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Explicit dimensional renormalization of quantum field theory in curved space-time

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The dimensional renormalization of scalar and spinor field theories quantized in a conformally flat ($k = 0$) Robertson-Walker space-time is discussed. The theories considered are conformally invariant save explicit mass terms. All calculations are performed on the level of the canonical, Heisenberg-picture field theory, and explicit formulas derived for renormalized Heisenberg-state expectation values, such as particle production. Some solvable models are presented as illustrations of the formalism.

I. INTRODUCTION

Although a mathematically consistent and physically viable approach to the quantization of the gravitational field has yet to be elucidated, it has recently become apparent that there are nevertheless quantum gravitational effects which are calculable in the absence of such a theory, and physically interesting. The archetype is, of course, Hawking's discovery[1] of black-hole radiance. The treatment of the gravitational field as classical with all other matter fields quantized, though noninterecacting, leads directly to the problem of regularizing and renormalizing linear quantum field theories on a curved space-time background.[2]

There are by now a wide variety of competing approaches to the regularization of quantum field theories in curved space. The four main candidates appear to be covariant point separation,[3,4] zeta-function regularization,[5] the use of Pauli-Villars regulator fields,[6] and dimensional regularization.[7] With the possible exception of a discrepancy in the $\Box R$ term in the spin-1 case,[8] all of these methods have now been shown to lead to identical results for the anomalous trace of the stress-energy tensor. Nevertheless, they are certainly not identical in regard to either the complexity of the calculations required, or their suitability for extension to more general theories (e.g., theories with matter self-interactions).

In both these respects, dimensional renormalization seems to offer pronounced advantages. It avoids the computational intricacies of the point-separation approach, which become particularly severe when one attempts to calculate renormalized quantities (the divergences, of course, are local, and have a much simpler structure), even in exactly solvable models. And the algebraic tedium of the Pauli-Villars method, which requires the introduction of a bevy of unphysical regulator fields, is also avoided. Secondly, it is difficult to see how to apply point separation to the regularization of interacting field theories, especially those involving local gauge symmetries. The Pauli-Villars method can be extended to interacting theories, but at the expense[6] of introducing additional covariant higher-derivative terms, so that the calculations inevitably become very tedious. On the other hand, no additional form of regularization is required in the dimensional approach when interactions are included.

In view of these remarks, one is led to ask whether there may not in fact be nontrivial curved-space field theories in which dimensional renormalization can be carried out explicitly, at the level of the canonical field theory, and in a way which facilitates the computation of physically interesting renormalized quantities. It will be shown below that such a procedure can in fact be implemented for scalar and spin-$\frac{1}{2}$ fields in arbitrary asymptotically flat ($k=0$) Robertson-Walker space-times. In Sec. II this is carried through for conformally coupled scalar fields: As an example of a specific physical result, an exact formula is obtained for particle production in terms of a single "reflection" coefficient arising from a one-dimensional Schrödinger equation. In Sec. III an analogous formalism is developed for the quantization and renormalization of spin-$\frac{1}{2}$ theories. The procedure here is identical in spirit to the scalar case, differing only in the (fairly substantial) kinematical complications introduced by spin. As specific illustrations of the applicability of the method, Sec. IV discusses briefly some analytically solvable models for both the scalar and spin-$\frac{1}{2}$ cases. Appendix A contains a derivation of some useful relations among the Bogoliubov coefficients in the spin-$\frac{1}{2}$ case, while Appendix B summarizes the relevant Dirac kinematics.

II. SPIN-0 THEORIES

Consider a conformally coupled massive scalar field theory in a d-dimensional space-time manifold specified by the metric $g_{\mu\nu}$. The matter field contribution to the action is

$$S_{\text{matter}}=\int d^dx\sqrt{g}\left[-\tfrac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi-\tfrac{1}{2}(m^2+\xi R)\phi^2\right], \tag{2.1}$$

where

$$\xi=\frac{1}{4}\frac{d-2}{1-d}. \tag{2.2}$$

It is well known that this theory is conformally invariant, for arbitrary d, in the massless limit. The action (2.1) leads to the equation of motion

$$\frac{1}{\sqrt{g}}\partial_\mu(\sqrt{g}\,g^{\mu\nu}\partial_\nu\phi)-(m^2+\xi R)\phi=0. \tag{2.3}$$

Now suppose that we are dealing with a conformally flat Robertson-Walker metric of the form ($\eta_{\mu\nu}$ is the Minkowski tensor)

$$g_{\mu\nu}(x)=C(t)\eta_{\mu\nu}. \tag{2.4}$$

At this stage, the only restriction on $C(t)$ is that it should tend to a constant for $t\to\pm\infty$ (asymptotic flatness). With this metric, one finds

$$R=(1-d)\frac{1}{C}\left[\frac{\ddot{C}}{C}+\left(\frac{d}{4}-\frac{3}{2}\right)\frac{\dot{C}^2}{C^2}\right], \tag{2.5}$$

$$\sqrt{g}=C^{d/2} \tag{2.6}$$

and the equation of motion becomes ($\Box\equiv\vec{\nabla}\cdot\vec{\nabla}-\partial^2/\partial t^2$)

$$\frac{1}{C}\Box\phi+\left(1-\frac{d}{2}\right)\frac{\dot{C}}{C^2}\dot{\phi}-\left\{m^2+\frac{d-2}{4C}\left[\frac{\ddot{C}}{C}+\left(\frac{d}{4}-\frac{3}{2}\right)\frac{\dot{C}^2}{C^2}\right]\right\}\phi=0. \tag{2.7}$$

