,. \qquad (73)$$

The condition on the individual energies for the Euler solution is

$$\frac{E_1}{E_2} = 1, \quad \frac{E_2}{E_3} = \frac{k}{2}, \quad \frac{E_3}{E_1} = \frac{2}{k}\,. \qquad (74)$$

In the gravitational case the fifth-order equation (31) has only one solution for arbitrary mass ratios since the left side monotonically changes from $-\infty$ to $+\infty$ as λ goes from 0 to 1. This is also true for (66). Now (66) can be written as

$$F(\beta\lambda) = k_2(\rho + k_2)\left(\frac{z_1}{\lambda^2} - k_1^{\,2}\right) - k_1(\lambda + k_1)\left(\frac{z_2}{\rho^2} - k_2^{\,2}\right) = 0\,. \qquad (75)$$

In this form it is easy to see that for all values of z_1 and z_2 each of these two terms decreases monotonically as λ increases from 0 to 1. Hence there is only one root just as with the gravitational potential. The existence of this one root does not in this case guarantee, however, that the orbit for u_3 will be bounded. This requires an additional constraint such as given in (68) and (69).

VII. A SUMMARY AND AN APPLICATION TO A QUANTUM SYSTEM

The equations of motion (10) for a three-body system under the influence of arbitrary interparticle potential can be derived from a Hamiltonian (13). This Hamiltonian was shown in the Appendix to be the c.m. Hamiltonian. One can treat the variables $\vec{u}_1$, $\vec{u}_2$, and $\vec{u}_3$ as independent. Their dependence follows from the equations of motion. One advantage of these variables is the clarity with which the exactly soluble configurations are identified with separability ($\vec{Z} = 0$) of the equations of motion (10). This was presented in this context in Ref. 3 for the equilateral-triangle case. For the collinear case, this separability was presented as a modification of the $\vec{Z} = 0$ condition to a $\vec{Z}' = 0$ condition.

In the gravitational case, the separability of the equations of motion for the Lagrange $\vec{Z} = 0$ solution and the Euler $\vec{Z}' = 0$ solution was expressed formally by rewriting the Hamiltonian (13) in the separable forms (22) and (33). The variables $\vec{u}_i$ are treated as independent and the conditions leading to $\vec{Z}=0$ or $\vec{Z}'=0$ are imposed on the solution of the equation of motion.

For scalar types of forces that are independent of the mass, condition (1) [below Eq. (37)] seems to require a relation between coupling constants and masses for an equilateral-triangle solution. However, for equal masses, there is no mass dependence on the coupling and nonhomogeneous potentials such as given in (41) allow such a configuration as well as the scalar $-1/u$ potential. These types of nongravitational scalar type of forces also allow general triangular solutions. In this case, unequal-mass combinations are allowed if the couplings are all equal and/or independent of the mass. If the potentials are nonhomogeneous, then the actual dimensions of the triangle are not arbitrary as with the homogeneous potentials. There are no triangular solutions of any kind for Coulomb or vector type of forces.

In the gravitational case there are no $\vec{Z}=0$ separable collinear solutions. This is in contrast with the nongravitational scalar type of forces which permit $\vec{Z}=0$ in addition to separable $\vec{Z}'=0$ solutions. As in the triangular case, nonhomogeneous potentials give rise to restrictions on the absolute size of the orbital configuration.

The Coulomb case leads to a $\vec{Z}'=0$ collinear solution similar to the gravitational case. The same type of fifth-order equation is obtained for the relative positions of three particles. However, unlike the gravitational case, there are additional restrictions. In particular, the charge ratios must satisfy certain constraints.

As mentioned in the Introduction, this paper is to be regarded as the classical starting point of a quantum solution. The most attractive feature concerning applicability to a quantum-mechanical system is the separability of the equations of motion. This would mean that the corresponding Schrödinger equation would be separable. An important question to be asked and answered is whether a spectrum derived from such a procedure is moderately accurate or in gross error.

