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Regularization, renormalization, and covariant geodesic point separation

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One of the techniques used in quantum field theory in curved space-times to eliminate divergences in the vacuum expectation value of the stress tensor for quantum fields propagating on a classical gravitational background is called covariant geodesic point separation. Beginning with the Schwinger-DeWitt proper-time method we show how to discard divergences in the effective action by renormalization of the coupling constants in a classical gravitational action functional. We then demonstrate how to determine which terms in the vacuum expectation value of the stress tensor vanish when this renormalization is carried out. This is done using the point-separation approach. We give the form of these terms for spin 0, 1/2, and 1 fields, massive or massless, on an arbitrary curved background. The procedure used is covariant and introduces no ambiguities beyond those inherent in any renormalization scheme. We note the appearance of trace anomalies which arise due to the breaking of conformal invariance by the renormalization process and give the form of the anomalies for arbitrary space-time dimension.

I. INTRODUCTION

Studying quantum field theory in curved space-times is not easy.[1] That is not to say that quantum field theory done in flat space *is* easy, but at least in flat space there are some experiments which can help guide us to the "correct" theory. In curved spaces there are no experiments (yet) to tell us if we are on the right track. We can only play theoretical games.

The games we play do have some rules. The main rule says that we must maintain general covariance in our calculations. This requirement leads to conservation laws for currents such as the stress tensor. There are also other invariances (gauge, conformal, etc.) that we may want to preserve. Of course we also know that breaking symmetries is a popular thing to do these days. The final determination of what invariances we should keep will have to wait for experimental verification.

The other rules and concepts of flat-space theory might carry over to curved spaces. For example, the notions of vacuum and multiparticle states are certainly valuable in studies of such things as the Hawking effect[2,3] (black-hole evaporation). Unfortunately, in many background gravitational fields we find that we are either very poor or too rich. We may not be able to define a vacuum state at all or we may find many possible definitions (as is the case in a Schwarzchild background).[4] In the last case the only recourse we have at present is to study them all.

The next problem we run into is the usual "curse" of quantum field theory—divergences. They appear nearly everywhere and the standard normal-ordering techniques of flat-space theory are not valid in curved spaces. Fortunately, in the past few years, manifestly covariant coordinate-space methods of regularization (identifying the infinities) and renormalization (eliminating them) have been developed. They are called dimensional regularization,[5] zeta-function regularization,[6] Pauli-Villars regularization,[7] and covariant geodesic point separation.[8,9] In this paper we will take a look at the last one. We will find that point separation is manifestly covariant, introduces no new ambiguities, and is a potentially highly useful way of treating the problem of divergences when studying the vacuum expectation value of the stress tensor for fields of spin 0, $\frac{1}{2}$, or 1 propagating on an arbitrary curved background.

We will see that the breaking of conformal invariance in this regularization technique leads to trace anomalies.[10] These anomalies may have important physical consequences in astrophysics[3] and even in high-energy particle physics.[11] We also begin to see that all of the regularization schemes will give (almost) equivalent results. This may not be surprising since it is possible to formulate the dimensional, zeta-function, and point-separation regularization methods beginning from the same initial point—Schwinger's proper-time technique.[12,13]

In the next section of this paper, we introduce the stress tensors for spins 0, $\frac{1}{2}$, and 1. We find we are able to express a special vacuum matrix element of these stress tensors in terms of the Hadamard function $G^{(1)}(x,x')$ which is calculated in Sec. III.

Section IV introduces the theory of regularization and renormalization as defined by DeWitt's[14,15] curved-space generalization of Schwinger's proper-time technique. We show how the divergences in the vacuum expectation value of the stress tensor may be renormalized away by adding counterterms to the classical gravitational action. We see that by using covariant geodesic point separation we can exhibit the terms which we subtract from the unrenormalized expectation value to achieve the finite result.

Section V presents the details of the point-separation techniques and gives the main result.

Section VI shows how massless fields can be treated and also that trace anomalies arise and gives their structure in any dimension.

Section VII contains a discussion of an alternative way to do point separation and reviews how to do specific calculations using the results of Sec. V. An Appendix contains a list of expansions which were used in the previous sections. We use Misner, Thorne, and Wheeler sign conventions.[16]

II. THE STRESS TENSORS

Spin 0. This case was discussed in Ref. 9 so only the most important equations will be repeated here. The action functional for the field $\phi(x)$ is

$$S[\phi]=-\tfrac{1}{2}\int g^{1/2}(\phi_{;\rho}\phi^{;\rho}+\xi R\phi^2+m^2\phi^2)d^4x\,, \tag{2.1}$$

where g is minus the determinant of the background metric $g_{\mu\nu}$, $\xi=\frac{1}{6}$ for a conformally invariant field, R is the scalar curvature, and m is the field's mass. The field equations for $\phi(x)$ are

$$\int F(x,x')\phi(x')d^4x'=0\,, \tag{2.2}$$

where

$$F(x,x')\equiv\frac{\delta^2 S}{\delta\phi(x)\delta\phi(x')} = -g^{1/2}[\delta_{;\rho}{}^{\rho}(x,x')-(\xi R+m^2)\delta(x,x')]\,. \tag{2.3}$$

$\delta(x,x')$ is the four-dimensional δ function and semicolons are the usual covariant derivatives.

Functionally differentiating Eq. (2.1) with respect to $g_{\mu\nu}$, multiplying the result by $2g^{-1/2}$, and setting $\xi=\frac{1}{6}$ gives

$$\begin{aligned}T^{\mu\nu}=\{&\tfrac{1}{3}[\phi^{;\mu},\phi^{;\nu}]_+-\tfrac{1}{12}g^{\mu\nu}[\phi_{;\rho},\phi^{;\rho}]_+\\&-\tfrac{1}{6}[\phi^{;\mu\nu},\phi]_+ +\tfrac{1}{6}g^{\mu\nu}[\phi_{;\rho}{}^{\rho},\phi]_+\\&+\tfrac{1}{12}G^{\mu\nu}[\phi,\phi]_+-\tfrac{1}{4}m^2g^{\mu\nu}[\phi,\phi]_+\}\,,\end{aligned} \tag{2.4}$$

the stress tensor. The symbol $[\,,\,]_+$ is the anticommutator and $G^{\mu\nu}$ is the Einstein tensor.