Making the change of variable[9]

$$\phi=C^{(2-d)/4}\chi, \tag{2.8}$$

we arrive at the much simplified equation

$$\Box\chi=m^2C\chi. \tag{2.9}$$

Essentially, in going from (2.7) to (2.9), we have utilized the conformal symmetry (up to mass terms) of the theory and the conformal flatness of the metric to make a field rescaling which reduces the equation of motion to its flat-space form. The general solution of (2.9) may be written

$$f_k(\vec{x},t)=e^{i\vec{k}\cdot\vec{x}}f_k(t), \tag{2.10}$$

where $f_k(t)$ satisfies a Schrödinger equation

$$\ddot{f}_k+[|\vec{k}|^2+m^2C(t)]f_k(t)=0. \tag{2.11}$$

The particular solution satisfying the pure positive-frequency boundary condition at $t\to-\infty$ $(+\infty)$ will be denoted f_k^{in} (f_k^{out}). In the standard fashion, we may now write down expressions for the interpolating field in terms of either "in" or "out" algebras of creation and destruction operators:

$$\begin{aligned}\phi&=C^{(2-d)/4}\int\frac{d^{d-1}k}{(2\pi)^{d-1}}\frac{1}{[2\omega_{\text{in}}(k)]^{1/2}}\times[f_k^{\text{in}}(\vec{x},t)a_{\text{in}}(\vec{k})+f_k^{\text{in}*}(\vec{x},t)a_{\text{in}}^\dagger(\vec{k})]\\&=C^{(2-d)/4}\int\frac{d^{d-1}k}{(2\pi)^{d-1}}\frac{1}{[2\omega_{\text{out}}(k)]^{1/2}}\times[f_k^{\text{out}}(\vec{x},t)a_{\text{out}}(\vec{k})+f_k^{\text{out}*}(\vec{x},t)a_{\text{out}}^\dagger(\vec{k})],\end{aligned} \tag{2.12}$$

where

$$\begin{aligned}\omega_{\text{in}}(k)&\equiv[|\vec{k}|^2+m^2C(-\infty)]^{1/2},\\\omega_{\text{out}}(k)&\equiv[|\vec{k}|^2+m^2C(+\infty)]^{1/2}.\end{aligned} \tag{2.13}$$

By the general theory of second-order ordinary differential equations, we must have

$$f_k^{\text{in}}(\vec{x},t)=\alpha(k)f_k^{\text{out}}(\vec{x},t)+\beta(k)f_{-\vec{k}}^{\text{out}*}(\vec{x},t). \tag{2.14}$$

By examining the Wronskian $W[f_k^{\text{in}}(t),f_k^{\text{in}*}(t)]$ one easily finds

$$|\alpha(k)|^2-|\beta(k)|^2=\frac{\omega_{\text{in}}(k)}{\omega_{\text{out}}(k)}. \tag{2.15}$$

This formula precisely implies the canonical character of the Bogoliubov transformation relating the "in" and "out" algebras. Namely, (2.12) and (2.14) imply

$$a_{\text{out}}(\vec{k})=\left(\frac{\omega_{\text{out}}}{\omega_{\text{in}}}\right)^{1/2}[\alpha(k)a_{\text{in}}(\text{k})+\beta^*(k)a_{\text{in}}^\dagger(-\vec{k})], \tag{2.16}$$

and it is trivial to check that

$$[a_{\text{in}}(\vec{k}),a_{\text{in}}^\dagger(\vec{k}')]=\delta^{d-1}(\vec{k}-\vec{k}')\Rightarrow[a_{\text{out}}(\vec{k}),a_{\text{out}}^\dagger(\vec{k}')]=\delta^{d-1}(\vec{k}-\vec{k}')$$

if and only if (2.15) holds.

We are now in a position to compute physical quantities, such as the expectation of the stress-energy tensor in the Heisenberg "in" vacuum $|0_{\text{in}}\rangle$, defined by $a_{\text{in}}(\vec{k})|0_{\text{in}}\rangle=0$ for all $\vec{k}$. This quantity, in the limit $t\to+\infty$, gives the particle production due to the external gravitational field. We shall need the expectation values

$$\langle 0_{\text{in}}|\phi^2(\vec{x},t)|0_{\text{in}}\rangle=C^{(2-d)/2}(t)\int\frac{d^{d-1}k}{(2\pi)^{d-1}}\frac{1}{2\omega_{\text{in}}(k)}\times|f_{\vec{k}}^{\text{in}}(\vec{x},t)|^2\xrightarrow[t\to+\infty]{}C^{(2-d)/2}(+\infty)(D+F), \tag{2.17a}$$

where

$$D \equiv \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{1}{2\omega_{\text{out}}(k)}$$
$$= \frac{\pi^{d/2}}{(2\pi)^d} m^{d-2} C^{(d-2)/2}(+\infty)\Gamma\left(1-\frac{d}{2}\right), \quad (2.17\text{b})$$
$$F \equiv \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{|\beta(k)|^2}{\omega_{\text{in}}(k)}. \quad (2.17\text{c})$$

Now semiclassical methods[10] allow us to relate the asymptotic behavior of the "reflection" coefficient $\beta(k)$ for large $|k|$ to smoothness and asymptotic properties of $C(t)$. Namely, we have

$$|\beta(k)| \underset{|k|\to\infty}{\sim} \frac{m^2}{4|k|^2} \left| \int_{-\infty}^{\infty} \dot{C}(t) e^{2i|k|t}\, dt \right|. \quad (2.18)$$

Thus, for example, if we require that $C(t)$ be C^∞ and that all derivatives vanish as $t\to\pm\infty$, then partial integration applied to (2.18) implies that $\beta(k)$ vanishes as $|k|\to\infty$ faster than any power of $|k|$. We shall assume this to be the case [although sensible physical results obtain with considerably weaker restrictions on $C(t)$]. With this assumption it is now evident that the quantity labeled F in (2.17c) is in fact finite for arbitrary physical space-time dimension d_{phys}. The divergence has been explicitly separated as the quantity D in (2.17b), appearing of course in this framework as a pole as $d\to d_{\text{phys}}$, where d_{phys} is any even positive integer ≥ 2.