For example, suppose one used the quantum analog of (73). In the Schrödinger representation the corresponding equation for stationary states would be

$$\left(-\frac{\hbar^2\vec{\nabla}_1^2}{2\mu_1}-\frac{\hbar^2\vec{\nabla}_2^2}{2\mu_2}-\frac{\hbar^2\vec{\nabla}_3^2}{2\mu_3}-\frac{\kappa_1}{|\vec{u}_1|}-\frac{\kappa_2}{|\vec{u}_2|}-\frac{\kappa_3}{|\vec{u}_3|}\right)\times\psi(\vec{u}_1,\vec{u}_2,\vec{u}_3)=E\psi(\vec{u}_1,\vec{u}_2,\vec{u}_3)\,. \qquad (76)$$

As in (73), the $\vec{u}_i$ are treated as independent. The energy eigenvalue is

$$E=E_1+E_2+E_3\,, \qquad (77)$$

where

$$E_i=-\frac{\mu_i\kappa_i^2}{2\hbar^2 n_i^2}\,,\quad i=1,2,3\,. \qquad (78)$$

To test the reasonableness of this spectrum, I shall work out the ground-state energy level of helium-like atoms using this method. Let $m_1=m_2\equiv m$ and $m_3=\infty$. Hence $\mu_1=\mu_2=m$ and $k=\infty$. This in turn implies $\kappa_1=\kappa_2=e^2(z-\frac{1}{4})$ and $\kappa_3=0$. For the ground state, (74) requires $n_1=n_2=1$. Hence

$$E_0=\frac{-me^4(z-\frac{1}{4})^2}{\hbar^2}=-2(z-\tfrac{1}{4})^2E_H\,, \qquad (79)$$

where $E_H=13.6$ eV. A comparison with the first-order perturbative and experimental values is given as

He(−74, −78.1, −83.3), Li^+(−193, −197.1, −205.7),

Be^{++}(−365.5, −370, −382.5),

where the first number is the perturbation result, the second number is the experimental value, and the third number is the result from (79). The results are comparable with the first-order values. Note, however, that the results of (79) are lower than the exact results. This is contrary to what one expects from a variational procedure which gives upper bounds.[7] It must be emphasized, however, that the results of (79) do not follow from an approximate solution of the exact Schrödinger equation as is true with the standard variational approach. Rather, they follow from an exact solution of an "approximate" Hamiltonian. As discussed earlier, the sense in which the Hamiltonian is approximate is related to classical considerations.

There are, of course, many well-established methods for obtaining far more accurate results for helium.[8] The purpose of this example is to show how an adaptation of the classical three-body Hamiltonian can lead to a moderately accurate spectrum prediction. This mild success serves as a further motivation for considering other applications. However, there are nontrivial questions of interpretation to be considered. For example, how is the separability condition to be reconciled with the constraint $\vec{u}_1+\vec{u}_2+\vec{u}_3=0$ and how is this in turn to be reflected in the eigenfunction? Can one obtain accurate corrections to the result (79) by using the exact Hamiltonian (13) in a quantum content with the trial wave function chosen as a separable form but with the constraint $\delta(\vec{u}_1+\vec{u}_2+\vec{u}_3)$ included in a variational calculation? These are difficult problems and will be discussed in a future paper concerned exclusively with quantum-mechanical aspects.

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APPENDIX: DERIVATION OF THE HAMILTONIAN (13)

The Lagrangian corresponding to the Hamiltonian (1) is

$$L=\tfrac{1}{2}\sum_{i=1}^{3} m_i\dot{\vec{r}}_i^{\,2}-V(\vec{r}_1-\vec{r}_2,\vec{r}_2-\vec{r}_3,\vec{r}_3-\vec{r}_1)\,. \quad \text{(A1)}$$

Substituting Eq. (7) into (A1) leads to

$$L=T-V=L(\dot{\vec{R}},\dot{\vec{u}}_1,\dot{\vec{u}}_2,\dot{\vec{u}}_3,\vec{u}_1,\vec{u}_2,\vec{u}_3)\,. \quad \text{(A2)}$$

If the constraints

$$\sum_{i=1}^{3}\vec{u}_2=0=\sum_{i=1}^{3}\dot{\vec{u}}_i$$

defining the $\vec{u}$'s as the sides of a triangle and the c.m. constraint $\dot{\vec{R}}=0=\vec{R}$ are not imposed one finds

$$\begin{aligned}T=&\tfrac{1}{2}M\dot{\vec{R}}^2+\frac{1}{2}\,\frac{m_1m_2(m_1+m_2)}{M^2}\dot{\vec{u}}_3^{\,2}\\&+\frac{1}{2}\,\frac{m_2m_3(m_3+m_2)}{M^2}\dot{\vec{u}}_1^{\,2}\\&+\frac{1}{2}\,\frac{m_3m_1(m_3+m_1)}{M^2}\dot{\vec{u}}_2^{\,2}\\&-\frac{m_1m_2m_3}{M^2}(\dot{\vec{u}}_1\cdot\dot{\vec{u}}_2+\dot{\vec{u}}_2\cdot\dot{\vec{u}}_3+\dot{\vec{u}}_3\cdot\dot{\vec{u}}_1)\,.\end{aligned} \quad \text{(A3)}$$