We now pass from classical field $\phi(x)$ to quantum operator $\underline{\phi}(x)$ (an underline means that the symbol above it is an operator) and introduce states $|\text{in, vac}\rangle$ and $|\text{out, vac}\rangle$ which represent vacuum states before and after any dynamics in the background gravitational field. DeWitt (Ref. 1) discusses these states in more detail. Define

$$\langle\underline{\Theta}\rangle_{\text{matrix}}\equiv\frac{\langle\text{out, vac}|\underline{\Theta}|\text{in, vac}\rangle}{\langle\text{out, vac}|\text{in, vac}\rangle} \tag{2.5}$$

for some operator Θ. Next write each set of brackets in Eq. (2.4) in the forms

$$\begin{aligned}[\phi^{;\mu},\phi^{;\nu}]_+&=\lim_{x'\to x}\tfrac{1}{2}\{[\phi^{;\mu'},\phi^{;\nu}]_+ +[\phi^{;\mu},\phi^{;\nu'}]_+\}\,,\\ [\phi^{;\mu\nu},\phi]_+&=\lim_{x'\to x}\tfrac{1}{2}\{[\phi^{;\mu'\nu'},\phi]_+ +[\phi^{;\mu\nu},\phi']_+\}\,,\end{aligned} \tag{2.6}$$

and

$$[\phi,\phi]_+=\lim_{x'\to x}[\phi,\phi']_+\,,$$

where x' is a point near x, $\phi'=\phi(x')$, and $\phi^{;\mu'}$ represents the covariant derivative at x'.

Equations (2.4), (2.5), and (2.6) are used to give

$$\begin{aligned}\langle\underline{T}^{\mu\nu}\rangle_{\text{matrix}}=\lim_{x'\to x}\,[&\tfrac{1}{6}(G^{(1);\mu'\nu}+G^{(1);\mu\nu'})-\tfrac{1}{12}g^{\mu\nu}G^{(1)}{}_{;\rho}{}^{\rho'}-\tfrac{1}{12}(G^{(1);\mu\nu}+G^{(1);\mu'\nu'})+\tfrac{1}{48}g^{\mu\nu}(G^{(1)}{}_{;\rho}{}^{\rho}+G^{(1)}{}_{;\rho'}{}^{\rho'})\\&+\tfrac{1}{12}(R^{\mu\nu}-\tfrac{1}{4}Rg^{\mu\nu})G^{(1)}-\tfrac{1}{8}m^2g^{\mu\nu}G^{(1)}]\,.\end{aligned} \tag{2.7}$$

The quantity $G^{(1)}$ is the Hadamard function defined by

$$G^{(1)}(x,x')\equiv\langle[\underline{\phi}(x),\underline{\phi}(x')]_+\rangle_{\text{matrix}}\,, \tag{2.8}$$

which satisfies

$$\int F(x,x'')G^{(1)}(x'',x')d^4x''=0\,.$$

Spin $\frac{1}{2}$. Following DeWitt,[17] the action for a neutral spinor field $\psi(x)$ is

$$S[\psi]=\tfrac{1}{2}i\int g^{1/2}\bar{\psi}(\gamma^{\rho}\psi_{;\rho}+m\psi)d^{4}x\,, \tag{2.9}$$

where ψ provides a spin representation of the vierbein group and $\bar{\psi}=\tilde{\psi}\gamma$. The Dirac matrices,[18] γ and γ^{μ}, satisfy

$$[\gamma^{\mu},\gamma^{\mu}]_{+}=2g^{\mu\nu}\underline{1}\,, \tag{2.10}$$

where $\underline{1}$ is the 4×4 unit matrix and ~ means transpose. The covariant derivative of a spinor $\mathfrak{A}$ obeys the commutation relations

$$\mathfrak{A}_{;\nu\mu}-\mathfrak{A}_{;\mu\nu}=\tfrac{1}{2}G_{[\alpha\beta]}R^{\alpha\beta}{}_{\mu\nu}\mathfrak{A}\,,$$
$$\mathfrak{A}_{;\mu\sigma\nu}-\mathfrak{A}_{;\mu\nu\sigma}=\tfrac{1}{2}G_{[\alpha\beta]}R^{\alpha\beta}{}_{\nu\sigma}\mathfrak{A}_{;\mu}+\mathfrak{A}_{;\rho}R_{\mu}{}^{\rho}{}_{\nu\sigma}\,, \tag{2.11}$$
$$\mathfrak{A}_{;\mu\nu\tau\sigma}-\mathfrak{A}_{;\mu\nu\sigma\tau}=\tfrac{1}{2}G_{[\alpha\beta]}R^{\alpha\beta}{}_{\sigma\tau}\mathfrak{A}_{;\mu\nu}+\mathfrak{A}_{\rho\nu}R_{\mu}{}^{\rho}{}_{\sigma\tau}+\mathfrak{A}_{;\mu\rho}R_{\nu}{}^{\rho}{}_{\sigma\tau}\,,$$

and so forth. In Eq. (2.11),

$$G_{[\alpha\beta]}\equiv\tfrac{1}{4}[\gamma_{\alpha},\gamma_{\beta}]_{-} \tag{2.12}$$

are the generators of the vierbein group, $[\ ,\ ]_{-}$ is the commutator bracket, and

$$R^{\alpha\beta}{}_{\mu\nu}=h^{\alpha}{}_{\sigma}h^{\beta}{}_{\tau}R^{\sigma\tau}{}_{\mu\nu}\,,$$

where $h^{\alpha}{}_{\sigma}$ is the vierbein which satisfies $h_{\alpha\mu}h^{\alpha}{}_{\nu}=g_{\mu\nu}$. The covariant derivatives of γ, γ^{μ}, and $G_{[\alpha\beta]}$ vanish.

The field equations are

$$\int F(x,x')\psi(x')\,d^{4}x'=0\,, \tag{2.13}$$

with

$$F(x,x')=-S\frac{\overleftarrow{\delta}}{\delta\tilde{\psi}}\frac{\overleftarrow{\delta}}{\delta\psi'}=ig^{1/2}\gamma[\gamma^{\rho}\delta_{;\rho}(x,x')+m\delta(x,x')]\,. \tag{2.14}$$

The stress tensor found from Eq. (2.9) is

$$T^{\mu\nu}=-\tfrac{1}{4}i(\bar{\psi}\gamma^{\mu}\psi^{;\nu}+\bar{\psi}\gamma^{\nu}\psi^{;\mu})\,,$$

which may be written as

$$T^{\mu\nu}=\tfrac{1}{4}i\,\mathrm{Tr}\gamma\gamma^{(\mu}[\psi^{;\nu)},\tilde{\psi}]_{-}\,, \tag{2.15}$$

where $A_{(\mu}B_{\nu)}=\tfrac{1}{2}(A_{\mu}B_{\nu}+A_{\nu}B_{\mu})$, $\tilde{\psi}\psi=-\psi\tilde{\psi}$ (i.e., ψ is an anticommuting field), and Tr means trace over the suppressed spinor indices. Now write

$$[\psi^{;\nu},\tilde{\psi}]_{-}=\lim_{x'\to x}\tfrac{1}{2}\{[\psi^{;\nu'},\tilde{\psi}]_{-}+[\psi^{;\nu},\tilde{\psi}']_{-}\}$$

in Eq. (2.15) and use the spinor Hadamard function

$$\mathcal{S}^{(1)}(x,x')=\langle[\underline{\psi}(x),\underline{\tilde{\psi}}(x')]_{-}\rangle_{\text{matrix}}\,, \tag{2.16}$$

which satisfies

$$\int F(x,x'')\mathcal{S}^{(1)}(x'',x')d^{4}x''=0\,,$$

and the property $\mathcal{S}^{(1)}(x',x)=-\mathcal{S}^{(1)}(x,x')$ to find

$$\langle\underline{T}^{\mu\nu}\rangle_{\text{matrix}}=\tfrac{1}{8}i\lim_{x'\to x}\mathrm{Tr}\gamma\gamma^{(\mu}(\mathcal{S}^{(1);\nu)}-\mathcal{S}^{(1);\nu')})\,. \tag{2.17}$$