We may similarly compute the expectation value

$$\langle 0_{\text{in}} | \partial_\rho \phi(\vec{x}, t) \partial_\sigma \phi(\vec{x}, t) | 0_{\text{in}} \rangle \underset{t\to+\infty}{\longrightarrow} C^{(2-d)/2}(+\infty)(D_{\rho\sigma} + F_{\rho\sigma}), \quad (2.19\text{a})$$

where

$$D_{\rho\sigma} \equiv \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{1}{2\omega_{\text{out}}(k)} k_\rho^{\text{out}} k_\sigma^{\text{out}}$$
$$= \frac{\pi^{d/2}}{2(2\pi)^d} m^d C^{d/2}(+\infty)\Gamma\left(-\frac{d}{2}\right)\eta_{\rho\sigma}, \quad (2.19\text{b})$$
$$F_{\rho\sigma} \equiv \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{|\beta(k)|^2}{\omega_{\text{in}}(k)} k_\rho^{\text{out}} k_\sigma^{\text{out}}. \quad (2.19\text{c})$$

Here $k_0^{\text{out}} = -\omega_{\text{out}}(k)$, $k_i^{\text{out}} = (\vec{k})_i$ for $1 \leq i \leq d-1$. Once again (with reasonable conditions on the metric) the entire divergence is isolated in the contribution labeled $D_{\rho\sigma}$. It is now trivial to obtain the expectation value of the full bare matter stress-energy tensor as $t\to+\infty$. Using the expression for $\theta^{\mu\nu}_{\text{matter}}$

$$\theta^{\mu\nu}_{\text{matter}} = g^{\mu\rho} g^{\nu\sigma} \partial_\rho\phi\partial_\sigma\phi - g^{\mu\nu}\left[\tfrac{1}{2} g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi + \tfrac{1}{2}(m^2+\xi R)\phi^2\right] + \xi\left[(g^{\mu\nu}g^{\rho\sigma} - g^{\mu\rho}g^{\nu\sigma})D_\rho D_\sigma(\phi^2) + R^{\mu\nu}\phi^2\right], \quad (2.20)$$

a simple calculation yields

$$\langle 0_{\text{in}} | \theta^{\mu\nu}_{\text{matter}}(\vec{x}, t) | 0_{\text{in}} \rangle \underset{t\to+\infty}{\longrightarrow} C^{-(d+2)/2}(+\infty)(D^{\mu\nu} + F^{\mu\nu}). \quad (2.21)$$

Of course, this is not yet finite—we have yet to include the contribution from the counterterms in the action. The only relevant counterterm (yielding a nonvanishing asymptotic contribution to $\theta^{\mu\nu}$) is the cosmological constant

$$S_{\text{ct}} = -2^{-(d+2)/2} \frac{1}{(2\pi)^{d/2}} m^d \Gamma\left(-\frac{d}{2}\right) \int d^dx\sqrt{g}. \quad (2.22)$$

The coefficient in (2.22) has been fixed by requiring

$$\lim_{t\to-\infty} \langle 0_{\text{in}} | \theta^{\mu\nu}_{\text{tot}}(\vec{x}, t) | 0_{\text{in}} \rangle = \lim_{t\to-\infty} \langle 0_{\text{in}} | \theta^{\mu\nu}_{\text{matter}}(\vec{x}, t) + \theta^{\mu\nu}_{\text{ct}}(\vec{x}, t) | 0_{\text{in}} \rangle = 0. \quad (2.23)$$

In the $t\to+\infty$ limit we find that the contribution arising from (2.22) exactly cancels the divergent piece $D^{\mu\nu}$ of the stress-energy expectation value, leaving us with a general formula for the (finite) particle production in terms of the Bogoliubov coefficient $\beta(k)$:

$$\lim_{t\to+\infty} \langle 0_{\text{in}} | \theta^{\mu\nu}_{\text{ren}}(\vec{x}, t) | 0_{\text{in}} \rangle \equiv \lim_{t\to+\infty} \lim_{d\to d_{\text{phys}}} \langle 0_{\text{in}} | \theta^{\mu\nu}_{\text{tot}}(\vec{x}, t) | 0_{\text{in}} \rangle$$
$$= C^{-(d_{\text{phys}}+2)/2}(+\infty) \int \frac{d^{d_{\text{phys}}-1}k}{(2\pi)^{d-1}} \frac{|\beta(k)|^2}{\omega_{\text{in}}(k)} k^\mu_{\text{out}} k^\nu_{\text{out}}. \quad (2.24)$$

The full renormalization of the theory will in general require additional counterterms such as $\int d^dx\sqrt{g}\,R$, $\int d^dx\sqrt{g}\,(R^2, R_{\mu\nu}{}^2, R_{\mu\nu}{}^2, R_{\mu\nu\rho\sigma}{}^2)$, etc. For example, it is known[11] that for $d_{\text{phys}} = 2$ the counterterm action in the massless limit is just (with a minimal subtraction)

$$-\frac{1}{24\pi}\frac{1}{d-2}\int d^dx\sqrt{g}\,R.$$

It might be thought that this coefficient could be determined by looking at the stress-tensor expectation at finite times, when the contribution of this counterterm, proportional to the Einstein tensor $G_{\mu\nu}$, would presumably appear. However, because of the high degree of symmetry characterizing the metrics under consideration here [Eq. (2.4)], one finds that $G_{\mu\nu}$ is *explicitly* proportional to $(d-2)$, so that this counterterm is not in fact reflected (for $d_{\text{phys}}=2$) in a pole term in the expectation value of the stress-energy tensor. Nevertheless, the counterterm must be added; otherwise, divergences in *products* of stress-energy tensors, e.g., in

$$\frac{\langle 0_{\text{out}}|\theta^{\mu_1\nu_1}(x_1)\cdots\theta^{\mu_n\nu_n}(x_n)|0_{\text{in}}\rangle}{\langle 0_{\text{out}}|0_{\text{in}}\rangle},$$

will not be canceled. In fact, Feynman-Dyson perturbation theory can conveniently be used to extract the counterterm here by a simple one-loop calculation.[11] In any event, caution should be exercised in making sure that the method used to extract the divergences is independent of the symmetries of the class of metrics considered.