In what follows, assume $\dot{\vec{R}}=0$. Since the constraints

$$\sum_{i=1}^{3}\dot{\vec{u}}_i=0=\sum_{i=1}^{3}\vec{u}_i\,.$$

are not imposed, one can define canonical momentum conjugate to $\vec{u}_i$ by

$$\vec{P}_i=\nabla_{\dot{\vec{u}}_2}L=\frac{m_jm_k\dot{\vec{u}}_i}{M}-\frac{m_1m_2m_3}{M^2}(\dot{\vec{u}}_1+\dot{\vec{u}}_2+\dot{\vec{u}}_3)\,,\quad i,j,k \text{ cyclic}\,. \quad \text{(A4)}$$

If the constraint is imposed, then $\vec{P}_i=m_jm_k\dot{\vec{u}}_i/M$, but in this case they are not independent canonical momenta.

As is customary, the Hamiltonian is defined by

$$H=\sum_{i=1}^{3}\vec{P}_i\cdot\dot{\vec{u}}_i-L\,. \quad \text{(A5)}$$

Using (A4) leads to

$$H=T+V(u_1,u_2,u_3)\,, \quad \text{(A6)}$$

where T is given by (A3). On the other hand, from (A4) follows

$$\begin{bmatrix}\vec{P}_1-\vec{P}_2\\ \vec{P}_2-\vec{P}_3\\ \vec{P}_3-\vec{P}_1\end{bmatrix}=\frac{1}{M}\begin{bmatrix}m_2m_3 & -m_3m_1 & 0\\ 0 & m_1m_3 & -m_2m_1\\ -m_2m_2 & 0 & m_1m_2\end{bmatrix}\begin{bmatrix}\dot{\vec{u}}_1\\ \dot{\vec{u}}_2\\ \dot{\vec{u}}_3\end{bmatrix}. \quad \text{(A7)}$$

Using these vectors, one finds that

$$\frac{(\vec{P}_1-\vec{P}_2)^2}{2m_3}+\frac{(\vec{P}_2-\vec{P}_3)^2}{2m_1}+\frac{(\vec{P}_3-\vec{P}_1)^2}{2m_2}+V(u_1,u_2,u_3)=H\,. \quad \text{(A8)}$$

Since we have made at an earlier point the assumption $\dot{\vec{R}}=0$, this is the Hamiltonian in the c.m. frame.

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[1]A. Roy and E. Roy, *The Foundations of Astrodynamics* (Macmillan, N.Y., 1965).

[2]A. Wintner, *The Analytical Foundations of Celestial Mechanics* (Princeton Univ. Press, Princeton, N.J., 1974), pp. 258–260.

[3]R. Brouchke and H. Lass, Celestial Mechanics 8, 5 (1973).

[4]R. Arenstorf (private communication).

[5]The corresponding quantum variables would satisfy the fundamental commutation relations and in the Schrödinger representation $\vec{P}_i=(\hbar/i)\vec{\nabla}_i$.

[6]The Hamiltonians (33) and (73) are like (22) and (38) in that they have the same type of kinetic energy terms. Therefore the triangle constraint is not built into them. They differ from (22) and (38) in that the potential energy terms are different from their form in the exact Hamiltonian (13). In fact, the potential energy term contains information about the special configuration they are restricted to. These separable Hamiltonians share the common feature that their solutions yield the same equations of motion as the exact Hamiltonian (13) for very special configurations.

[7]The standard variational procedure gives $E_0=2(Z-\frac{5}{16})^2E_H$ and is more accurate than our simple model.

[8]The most recent work along these lines involves using the Feshbach-Rubinow approximation. R. K. Bhadini and Y. Nogami. Phys. Rev. A 13, 1986 (1976).