Later we will find it necessary to write

$$\mathcal{S}^{(1)}(x,x')=-i(\gamma^{\rho}\mathcal{G}^{(1)}{}_{;\rho}-m\mathcal{G}^{(1)})\gamma^{-1}\,,$$

where $\mathcal{G}^{(1)}(x,x')$ is a Hadamard function and satisfies

$$\int\mathfrak{F}(x,x'')\mathcal{G}^{(1)}(x'',x')d^{4}x''=0\,,$$

where

$$\mathfrak{F}(x,x')=-g^{1/2}[\delta_{;\rho}{}^{\rho}(x,x')-(\tfrac{1}{4}R+m^{2})\delta(x,x')]\underline{1}\,. \tag{2.18}$$

Equation (2.17) then takes the new form

$$\langle\underline{T}^{\mu\nu}\rangle_{\text{matrix}}=\tfrac{1}{8}\lim_{x'\to x}\mathrm{Tr}\gamma^{(\mu}\gamma^{\rho}(\mathcal{G}^{(1)}{}_{;\rho}{}^{\nu)}-\mathcal{G}^{(1)}{}_{;\rho}{}^{\nu')})\,. \tag{2.19}$$

Spin 1.[19] This case presents a new complication, the existence of a gauge invariance. The vector field $A_{\mu}(x)$ has the action functional

$$S[A_{\mu}]=-\tfrac{1}{4}\int g^{1/2}F_{\mu\nu}F^{\mu\nu}d^{4}x\,,$$

where $F_{\mu\nu}=A_{\nu;\mu}-A_{\mu;\nu}$. Because of the gauge invariance of the action, the wave operator

$$\frac{\delta^{2}S}{\delta A_{\mu}\delta A_{\nu'}}$$

is singular. To remedy this problem we add a gauge-breaking term [which leads to a nonsingular operator in Eq. (2.23)]

$$-\tfrac{1}{2}(A^{\alpha}{}_{;\alpha})^{2}$$

(the semicolon is now the usual covariant derivative) and ghost term

$$c^{*;\alpha}c_{;\alpha}\,,$$

where c is a complex scalar field (c^* is its complex conjugate). So that we may use the Schwinger-DeWitt proper-time method for finding the $G^{(1)}$'s we also introduce mass terms

$$-\tfrac{1}{2}m^{2}A^{\alpha}A_{\alpha}+m^{2}c^{*}c\,.$$

The total action is

$$S[A_{\mu},c]=\int g^{1/2}[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu}-\tfrac{1}{2}(A^{\alpha}{}_{;\alpha})^{2}-\tfrac{1}{2}m^{2}A^{\alpha}A_{\alpha}+c^{*;\alpha}c_{;\alpha}+m^{2}c^{*}c]d^{4}x\,, \tag{2.20}$$

and the field equations are

$$\int F^{\alpha\beta'}A_{\beta'}d^{4}x'=0 \tag{2.21}$$

and

$$\int F_g(x,x')c(x')d^4x' = 0\,, \tag{2.22}$$

with

$$F^{\alpha\beta'} = \frac{\delta^2 S}{\delta A_\alpha \delta A_{\beta'}} = g^{1/2}(\delta^{\alpha\beta'}{}_{;\rho}{}^{\rho} - R^\alpha{}_\rho \delta^{\rho\beta'} - m^2\delta^{\alpha\beta'}) \tag{2.23}$$

and

$$F_g(x,x') = \frac{\delta^2 S}{\delta c^* \delta c'} = -g^{1/2}[\delta_{;\rho}{}^{\rho}(x,x') - m^2\delta(x,x')]\,, \tag{2.24}$$

where $\delta^{\alpha\beta'} \equiv g^{\alpha\beta}\delta(x,x')$.

The stress tensor is

$$T^{\mu\nu} = 2g^{-1/2}\frac{\delta S}{\delta g_{\mu\nu}} = T^{\mu\nu}_{\text{Maxwell}} + T^{\mu\nu}_{\text{gauge}} + T^{\mu\nu}_{\text{ghost}} + T^{\mu\nu}_{\text{mass}}\,, \tag{2.25}$$

where

$$\begin{aligned} T^{\mu\nu}_{\text{Maxwell}} &= F^\mu{}_\alpha F^{\alpha\nu} - \tfrac{1}{4}F^{\alpha\beta}F_{\alpha\beta}g^{\mu\nu}\,,\\ T^{\mu\nu}_{\text{gauge}} &= -A^\alpha{}_{;\alpha}{}^{\mu}A^\nu - A^\alpha{}_{;\alpha}{}^{\nu}A^\mu + [A^\alpha{}_{;\alpha\beta}A^\beta + \tfrac{1}{2}(A^\alpha{}_{;\alpha})^2]g^{\mu\nu}\,,\\ T^{\mu\nu}_{\text{ghost}} &= -c^{*;\mu}c^{;\nu} - c^{*;\nu}c^{;\mu} + c^{*;\alpha}c_{;\alpha}g^{\mu\nu}\,,\\ T^{\mu\nu}_{\text{mass}} &= m^2(A^\mu A^\nu - \tfrac{1}{2}A^\alpha A_\alpha g^{\mu\nu}) + m^2 c^* c\, g^{\mu\nu}\,. \end{aligned} \tag{2.26}$$

Consider $T^{\mu\nu}_{\text{Maxwell}}$ which is constructed from products of the form $F_{\rho\alpha}F_{\tau\beta}$. We may write

$$F_{\rho\sigma}F_{\tau\beta} = \tfrac{1}{2}\{[A_{\alpha;\rho}, A_{\beta;\tau}]_+ - [A_{\alpha;\rho}, A_{\tau;\beta}]_+ - [A_{\rho;\alpha}, A_{\beta;\tau}]_+ + [A_{\rho;\alpha}, A_{\tau;\beta}]_+\}\,.$$