A related point is the rather obvious criticism which may be leveled at the method just outlined for regularizing the theory by extension to arbitrary integer dimensions followed by analytic contribution to complex dimensions. Namely, the extension procedure seems hardly unique: One might, for example, instead have added on $(d-d_{\text{phys}})$ extra "flat" dimensions to the original Robertson-Walker metric. I have no rigorous proof that this regularization leads to a satisfactory (i.e., generally covariant, local, and unitary) quantum field theory. However, the fact that the results obtained for a specific finite renormalized quantity such as particle production agree precisely (cf. Sec. IV) with those obtained by means of the Pauli-Villars approach (in which general covariance, locality, and unitarity are *manifest*) strongly suggests that the method followed above is both natural and correct. It must be admitted, however, that finding a "natural" extension of the theory to d dimensions depended crucially on the high degree of symmetry manifested by the class of metrics elected for study. Of course, it is very difficult, in the absence of such symmetry, to perform the calculations analytically in the first place, so that this limitation of the method may not be too significant in practice.

III. SPIN-$\frac{1}{2}$ THEORIES

The quantization and renormalization of spin-$\frac{1}{2}$ fermions in a curved spacetime background follows a similar path to that outlined above for scalar fields. However, the complication of spin makes somewhat more difficult the disentangling of the purely kinematical (i.e., metric independent) from the specifically dynamical parts of the calculation. We begin with the expression for the matter action

$$S_{\text{matter}}=-\int d^dx\det(V)\{\tfrac{1}{2}V^{\alpha\mu}[\bar\psi\gamma_\alpha\mathfrak{D}_\mu\psi-(\mathfrak{D}_\mu\bar\psi)\gamma_\alpha]\psi+m\bar\psi\psi\},\tag{3.1}$$

where $V^{\alpha\mu}$, the vierbein field, satisfies $V^{\alpha\mu}V^{\beta\nu}\eta_{\alpha\beta}=g^{\mu\nu}$, and the covariant derivative on the fermion field is

$$\mathfrak{D}_\mu\psi\equiv\partial_\mu\psi+\tfrac{1}{2}\sigma^{\alpha\beta}V_\alpha^\nu(D_\mu V_{\beta\nu})\psi,\tag{3.2}$$

$$D_\mu V_{\beta\nu}=\partial_\mu V_{\beta\nu}-\Gamma^\lambda_{\mu\nu}V_{\beta\lambda}.\tag{3.3}$$

With the metric (2.4), we may take

$$V_{\alpha\mu}=C^{1/2}(t)\eta_{\alpha\mu}.\tag{3.4}$$

The action (3.1) leads to the Dirac equation in covariant form

$$(V_\alpha^\mu\gamma^\alpha\mathfrak{D}_\mu+m)\psi=0.\tag{3.5}$$

Our procedure will be to analytically continue to complex dimension formulas derived in an arbitrary *even*-integer-dimensional space-time manifold. Thus, by γ^α in Eq. (3.5), we mean the natural $2^{d/2}\times2^{d/2}$ matrix representation of the Dirac algebra, with ψ a $2^{d/2}$-dimensional spinor. [Both α and μ in Eq. (3.5) run, of course, from 0 to $d-1$.] Writing now

$$\psi=C^{(1-d)/4}(t)(\gamma^0\partial_t+i\vec{k}\cdot\vec{\gamma}-mC^{1/2})e^{i\vec{k}\cdot\vec{x}}\phi_k(t),\tag{3.6}$$

we find that ϕ_k satisfies

$$\ddot\phi_k+\left(|\vec{k}|^2+m^2C+m\gamma^0\frac{\dot C}{2C^{1/2}}\right)\phi_k=0.\tag{3.7}$$

On the $\pm i$ eigenspaces of γ^0

$$\ddot\phi_k^{(\pm)}+\left(|\vec{k}|^2+m^2C\pm im\frac{\dot C}{2C^{1/2}}\right)\phi_k^{(\pm)}=0.\tag{3.8}$$

So again we encounter a Schrödinger equation—but this time, with a complex potential. We now state various properties of the solutions of (3.8) which will be needed later. Let $\phi_k^{\text{in}(\pm)}$ ($\phi_k^{\text{out}(\pm)}$) denote the solution of (3.8) behaving as $t\to-\infty$ $(+\infty)$ like $e^{-i\omega_{\text{in}}(k)t}$ $(e^{-i\omega_{\text{out}}(k)t})$, where

$$\begin{aligned}\omega_{\text{in}}(k)&\equiv(|\vec{k}|^2+\mu_{\text{in}}^2)^{1/2},\\ \omega_{\text{out}}(k)&\equiv(|\text{k}|^2+\mu_{\text{out}}^2)^{1/2},\\ \mu_{\text{in}}&\equiv mC^{1/2}(-\infty),\\ \mu_{\text{out}}&\equiv mC^{1/2}(+\infty).\end{aligned}\tag{3.9}$$

The corresponding negative-frequency solutions are then $\phi_k^{\text{in}(\mp)*}$, $\phi_k^{\text{out}(\mp)*}$ [note the sign flip, due to the explicit factor of i in (3.8)]. Define Bogoliubov coefficients $\alpha^{(\pm)}, \beta^{(\pm)}$ as follows:

$$\phi_k^{\mathrm{in}(\pm)}(t) = \alpha^{(\pm)}(k)\phi_k^{\mathrm{out}(\pm)}(t) + \beta^{(\pm)}(k)\phi_k^{\mathrm{out}(\mp)*}(t) . \tag{3.10}$$

Then the following relations are readily established (see Appendix A for derivation):

$$\frac{\alpha^{(+)}}{\alpha^{(-)}} = \frac{\omega_{\mathrm{in}} - \mu_{\mathrm{in}}}{\omega_{\mathrm{out}} - \mu_{\mathrm{out}}} = \frac{\omega_{\mathrm{out}} + \mu_{\mathrm{out}}}{\omega_{\mathrm{in}} + \mu_{\mathrm{in}}} ,$$
$$\frac{\beta^{(+)}}{\beta^{(-)}} = -\frac{\omega_{\mathrm{in}} - \mu_{\mathrm{in}}}{\omega_{\mathrm{out}} + \mu_{\mathrm{out}}} = -\frac{\omega_{\mathrm{out}} - \mu_{\mathrm{out}}}{\omega_{\mathrm{in}} + \mu_{\mathrm{in}}} , \tag{3.11}$$

$$\alpha^{(-)}\alpha^{(+)*} - \beta^{(-)}\beta^{(+)*} = \frac{\omega_{\mathrm{in}}}{\omega_{\mathrm{out}}} . \tag{3.12}$$