In the now familar way, separate points symmetrically,

$$[A_{\alpha;\rho}, A_{\beta;\tau}]_+ = \lim_{x'\to x}\tfrac{1}{2}\{[A_{\alpha';\rho'}, A_{\beta;\tau}]_+ + [A_{\alpha;\rho}, A_{\beta';\tau'}]_+\}\,,$$

and use the definition for the vector Hadamard function

$$G^{(1)}{}_{\mu\nu'} \equiv \langle[\underline{A}_\mu, \underline{A}_{\nu'}]_+\rangle_{\text{matrix}} \tag{2.27}$$

to derive

$$\begin{aligned}\langle \underline{F}_{\rho\alpha}\underline{F}_{\tau\beta}\rangle_{\text{matrix}} = \lim_{x'\to x}\tfrac{1}{4}(&G^{(1)}{}_{\alpha\beta';\rho\tau'} + G^{(1)}{}_{\beta\alpha';\tau\rho'} - G^{(1)}{}_{\alpha\tau';\rho\beta'}\\ &- G^{(1)}{}_{\tau\alpha';\beta\rho'} - G^{(1)}{}_{\rho\beta';\alpha\tau'} - G^{(1)}{}_{\beta\rho';\tau\alpha'}\\ &+ G^{(1)}{}_{\rho\tau';\alpha\beta'} + G^{(1)}{}_{\tau\rho';\beta\alpha'})\,.\end{aligned} \tag{2.28}$$

Applying this procedure to each expression in Eq. (2.26) gives

$$\langle \underline{T}^{\mu\nu}\rangle_{\text{matrix}} = \langle \underline{T}^{\mu\nu}\rangle_{\text{Maxwell}} + \langle \underline{T}^{\mu\nu}\rangle_{\text{gauge}} + \langle \underline{T}^{\mu\nu}\rangle_{\text{ghost}} + \langle \underline{T}^{\mu\nu}\rangle_{\text{mass}}\,, \tag{2.29}$$

where

$$\langle \underline{T}^{\mu\nu}\rangle_{\text{Maxwell}} = \lim_{x'\to x}[(g^{\mu\rho}g^{\nu\tau} - \tfrac{1}{4}g^{\rho\tau}g^{\mu\nu}) \times g^{\alpha\beta}\langle \underline{F}_{\rho\alpha}\underline{F}_{\tau\beta}\rangle_{\text{matrix}}]\,, \tag{2.30a}$$

$$\begin{aligned}\langle \underline{T}^{\mu\nu}\rangle_{\text{gauge}} = \lim_{x'\to x}[&-\tfrac{1}{4}g^{\alpha\beta}(g^{\mu\rho}g^{\nu\tau} + g^{\mu\tau}g^{\nu\rho} - g^{\mu\nu}g^{\rho\tau})\\ &\times(G^{(1)}{}_{\beta\tau';\alpha\rho} - G^{(1)}{}_{\tau\beta';\alpha'\rho'})\\ &+ \tfrac{1}{8}g^{\alpha\beta}g^{\mu\nu}g^{\rho\tau}(G^{(1)}{}_{\beta\tau';\alpha\rho'} + G^{(1)}{}_{\tau\beta';\rho\alpha'})]\,,\end{aligned} \tag{2.30b}$$

$$\langle \underline{T}^{\mu\nu}\rangle_{\text{ghost}} = \lim_{x'\to x}[-\tfrac{1}{4}(g^{\mu\alpha}g^{\nu\beta} + g^{\mu\beta}g^{\nu\alpha} - g^{\mu\nu}g^{\alpha\beta}) \times(G^{(1)}{}_{g;\alpha\beta'} + G^{(1)}{}_{g;\beta\alpha'})]\,, \tag{2.30c}$$

$$\langle \underline{T}^{\mu\nu}\rangle_{\text{mass}} = \lim_{x'\to x}[\tfrac{1}{4}m^2(g^{\mu\alpha}g^{\nu\beta} - \tfrac{1}{2}g^{\mu\nu}g^{\alpha\beta}) \times(G^{(1)}{}_{\alpha\beta'} + G^{(1)}{}_{\beta\alpha'}) + \tfrac{1}{2}m^2 g^{\mu\nu}G^{(1)}{}_g]\,, \tag{2.30d}$$

where

$$G^{(1)}{}_g(x,x') = \langle[\underline{c}^*(x), \underline{c}(x')]_+\rangle_{\text{matrix}}\,, \tag{2.31}$$

the scalar Hadamard function with $\xi = 0$ in Eq. (2.1).

III. THE HADAMARD FUNCTION

The Hadamard functions $G^{(1)}$, $\mathcal{S}^{(1)}$, $\mathcal{G}^{(1)}$, $G^{(1)}{}_{\mu\nu'}$, and $G^{(1)}$ and their derivatives appeared in the preceding section as the key structures in the $\langle \underline{T}^{\mu\nu}\rangle_{\text{matrix}}$'s derived there. We will now find the Hadamard functions for the three spins. This will be done in arbitrary dimension then for the purposes of this paper specialized to four dimensions. The two-dimensional case will be discussed elsewhere.

The Hadamaru functions are found by studying the Feynman function $G(x,x')$. The two are related by

$$G(x,x') = \overline{G}(x,x') + \tfrac{1}{2}iG^{(1)}(x,x')\,, \tag{3.1}$$

where $\overline{G}$ is the principal-value function. (We will suppress all indices since the mathematics is the same for all spins.) Let $F(x,x')$ be one of the operators in Eqs. (2.3), (2.18), (2.22), or (2.24). The Feynman function $G(x,x')$ satisfies

$$\int F(x,x'')G(x'',x')d^nx'' = -\delta(x,x')\,, \tag{3.2}$$

in n dimensions. Using the Schwinger-DeWitt proper-time approach, we find that[20]

$$G(x,x') = -\frac{\Delta^{1/2}}{(4\pi i)^{n/2}} \int_0^\infty \frac{ds}{s^{n/2}} \exp\left[-i\left(m^2 s - \frac{\sigma}{2s}\right)\right] \times \Omega(x,x',s)\,, \qquad (3.3)$$

where $\Delta(x,x') \equiv g^{-1/2}(x) D(x,x') g^{-1/2}(x')$, $g(x) \equiv -\det(g_{\mu\nu})$, $D(x,x') \equiv -\det(-\sigma_{;\mu\nu'})$ (the Van Vleck-Morette determinant), and $\sigma(x,x')$ is the biscalar of geodetic interval. The covariant derivative of σ, $\sigma^{;\mu}$, is a vector tangent to the geodesic between x and x' at the point x which is oriented in the $x' \to x$ direction and has length equal to the geodesic distance between x and x'. The biscalar σ is related to $\sigma^{;\mu}$ by $2\sigma = \sigma^{;\mu}\sigma_{;\mu}$.[21]

The function $\Omega(x,x',s)$ in Eq. (3.3) satisfies the equation

$$i\frac{\partial}{\partial s}\Omega + \frac{i}{s}\Omega^{;\rho}\sigma_{;\rho} = -D^{-1/2}\int F(x,x'')\Big|_{m=0} D^{1/2}(x'') \times \Omega(x'',x',s)\,d^n x''\,, \qquad (3.4)$$

with the boundary condition $\Omega(x,x',0) = \delta(x,x')$. This differential equation may be solved by writing the power series

$$\Omega(x,x',s) = \sum_{k=0}^{\infty} a_k(x,x')(is)^k\,, \qquad (3.5)$$

which gives the recursion relations

$$\sigma^{;\rho} a_{k+1;\rho} + (k+1)a_{k+1} = \Delta^{-1/2}\int F(x,x')\Big|_{m=0} \Delta^{1/2}(x'') \times a_k(x'',x')\,d^n x'' \qquad (3.6)$$

and

$$\sigma^{;\rho} a_{0;\rho} = 0\,, \qquad (3.7a)$$

along with the boundary condition

$$a_0(x,x) = 1\,. \qquad (3.7b)$$

Note that in applying the Schwinger-DeWitt technique we must have an $F(x,x')$ with derivatives of the form $\delta_{;\rho}{}^{\rho}(x,x')$. Since the spinor operator $F(x,x')$ in Eq. (2.14) is not of this form we perform the $\mathcal{S}^{(1)}$ to $\mathcal{G}^{(1)}$ transformation and use $\mathcal{F}(x,x')$ which as we can see from Eq. (2.18) has the correct form. Also in the scalar case,

$$\xi(n) = \frac{1}{4}\frac{(n-2)}{(n-1)} \qquad (3.8)$$

if we want a scalar field which is conformally invariant in n dimensions. In four dimensions we have $\xi = \frac{1}{6}$.

Using Eqs. (3.3) and (3.5) we find that

$$G(x,x') = -\frac{\Delta^{1/2}}{(4\pi i)^{n/2}} \sum_{k=0}^{\infty} a_k(x,x')\left(-\frac{\partial}{\partial m^2}\right)^k \int_0^\infty \frac{ds}{s^{n/2}} \exp\left[-i\left(m^2 s - \frac{\sigma}{2s}\right)\right]. \qquad (3.9)$$

A change of variables

$$z^2 = -2m^2\sigma, \quad u = -2im^2 s/z$$

allows us to write the integral in Eq. (3.9) as

$$\left(-\frac{z}{2im^2}\right)^{1-n/2} \int_0^{-i\infty} \frac{du}{u^{n/2}} \exp[\tfrac{1}{2}z(u - 1/u)]\,,$$

which is[22]

$$-i\pi(-z/2im^2)^{1-n/2} H^{(2)}_{n/2-1}(z)\,,$$

where $H^{(2)}_{n/2-1}(z)$ is a Hankel function of the second kind of order $n/2-1$. Thus Eq. (3.9) becomes

$$G(x,x') = \frac{i\pi\Delta^{1/2}}{(4\pi i)^{n/2}} \sum_{k=0}^{\infty} a_k(x,x')\left(-\frac{\partial}{\partial m^2}\right)^k \times \left(-\frac{z}{2im^2}\right)^{1-n/2} H^{(2)}_{n/2-1}(z)\,. \qquad (3.10)$$

Now in order to find the small-distance behavior of G in four dimensions, we expand $H^{(2)}_1(z)$ in an asymptotic series[23] and follow the procedure given in Ref. 9 to obtain

$$G^{(1)}(x,x') = \frac{\Delta^{1/2}}{4\pi^2}\left\{a_0\left[\frac{1}{\sigma} + m^2 L(1 + \tfrac{1}{4}m^2\sigma + \cdots) - \tfrac{1}{2}m^2 - \tfrac{5}{16}m^4\sigma + \cdots\right] - a_1[L(1 + \tfrac{1}{2}m^2\sigma + \cdots) - \tfrac{1}{2}m^2\sigma - \cdots] \right.$$
$$\left. a_2\sigma[L(\tfrac{1}{2} + \tfrac{1}{8}m^2\sigma + \cdots) - \tfrac{1}{4} - \cdots] + \cdots + \frac{1}{2m^2}[a_2 + \cdots] + \frac{1}{2m^4}[a_3 + \cdots] + \cdots\right\}, \qquad (3.11)$$

where $L \equiv (\gamma + \frac{1}{2}\ln|\frac{1}{2}m^2\sigma|)$ and γ is Euler's constant. Note again that above[24]

$$G^{(1)}(x,x') = \begin{cases} G^{(1)}(x,x'), \text{ a biscalar for spin } 0, \\ \mathcal{G}^{(1)}(x,x'), \text{ a bispinor for spin } \frac{1}{2}, \\ G^{(1)}{}_{\mu\nu'}, \text{ a bivector for spin } 1; \end{cases} \tag{3.12}$$

$$a_k(x,x') = \begin{cases} a_k(x,x'), \text{ a biscalar for spin } 0, \\ \mathfrak{a}_k(x,x'), \text{ a bispinor for spin } \frac{1}{2}, \\ a_{k\mu\nu'}, \text{ a bivector for spin } 1. \end{cases} \tag{3.13}$$

We now can see that $\langle \underline{T}^{\mu\nu}\rangle_{\text{matrix}}$ is divergent. As $x' \to x$, the length of the geodesic goes to zero, so that $\sigma \to 0$. Thus, $G^{(1)} \to \infty$ because of terms such as σ^{-1} and L. Since $\langle \underline{T}^{\mu\nu}\rangle_{\text{matrix}}$ is constructed from $G^{(1)}$ and its derivatives it is also divergent.

IV. REGULARIZATION AND RENORMALIZATION

In Sec. II we looked at $\langle \underline{T}^{\mu\nu}\rangle_{\text{matrix}}$ defined using Eq. (2.5). Here we are really interested in finding

$$\langle \underline{T}^{\mu\nu}\rangle_{\text{vac}} \equiv \langle \text{in, vac}|\underline{T}^{\mu\nu}|\text{in, vac}\rangle, \tag{4.1}$$

the vacuum expectation value of the stress tensor in the vacuum state defined prior to any dynamics in the background gravitational field. This quantity, properly regularized and renormalized, gives us all the information we want about particle production and vacuum polarization. It is the object we choose to use as a source in the semiclassical gravitational field equations,

$$G^{\mu\nu} = \langle \underline{T}^{\mu\nu}\rangle_{\text{vac}}, \tag{4.2}$$

when doing a back-reaction problem.[25] So why then do we need to calculate $\langle \underline{T}^{\mu\nu}\rangle_{\text{matrix}}$?