Given a complete set of zero-momentum, flat-space spinors $u(0, \lambda), v(0, \lambda)$ satisfying

$$\left.\begin{aligned} \gamma^0 u(0, \lambda) &= -iu(0, \lambda) \\ \gamma^0 v(0, \lambda) &= iv(0, \lambda) \end{aligned}\right\} \quad 1 \leq \lambda \leq 2^{d/2-1} , \tag{3.13}$$

we can construct the appropriate curved-space spinor solutions of the Dirac equation:

$$\begin{aligned} U_{\mathrm{in}}(\vec{k}, \lambda; \vec{x}, t) &\equiv K_{\mathrm{in}}(k) C^{(1-d)/4}(t)(-i\partial_t + i\vec{k}\cdot\vec{\gamma} - mC^{1/2}) \\ &\quad \times \phi_k^{\mathrm{in}(-)}(t) e^{i\vec{k}\cdot\vec{x}} u(0, \lambda) , \\ V_{\mathrm{in}}(\vec{k}, \lambda; \vec{x}, t) &\equiv K_{\mathrm{in}}(k) C^{(1-d)/4}(t)(i\partial_t - i\vec{k}\cdot\vec{\gamma} - mC^{1/2}) \\ &\quad \times \phi_k^{\mathrm{in}(+)*}(t) e^{-i\vec{k}\cdot\vec{x}} v(0, \lambda) , \end{aligned} \tag{3.14}$$

where

$$K_{\mathrm{in}}(k) \equiv -\frac{1}{|k|}\left[\frac{\omega_{\mathrm{in}}(k) - \mu_{\mathrm{in}}}{2\mu_{\mathrm{in}}}\right]^{1/2} , \tag{3.15}$$

with corresponding equations for $U_{\mathrm{out}}(\vec{k}, \lambda; \vec{x}, t)$, $V_{\mathrm{out}}(\vec{k}, \lambda; \vec{x}, t)$, and $K_{\mathrm{out}}(k)$. These spinors go over asymptotically to the corresponding flat-space solutions as follows:

$$\begin{aligned} U_{\mathrm{in}}(\vec{k}, \lambda; \vec{x}, t) &\xrightarrow[t\to-\infty]{} C^{(1-d)/4}(-\infty) e^{ik\cdot x} u_{\mathrm{in}}(\vec{k}, \lambda) , \\ U_{\mathrm{out}}(\vec{k}, \lambda; \vec{x}, t) &\xrightarrow[t\to+\infty]{} C^{(1-d)/4}(+\infty) e^{ik\cdot x} u_{\mathrm{out}}(\vec{k}, \lambda) , \\ V_{\mathrm{in}}(\vec{k}, \lambda; \vec{x}, t) &\xrightarrow[t\to-\infty]{} C^{(1-d)/4}(-\infty) e^{-ik\cdot x} v_{\mathrm{in}}(\vec{k}, \lambda) , \\ V_{\mathrm{out}}(\vec{k}, \lambda; \vec{x}, t) &\xrightarrow[t\to+\infty]{} C^{(1-d)/4}(+\infty) e^{-ik\cdot x} v_{\mathrm{out}}(\vec{k}, \lambda) . \end{aligned} \tag{3.16}$$

The interpolating field takes the form, in terms of the algebra of "in" operators,

$$\psi = (2\pi)^{-(d-1)/2} \int d^{d-1}k \left[\frac{\mu_{\mathrm{in}}}{\omega_{\mathrm{in}}(k)}\right]^{1/2} \sum_{\lambda=1}^{2^{d/2-1}} \{a_{\mathrm{in}}(\vec{k}, \lambda) U_{\mathrm{in}}(\vec{k}, \lambda; \vec{x}, t) + b_{\mathrm{in}}^\dagger(\vec{k}, \lambda) V_{\mathrm{in}}(\vec{k}, \lambda; \vec{x}, t)\} \tag{3.17}$$

with a completely analogous expression in terms of "out" operators. By equating coefficients of the various curved-space spinors of (3.14) one obtains the Bogoliubov transformation

$$\begin{aligned} a_{\mathrm{out}}(\vec{k}, \lambda) &= \left(\frac{\mu_{\mathrm{in}}}{\mu_{\mathrm{out}}} \frac{\omega_{\mathrm{out}}}{\omega_{\mathrm{in}}}\right)^{1/2} \frac{K_{\mathrm{in}}}{K_{\mathrm{out}}} \left(\alpha^{(-)}(\vec{k}) a_{\mathrm{in}}(\vec{k}, \lambda) + \beta^{(-)}(k)^* \sum_{\lambda'} X_{\lambda'\lambda}(-\vec{k}) b_{\mathrm{in}}^\dagger(-\vec{k}, \lambda')\right) , \\ b_{\mathrm{out}}(\vec{k}, \lambda) &= \left(\frac{\mu_{\mathrm{in}}}{\mu_{\mathrm{out}}} \frac{\omega_{\mathrm{out}}}{\omega_{\mathrm{in}}}\right)^{1/2} \frac{K_{\mathrm{in}}}{K_{\mathrm{out}}} \left(\alpha^{(-)}(k) b_{\mathrm{in}}(\vec{k}, \lambda) - \beta^{(-)}(k)^* \sum_{\lambda'} X_{\lambda\lambda'}(\vec{k}) a_{\mathrm{in}}^\dagger(-\vec{k}, \lambda')\right) , \end{aligned} \tag{3.18}$$

where the polarization tensor $X_{\lambda\lambda'}$ is purely kinematical and given explicitly by

$$X_{\lambda\lambda'}(\vec{k}) = -2\mu_{\mathrm{out}} K_{\mathrm{out}}(k) \bar{u}_{\mathrm{out}}(-\vec{k}, \lambda') v(0, \lambda) . \tag{3.19}$$

The canonical character of the transformation (3.18) is ensured, as a short calculation shows, once we verify the relation

$$|\alpha^{(-)}(k)|^2 - 2\mu_{\mathrm{out}}^2 K_{\mathrm{out}}^2(k)\left(1 - \frac{\omega_{\mathrm{out}}(k)}{\mu_{\mathrm{out}}}\right) |\beta^{(-)}(k)|^2 = \frac{\mu_{\mathrm{out}}}{\mu_{\mathrm{in}}} \frac{\omega_{\mathrm{in}}(k)}{\omega_{\mathrm{out}}(k)} \left(\frac{K_{\mathrm{out}}(k)}{K_{\mathrm{in}}(k)}\right)^2 , \tag{3.20}$$

but this is a trivial consequence of (3.11) and (3.12). From (3.18) we obtain the formal connection[12] between the "in" and "out" vacuums

$$|0_{\mathrm{in}}\rangle = \langle 0_{\mathrm{out}} | 0_{\mathrm{in}}\rangle \exp\left[\int d^{d-1}k \frac{\beta^{(-)*}(k)}{\alpha^{(-)*}(k)} X_{\lambda'\lambda}(-\vec{k}) a_{\mathrm{out}}^\dagger(\vec{k}, \lambda) b_{\mathrm{out}}^\dagger(-\vec{k}, \lambda')\right] |0_{\mathrm{out}}\rangle . \tag{3.21}$$

Indeed, the right-hand side is easily seen by a standard commutator identity to be annihilated by $a_{\mathrm{in}}(\vec{k}, \lambda)$, and the normalization follows immediately by applying $\langle 0_{\mathrm{out}}|$ to both sides.

The calculation of renormalized physical quantities can now be carried out in complete analogy to the scalar field case. We will again consider the stress-energy tensor expectation value $\langle 0_{\text{in}} | \theta^{\mu\nu}(\vec{x}, t) | 0_{\text{in}} \rangle$ for large positive times. By varying the action (3.1), one finds

$$\theta^{\mu\nu}_{\text{matter}} = -\tfrac{1}{4}[(\mathfrak{D}^\nu \bar{\psi})\gamma^\mu \psi - \bar{\psi}\gamma^\mu \mathfrak{D}^\nu \psi + (\mu \leftrightarrow \nu)]. \tag{3.22}$$

Inserting the expression (3.17) for the interpolating field,

$$\langle 0_{\text{in}} | \theta^{\mu\nu}_{\text{matter}}(\vec{x}, t) | 0_{\text{in}} \rangle = \frac{1}{4C^{3/2}} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{\mu_{\text{in}}}{\omega_{\text{in}}(k)} \sum_\lambda [\bar{V}_{\text{in}}(\vec{k}\lambda; \vec{x}t)\gamma^\nu \overleftrightarrow{\partial}^\mu V_{\text{in}}(\vec{k}\lambda; \vec{x}t) + (\mu \leftrightarrow \nu)], \tag{3.23}$$

where μ, ν on the right-hand side of (3.23) are "flat-space" indices ($\partial^0 = -\partial/\partial t$, etc.). From (3.10) and (3.14) it follows that

$$V_{\text{in}}(\vec{k}\lambda; \vec{x}t) \underset{t\to+\infty}{\longrightarrow} K_{\text{in}}(k) C^{(1-d)/4}(+\infty)(i\partial_t - i\vec{k}\cdot\vec{\gamma} - mC^{1/2})[\alpha^{(+)*}(k)e^{i\omega_{\text{out}}t} + \beta^{(+)*}(k)e^{-i\omega_{\text{out}}t}]e^{-i\vec{k}\cdot\vec{x}}v(0,\lambda). \tag{3.24}$$

Substituting this in (3.23) one finds, using again (3.11) and (3.12),

$$\langle 0_{\text{in}} | \theta^{\mu\nu}_{\text{matter}}(\vec{x}, t) | 0_{\text{in}} \rangle \underset{t\to+\infty}{\longrightarrow} C^{-(d+2)/2}(+\infty)(D^{\mu\nu} + F^{\mu\nu}), \tag{3.25a}$$

where

$$D^{\mu\nu} \equiv -2^{d/2} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{k^\mu_{\text{out}} k^\nu_{\text{out}}}{2\omega_{\text{out}}(k)} = -\frac{1}{2}\frac{1}{(2\pi)^{d/2}} m^d C^{d/2}(+\infty)\Gamma\left(-\frac{d}{2}\right)\eta^{\mu\nu}, \tag{3.25b}$$

$$F^{\mu\nu} \equiv 2^{d/2} \int \frac{d^{d-1}k}{(2\pi)^{d-1}} \frac{|\beta^{(+)}(k)|^2}{\omega_{\text{in}}(k)} \frac{[\omega_{\text{in}}(k) - \mu_{\text{in}}]}{[\omega_{\text{out}}(k) + \mu_{\text{out}}]} k^\mu_{\text{out}} k^\nu_{\text{out}}. \tag{3.25c}$$