The answer to this question (and many others) can be found in DeWitt's work.[26] He shows that

$$\langle \underline{T}^{\mu\nu}\rangle_{\text{vac}} = \langle \underline{T}^{\mu\nu}\rangle_{\text{matrix}} + \langle \underline{T}^{\mu\nu}\rangle_{\text{finite}}, \tag{4.3}$$

where $\langle \underline{T}^{\mu\nu}\rangle_{\text{finite}}$ is zero when there is no particle production, is *always* finite, and satisfies the conversation equation $\langle \underline{T}^{\mu\nu}\rangle_{\text{finite};\nu} = 0$. The divergences appearing in $\langle \underline{T}^{\mu\nu}\rangle_{\text{vac}}$ and $\langle \underline{T}^{\mu\nu}\rangle_{\text{matrix}}$ are identical. Regularize $\langle \underline{T}^{\mu\nu}\rangle_{\text{matrix}}$ and you have regularized $\langle \underline{T}^{\mu\nu}\rangle_{\text{vac}}$. Regularizing $\langle \underline{T}^{\mu\nu}\rangle_{\text{matrix}}$ gives

$$\langle \underline{T}^{\mu\nu}\rangle_{\text{matrix}} = \langle \underline{T}^{\mu\nu}\rangle_{\text{div}} + \langle \underline{T}^{\mu\nu}\rangle_{\text{matrix, ren}}, \tag{4.4}$$

where $\langle \underline{T}^{\mu\nu}\rangle_{\text{div}}$ contains the infinite pieces which we will renormalize away by adding infinite counterterms onto the classical action for the gravitational field and $\langle \underline{T}^{\mu\nu}\rangle_{\text{matrix,ren}}$ is the remaining finite physical part of the matrix element. Renormalizing $\langle \underline{T}^{\mu\nu}\rangle_{\text{div}}$ away also gives us a renormalized $\langle \underline{T}^{\mu\nu}\rangle_{\text{vac}}$,

$$\begin{aligned}\langle \underline{T}^{\mu\nu}\rangle_{\text{vac, ren}} &\equiv \langle \underline{T}^{\mu\nu}\rangle_{\text{vac}} - \langle \underline{T}^{\mu\nu}\rangle_{\text{div}} \\ &= \langle \underline{T}^{\mu\nu}\rangle_{\text{matrix, ren}} + \langle \underline{T}^{\mu\nu}\rangle_{\text{finite}},\end{aligned} \tag{4.5}$$

to be used as the source in Eq. (4.2).

How do we find out what $\langle \underline{T}^{\mu\nu}\rangle_{\text{div}}$ is? We do this by applying Schwinger's regularization prescription using the covariant geodesic point-separation methods proposed by DeWitt.[27] He shows that

$$\langle \underline{T}^{\mu\nu}\rangle_{\text{matrix}} = 2g^{-1/2}\frac{\delta W_{\text{eff}}}{\delta g_{\mu\nu}}, \tag{4.6}$$

where

$$W_{\text{eff}} \equiv -i\ln\langle \text{out, vac}|\text{in, vac}\rangle \tag{4.7}$$

is the so-called effective action. He also shows that

$$\begin{aligned}2g^{-1/2}\frac{\delta W_{\text{eff}}}{\delta g_{\mu\nu}} &= 2g^{-1/2}\frac{\delta}{\delta g_{\mu\nu}}\int L_{\text{eff}}\,d^4x \\ &= \tfrac{1}{2}g^{-1/2}\,\text{Tr}\int d^4x''\,\frac{\delta F(x,x'')}{\delta g_{\mu\nu}} \\ &\quad \times \langle[\underline{\phi}(x''),\underline{\phi}(x)]_{+}\rangle_{\text{matrix}},\end{aligned} \tag{4.8}$$

where Tr means trace over the suppressed spinor and vector indices and the anticommutator is replaced by a commutator for spinors. If we perform point separation on the right-hand side of Eq. (4.8) we will obtain Eqs. (2.7), (2.19), or (2.30). Thus we see that the divergences which will appear in the right-hand side of Eq. (4.8), found by studying $G^{(1)}$, are *exactly* the same as those obtained by functionally differentiating W_{eff}. Eliminating the divergences on one side of Eq. (4.8) will make the other side finite as well.

DeWitt (see also Ref. 5) goes on to find that

$$L_{\text{eff}} = -\lim_{x'\to x}\tfrac{1}{2}\,\text{Tr}\frac{\partial}{\partial\sigma}[g^{1/4}(x)G^{(1)}(x,x')g^{1/4}(x')] \tag{4.9}$$

($\Delta^{1/2}$ and Ω held fixed), which may be written in terms of G,

$$L_{\text{eff}} = \text{Im}\lim_{x'\to x}\,\text{Tr}\frac{\partial}{\partial\sigma}[g^{1/4}(x)G(x,x')g^{1/4}(x')]. \tag{4.10}$$

Now using Eq. (3.3) this becomes

$$L_{\text{eff}} = -\lim_{x'\to x} \mathrm{Tr}\frac{D^{1/2}}{32\pi^2}\int_0^\infty \frac{ds}{s^3}\exp\left[-i\left(m^2 s - \frac{\sigma}{2s}\right)\right] \times \Omega(x,x',s)\,, \tag{4.11}$$

in four dimensions.

Schwinger's prescription tells us that the renormalized effective Lagrangian is[28]

$$L_{\text{eff, ren}} = L_{\text{eff}} - L_{\text{div}}\,, \tag{4.12}$$

where

$$L_{\text{div}} = -\lim_{x'\to x} \mathrm{Tr}\frac{D^{1/2}}{32\pi^2}\int_0^\infty \frac{ds}{s^3}\exp\left[-i\left(m^2 s - \frac{\sigma}{2s}\right)\right] \times [a_0 + a_1(is) + a_2(is)^2]\,. \tag{4.13}$$

The a_0, a_1, and a_2 are the expansion coefficients determined by Eqs. (3.6) and (3.7).

From Eqs. (4.9) and (3.11) we can determine the form of L_{div}. We find

$$L_{\text{div}} = \lim_{x'\to x} \mathrm{Tr}\frac{D^{1/2}}{8\pi^2}\left\{a_0\left[\frac{1}{\sigma^2} - \frac{m^2}{2\sigma} - \frac{m^4}{4}(\gamma + \tfrac{1}{2}\ln|\tfrac{1}{2}m^2\sigma|) + \tfrac{3}{16}m^4\right] + a_1\left[\frac{1}{2\sigma} + \frac{m^2}{2}(\gamma + \tfrac{1}{2}\ln|\tfrac{1}{2}m^2\sigma|) - \tfrac{1}{4}m^2\right] - \tfrac{1}{2}a^2(\gamma + \tfrac{1}{2}\ln|\tfrac{1}{2}m^2\sigma|)\right\}, \tag{4.14}$$

which we can use to find

$$W_{\text{div}} \equiv \int L_{\text{div}} d^4x\,, \tag{4.15}$$

and then

$$\langle T^{\mu\nu}\rangle_{\text{div}} \equiv 2g^{-1/2}\delta W_{\text{div}}/\delta g_{\mu\nu}\,. \tag{4.16}$$

The normal procedure now would be to absorb W_{div} into a classical gravitational action of the form

$$S_{\text{grav}} = \int g^{1/2}[\lambda_0 + \kappa_0 R + \alpha_0(R^{\lambda\xi}R_{\lambda\xi} - \tfrac{1}{3}R^2) + \beta_0 R^2]d^4x \tag{4.17}$$

by renormalization of the bare coupling constants λ_0, κ_0, α_0, and β_0. Unfortunately, Eq. (4.14) is not obviously of the same form as Eq. (4.17). If we expand $D^{1/2}$, $\mathrm{Tr}a_0$, $\mathrm{Tr}a_1$, and $\mathrm{Tr}a_2$ we find that

$$\begin{aligned} L_{\text{div}} = \frac{1}{8\pi^2}g^{1/2}\Bigg\{&\frac{C_1(\bar{s})}{\sigma^2} + \frac{1}{\sigma}\left[C_2(\bar{s})m^2 + C_3(\bar{s})R + C_4(\bar{s})R_{\lambda\xi}\frac{\sigma^\lambda\sigma^\xi}{\sigma}\right] \\ &+ [C_5(\bar{s})m^4 + C_6(\bar{s})m^2R + C_7(\bar{s})R^{\lambda\xi\kappa\iota}R_{\lambda\xi\kappa\iota} + C_8(\bar{s})R^{\lambda\xi}R_{\lambda\xi} + C_9(\bar{s})R^2 + C_{10}(\bar{s})\Box R](\gamma + \tfrac{1}{2}\ln|\tfrac{1}{2}m^2\sigma|) \\ &+ \Bigg[C_{11}(\bar{s})m^4 + C_{12}(\bar{s})m^2R + C_{13}(\bar{s})m^2R_{\lambda\xi}\frac{\sigma^\lambda\sigma^\xi}{\sigma} \\ &+ C_{14}(\bar{s})(\tfrac{1}{288}R_{\lambda\xi}R_{\kappa\iota} + \tfrac{1}{360}R^\rho{}_\lambda{}^\tau{}_\xi R_{\rho\kappa\tau\iota} + \tfrac{1}{80}R_{\lambda\xi;\kappa\iota})\frac{\sigma^\lambda\sigma^\xi\sigma^\kappa\sigma^\iota}{\sigma^2} \\ &+ C_{15}(\bar{s})[-\tfrac{1}{180}R_{\lambda\rho}R^\rho{}_\xi + \tfrac{1}{860}R^{\rho\tau}R_{\rho\lambda\tau\xi} + \tfrac{1}{240}\Box R_{\lambda\xi} + C_{16}(\bar{s})R^{\rho\tau\kappa}{}_\lambda R_{\rho\tau\kappa\xi} + C_{17}(\bar{s})R_{;\lambda\xi}]\frac{\sigma^\lambda\sigma^\xi}{\sigma}\Bigg]\Bigg\}, \end{aligned} \tag{4.18}$$

where the constants $C_i(\bar{s})$ are functions of the spin $\bar{s}$. The direction-dependent terms cannot be absorbed into Eq. (4.17) without special treatment.

One of the main advantages of the point-separation approach is that, unlike dimensional regularization, we never need to know the mode functions outside of four dimensions. However, this advantage is also the cause of the complicated form of L_{div} in Eq. (4.18). In dimensional regularization there is one regularization parameter $\epsilon \equiv n-4$, while in point separation there are four. If we write $\sigma^\rho \equiv \epsilon t^\rho$, where ϵ is the geodesic length and t^ρ is parallel to σ^ρ but normalized to ± 1, we see that the regularization parameters are ϵ, t^1, t^2, and t^3. To eliminate three of the parameters we can average over direction in the manner of Adler *et al.* (Ref. 19). With the averaged products

$$\langle\sigma^\lambda\sigma^\xi\rangle_{av} = \tfrac{1}{2}\sigma g^{\lambda\xi}$$

and

$$\langle\sigma^\lambda\sigma^\xi\sigma^\kappa\sigma^\iota\rangle_{av} = \tfrac{1}{6}\sigma^2(g^{\lambda\xi}g^{\kappa\iota}+g^{\lambda\kappa}g^{\xi\iota}+g^{\lambda\iota}g^{\xi\kappa}), \tag{4.19}$$

the direction-dependent terms become combinations of R, R^2, $R^{\lambda\xi}R_{\lambda\xi}-\frac{1}{3}R^2$, and $R^{\lambda\xi\kappa\iota}R_{\lambda\xi\kappa\iota}$ and may, after use of the four-dimensional Gauss-Bonnet theorem, be absorbed into Eq. (4.17) to produce

$$S_{\text{grav, ren}} \equiv \int g^{1/2}[\lambda_{\text{ren}}+\kappa_{\text{ren}}R + \alpha_{\text{ren}}(R^{\lambda\xi}R_{\lambda\xi}-\tfrac{1}{3}R^2)+\beta_{\text{ren}}R^2]d^4x, \tag{4.20}$$

where the renormalized constants must be determined experimentally.

Note that we will never do an averaging in the actual calculations. It is presented here only to show the underlying similarity between regularization methods.

This process leaves us with the finite quantum action $W_{\text{eff, ren}}$ whose functional derivative will give $\langle \underline{T}^{\mu\nu}\rangle_{\text{matrix,ren}}$ and then using Eq. (4.5) give $\langle \underline{T}^{\mu\nu}\rangle_{\text{vac,ren}}$. We have eliminated $\langle \underline{T}^{\mu\nu}\rangle_{\text{div}}$ by renormalization. This whole process is totally equivalent to simply subtracting $\langle \underline{T}^{\mu\nu}\rangle_{\text{div}}$ from $\langle \underline{T}^{\mu\nu}\rangle_{\text{vac}}$. Since both $\langle \underline{T}^{\mu\nu}\rangle_{\text{matrix,ren}}$ and $\langle \underline{T}^{\mu\nu}\rangle_{\text{finite}}$ are conserved by construction, $\langle \underline{T}^{\mu\nu}\rangle_{\text{vac,ren}}$ will, by definition, be conserved.

How do we find the detailed structure $\langle \underline{T}^{\mu\nu}\rangle_{\text{div}}$? There are two possible paths. We may functionally differentiate W_{div} using Eqs. (4.14)–(4.16) or we can put the expression for $G^{(1)}$ in Eq. (3.11) into Eq. (2.7), (2.19), or (2.30) and then pick out the pieces which are constructed from a_0, a_1, and a_2 *and* of order m^0, m^2, m^4, and $\ln m^2$. These are the only terms which can arise from the functional derivative of Eqs. (4.14) and (4.15). The only reason we will choose the second path is because it is conceptually simpler than trying to functionally differentiate W_{eff}. In Sec. VII we will show how one might carry out the first method.