Once again, we may now add the contribution from the cosmological-constant counterterm [again fixed by requiring $\langle 0_{\text{in}} | \theta^{\mu\nu}_{\text{tot}}(\vec{x}, t) | 0_{\text{in}} \rangle \to 0$ as $t \to -\infty$]

$$S_{\text{ct}} = \frac{1}{2}\frac{1}{(2\pi)^{d/2}} m^d \Gamma\left(-\frac{d}{2}\right) \int d^d x \det(V), \tag{3.26}$$

which exactly cancels the explicitly divergent part $D^{\mu\nu}$ of the bare matter stress-energy tensor. Thus we obtain the final formula for particle production in terms of the Bogoliubov coefficient $\beta^{(+)}(k)$:

$$\lim_{t\to+\infty} \langle 0_{\text{in}} | \theta^{\mu\nu}_{\text{ren}}(\vec{x}, t) | 0_{\text{in}} \rangle = C^{-(d_{\text{phys}}+2)/2}(+\infty) 2^{d_{\text{phys}}/2} \int \frac{d^{d_{\text{phys}}-1}k}{(2\pi)^{d_{\text{phys}}-1}} \frac{|\beta^{(+)}(k)|^2}{\omega_{\text{in}}(k)} \frac{\omega_{\text{in}}(k) - \mu_{\text{in}}}{\omega_{\text{out}}(k) + \mu_{\text{out}}} k^\mu_{\text{out}} k^\nu_{\text{out}}. \tag{3.27}$$

IV. SOME SOLVABLE EXAMPLES

The formalism presented in the preceding two sections is valid for general conformally flat Robertson-Walker metrics which are asymptotically flat and sufficiently smooth [recall the discussion following (2.18)]. In this section we display metrics for both the scalar and spin-$\frac{1}{2}$ cases which lead to explicitly solvable models.

A solvable model of Robertson-Walker type for scalar particles has already been discussed in the literature.[5] The metric is

$$C(t) = A + B \tanh \rho t. \tag{4.1}$$

It suffices to note here that the formalism of Sec. II extends the discussion of Ref. 5 (in two space-time dimensions) to arbitrary dimension. Also, it is trivial to check that (2.24), for $d_{\text{phys}} = 2$, agrees precisely with the formula [Eq. (3.21) of Ref. 5] for particle production in this model derived previously employing Pauli-Villars regularization. In fact, it is possible to obtain the results (2.24) and (3.27) by a Pauli-Villars approach—at the cost of considerably increased algebraic tedium.

A solvable model for the fermionic case is obtained by taking the vierbein field proportional to $(A + B \tanh \rho t)$, namely

$$C(t) = (A + B \tanh \rho t)^2. \tag{4.2}$$

It is interesting to note that in the limit $A \to 0$, this metric behaves, for $0 < t \ll 1/\rho$, precisely like the radiation-dominated, $k = 0$ Friedmann universe, with an exponentially rapid approach to asymptotic flatness for $t \gg 1/\rho$. The latter we may regard as simply a convenient way of "freezing" the red-shift in order to define asymptotic particle states. Thus, if the question of the appropriate initial state in a universe expanding from a singularity can be resolved, this model offers both a fairly realistic and analytically tractable description.[13]

Substituting (4.2) into (3.8), we must solve

$$\ddot{\phi}_k^{(\pm)}+[\,|\vec{k}|^2+m^2(A+B\tanh\rho t)^2\pm imB\rho\,\mathrm{sech}^2(\rho t)]\phi_k^{(\pm)}=0\,. \tag{4.3}$$

After some algebra, one finds for the solution behaving like $e^{-i\omega_{\rm in}(k)t}$ as $t\to-\infty$

$$\phi_k^{\rm in(\pm)}(t)=\exp\left\{-i\left[\omega_+(k)t+\frac{1}{\rho}\,\omega_-(k)\ln(2\cosh\rho t)\right]\right\}{}_2F_1\left(1+\frac{i}{\rho}\,\omega_-\pm\frac{imB}{\rho},\ \frac{i}{\rho}\omega_-\mp\frac{imB}{\rho};\ 1-\frac{i}{\rho}\omega_{\rm in};\ \frac{1+\tanh\rho t}{2}\right). \tag{4.4}$$

The solution behaving like $e^{-i\omega_{\rm out}(k)t}$ as $t\to+\infty$ is

$$\phi_k^{\rm out(\pm)}(t)=\exp\left\{-i\left[\omega_+(k)t+\frac{1}{\rho}\,\omega_-(k)\ln(2\cosh\rho t)\right]\right\}{}_2F_1\left(1+\frac{i}{\rho}\omega_-\pm\frac{imB}{\rho},\ \frac{i}{\rho}\omega_-\mp\frac{imB}{\rho};\ 1+\frac{i}{\rho}\omega_{\rm out};\ \frac{1-\tanh\rho t}{2}\right). \tag{4.5}$$

Here $\omega_\pm(k)\equiv\frac{1}{2}[\omega_{\rm out}(k)\pm\omega_{\rm in}(k)]$. The Bogoliubov coefficients defined by (3.10) now follow directly from the linear transformation properties of the hypergeometric function:

$$\begin{aligned}\alpha^{(\pm)}(k)&=\frac{\Gamma(1-(i/\rho)\omega_{\rm in})\Gamma(-(i/\rho)\omega_{\rm out})}{\Gamma(1-(i/\rho)\omega_+\pm imB/\rho)\Gamma(-(i/\rho)\omega_+\mp imB/\rho)}\,,\\ \beta^{(\pm)}(k)&=\frac{\Gamma(1-(i/\rho)\omega_{\rm in})\Gamma((i/\rho)\omega_{\rm out})}{\Gamma(1+(i/\rho)\omega_-\pm imB/\rho)\Gamma((i/\rho)\omega_-\mp imB/\rho)}\,.\end{aligned} \tag{4.6}$$

The reader may easily verify directly the relations (3.11) and (3.12) among these coefficients. Finally, the explicit formula for particle production is obtained by substituting (4.6) into (3.27):

$$\langle 0_{\rm in}|\theta_{\rm ren}(\vec{x},t)|0_{\rm in}\rangle\underset{t\to+\infty}{\longrightarrow}2^{d_{\rm phys}/2-1}(A+B)^{-d_{\rm phys}-2}\times\int\frac{d^{d_{\rm phys}-1}k}{(2\pi)^{d_{\rm phys}-1}}\frac{k^\mu_{\rm out}k^\nu_{\rm out}}{\omega_{\rm out}(k)}\,\frac{\cosh(2\pi mB/\rho)-\cosh(2\pi\omega_-/\rho)}{\sinh(\pi\omega_{\rm in}/\rho)\sinh(\pi\omega_{\rm out}/\rho)}\,. \tag{4.7}$$

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APPENDIX A

In order to derive the relations (3.11), it is convenient to factorize the Schrödinger equation (3.8) by defining first-order operators

$$\mathfrak{D}_\pm\equiv i\frac{\partial}{\partial t}\pm mC^{1/2}(t) \tag{A1}$$

in terms of which (3.8) becomes

$$\mathfrak{D}_\pm\mathfrak{D}_\mp\,\phi_k^{(\pm)}=|\vec{k}|^2\phi_k^{(\pm)}\,. \tag{A2}$$

Applying $\mathfrak{D}_\mp$ to both sides of (A2) gives an intertwining formula

$$\mathfrak{D}_\mp\,\mathfrak{D}_\pm\,(\mathfrak{D}_\mp\,\phi_k^{(\pm)})=|\vec{k}|^2(\mathfrak{D}_\mp\,\phi_k^{(\pm)})\,. \tag{A3}$$

Thus $\mathfrak{D}_-\phi_k^{(+)}$ is a solution of (3.8) of "minus" type, $\mathfrak{D}_+\phi_k^{(-)}$ a solution of "plus" type. By looking at the region $t\to-\infty$, we conclude that

$$\mathfrak{D}_\mp\phi_k^{\rm in(\pm)}=(\omega_{\rm in}\mp\mu_{\rm in})\phi_k^{\rm in(\mp)}\,. \tag{A4}$$

By taking now $t\to+\infty$ in (A4) and recalling

$$\phi_k^{\rm in(\pm)}(t)\underset{t\to+\infty}{\sim}\alpha^{(\pm)}e^{-i\omega_{\rm out}t}+\beta^{(\pm)}e^{i\omega_{\rm out}t}\,,$$

we arrive at the required results:

$$\begin{aligned}\frac{\alpha^{(+)}}{\alpha^{(-)}}&=\frac{\omega_{\rm in}-\mu_{\rm in}}{\omega_{\rm out}-\mu_{\rm out}}=\frac{\omega_{\rm out}+\mu_{\rm out}}{\omega_{\rm in}+\mu_{\rm in}}\,,\\ \frac{\beta^{(+)}}{\beta^{(-)}}&=-\frac{\omega_{\rm in}-\mu_{\rm in}}{\omega_{\rm out}+\mu_{\rm out}}=-\frac{\omega_{\rm out}-\mu_{\rm out}}{\omega_{\rm in}+\mu_{\rm in}}\,.\end{aligned} \tag{3.11}$$

Equation (3.12) follows directly by comparing the values of the constant Wronskian

$$W[\phi_k^{\text{in}(-)}, \phi_k^{\text{in}(+)*}] \equiv \phi_k^{\text{in}(-)} \frac{\overleftrightarrow{\partial}}{\partial t} \phi_k^{\text{in}(+)*}$$

in the asymptotic $t \to \pm\infty$.

APPENDIX B

The spinor formalism [for d(even)-dimensional space-time] needed in the developments of Sec. III will now be presented. Our sign convention is $\eta_{00} = -1$, $\eta_{ii} = +1$, $1 \le i \le d-1$. Choose a representation of the Dirac algebra in terms of $2^{d/2} \times 2^{d/2}$ matrices, and with $\gamma^{0\dagger} = -\gamma^0$, $\gamma^{i\dagger} = \gamma^i$, $1 \le i \le d-1$. Let $u(0, \lambda), v(0, \lambda)$ be an orthonormal complete set of eigenvectors of γ^0:

$$\left.\begin{aligned} \gamma^0 u(0,\lambda) &= -iu(0,\lambda) \\ \gamma^0 v(0,\lambda) &= +iv(0,\lambda) \end{aligned}\right\} \quad (1 \le \lambda \le 2^{d/2-1}). \tag{B1}$$

Then the corresponding (flat-space) spinors with polarization λ, momentum $\vec{k}$, and energy $\omega(|k|) = (|k|^2 + \mu^2)^{1/2} = k^0$ are

$$\begin{aligned} u(\vec{k},\lambda) &= K(|k|)(i\not{k} - \mu)u(0,\lambda), \\ v(\vec{k},\lambda) &= -K(|k|)(i\not{k} + \mu)v(0,\lambda) \end{aligned} \tag{B2}$$

where

$$K(|k|) \equiv -\frac{1}{|k|}\left(\frac{\omega-\mu}{2\mu}\right)^{1/2}.$$

These spinors satisfy the following completeness and normalization relations ($\bar{u} = iu^\dagger\gamma^0$):

$$\begin{aligned} \sum_{\lambda=1}^{2^{d/2-1}} u(\vec{k},\lambda)\bar{u}(\vec{k},\lambda) &= \frac{1}{2\mu}(\mu - i\not{k}), \\ \sum_{\lambda=1}^{2^{d/2-1}} v(\vec{k},\lambda)\bar{v}(\vec{k},\lambda) &= -\frac{1}{2\mu}(\mu - i\not{k}), \\ \bar{u}(\vec{k},\lambda)u(\vec{k},\lambda') = -\bar{v}(\vec{k},\lambda)v(\vec{k},\lambda') &= \delta_{\lambda\lambda'}. \end{aligned} \tag{B3}$$

Of some use in the computations are the formulas

$$\begin{aligned} \bar{u}(\vec{k},\lambda)\gamma^\rho u(\vec{k},\lambda') &= \bar{v}(\vec{k},\lambda)\gamma^\rho v(\vec{k},\lambda') \\ &= -\frac{ik^\rho}{\mu}\delta_{\lambda\lambda'} \end{aligned} \tag{B4}$$

and [with $X_{\lambda\lambda'}$ defined in (3.19)]

$$\sum_{\kappa=1}^{2^{d/2-1}} X_{\lambda\kappa}(k)X^*_{\lambda'\kappa}(k) = 2\mu K(|k|)^2[\omega(|k|) - \mu]\delta_{\lambda\lambda'}. \tag{B5}$$

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