Recently, Wald,[29] using a very general axiomatic approach, has shown that the point-separation approach will give the unique vacuum expectation value up to the addition of arbitrary conserved local geometrical tensors (the usual renormalization ambiguity).

V. FINDING $\langle \underline{T}^{\mu\nu}\rangle_{\text{div}}$

Now we will find the explicit form for $\langle \underline{T}^{\mu\nu}\rangle_{\text{div}}$ for the three spins.

Spin 0. As was done in Ref. 9, we use Eq. (2.7). We want to find the expansion of $G^{(1)}$ and its derivatives in terms of functions at the point x and the tangent to the geodesic $\sigma^{;\mu}\equiv\sigma^\mu$. To expand a bivector such as $G^{(1);\mu\nu'}$, for example, we form

$$g^{\nu}{}_{\lambda'}G^{(1);\mu\lambda'}, \tag{5.1}$$

which is a contravariant tensor of rank two at x and a scalar at x'. The object $g^{\nu}{}_{\lambda'}$ is the bivector of parallel displacement which has the effect of transporting in parallel the vector-at-x' part of $G^{(1);\mu\lambda'}$ along the geodesic between x and x' back to x. When this is done we can expand expression (5.1) in powers of σ^ρ.

The procedure we actually follow is to substitute $G^{(1)}$ from Eq. (3.11) into Eq. (2.7) and then expand each bitensor using the series expansions on page 2497 of Ref. 9 and in the Appendix of this paper. We then collect terms in powers of σ^ρ. Finally we pick out the terms built from a_0, a_1, or a_2 and of order m^0, m^2, m^4, or $\ln m^2$. We call the collection of these terms $T^{\mu\nu}_{\text{div}}[x,\sigma^\rho]$ and we have

$$\langle \underline{T}^{\mu\nu}\rangle_{\text{div}} = \lim_{x'\to x} T^{\mu\nu}_{\text{div}}[x,\sigma^\rho].$$

See Eqs. (5.5)–(5.8) for the results.

Spin $\frac{1}{2}$. The procedure in this case is almost the same as the spin-0 case. We now have $\mathcal{G}^{(1)}$ and $\mathcal{a}_0$, $\mathcal{a}_1$, and $\mathcal{a}_2$ in Eq. (3.11). We substitute Eq. (3.11) into Eq. (2.19) and expand the parallel-transported bispinors. The expansions are different of course.

In the scalar case, Eq. (3.7) implies $a_0(x,x')=1$ for all x and x'. But in the spinor case the solution to Eq. (3.7) is the bispinor of parallel displacement $\mathcal{I}(x,x')$. This object parallel transports spinors from x' to x. We have $\mathcal{a}_0(x,x')=\mathcal{I}(x,x')$ and $\mathcal{I}(x,x)=\underline{1}$. Equation (3.6) for spinors is

$$\sigma^\rho\mathcal{a}_{k+1;\rho}+(k+1)\mathcal{a}_{k+1}=\Delta^{-1/2}(\Delta^{1/2}\mathcal{a}_k)_{;\rho}{}^{\rho}-\tfrac{1}{4}R\mathcal{a}_k. \tag{5.2}$$

Following the expansion procedure in Ref. 9 and using Eq. (2.11), we obtain the expansions of the bispinors listed in the Appendix. Note that in calculating an object such as $\mathcal{a}_1{}^{;\mu\nu'}$ say, we form

$$\mathcal{I}g^{\nu}{}_{\lambda'}\mathcal{a}_1{}^{;\mu\lambda'}$$

first. The $\mathcal{I}$ parallel transports the spinor-at-x' part of $\mathcal{a}_1$ back to x and the $g^{\nu}{}_{\lambda'}$ does the same for the vector-at-x' in the covariant derivative.

Now collect powers of σ^ρ and perform the trace on the γ^μ's and $G_{[\alpha\beta]}$ matrices which appear. We need the relations $\mathrm{Tr}\gamma^\mu\gamma^\rho=4g^{\mu\rho}$,

$$\mathrm{Tr}\gamma^\mu\gamma^\rho G_{[\alpha\beta]}R^{\alpha\beta}{}_{\sigma\tau}=-4R^{\mu\rho}{}_{\sigma\tau},$$

and

$$\mathrm{Tr}\gamma^{\mu}\gamma^{\rho}G_{[\alpha\beta]}G_{[\gamma\delta]}R^{\alpha\beta}{}_{\sigma\tau}R^{\gamma\delta}{}_{\kappa\iota}$$
$$=4(R^{\mu\lambda}{}_{\sigma\tau}R^{\rho}{}_{\lambda\kappa\iota}-R^{\rho\lambda}{}_{\sigma\tau}R^{\mu}{}_{\lambda\kappa\iota}-\tfrac{1}{2}g^{\mu\rho}R^{\lambda\xi}{}_{\sigma\tau}R_{\lambda\xi\kappa\iota}),$$

when we take the traces. Once again, pick out the terms which come from $L_{\rm div}$ to obtain

$$\langle \underline{T}^{\mu\nu}\rangle_{\rm div}=\lim_{x'\to x}T^{\mu\nu}_{\rm div}[x,\sigma^{\rho}]$$

given in Eqs. (5.5)–(5.8).

Spin 1. This is the most difficult case. In Eq. (3.11), we now have the bivectors $G^{(1)}{}_{\mu\nu'}$ and $a_{k\mu\nu'}$. Consider Eqs. (2.30b) and (2.30c). To expand these objects we must form $g_{\tau}{}^{\tau'}G^{(1)}{}_{\beta\tau';\alpha\rho}$, $g_{\beta}{}^{\beta'}g_{\rho}{}^{\rho'}G^{(1)}{}_{\tau\beta';\alpha'\rho'}$, $g_{\tau}{}^{\tau'}g_{\rho}{}^{\rho'}G^{(1)}{}_{\beta\tau';\alpha\rho'}$, $g_{\beta}{}^{\beta'}g_{\alpha}{}^{\alpha'}G^{(1)}{}_{\tau\beta';\rho\alpha'}$, $g_{\beta}{}^{\beta'}G^{(1)}{}_{;\alpha\beta'}$, and $g_{\alpha}{}^{\alpha'}G^{(1)}{}_{;\beta\alpha'}$. But there are two relationships between $G^{(1)}{}_{;\mu\nu'}$ and $G^{(1)}$,[30]

$$G^{(1)}{}_{\mu\nu'}{}^{;\mu}=-G^{(1)}{}_{;\nu'}$$

and

$$G^{(1)}{}_{\mu\nu'}{}^{;\nu'}=-G^{(1)}{}_{;\mu},$$

which is we put then into the parallel-transported forms of Eqs. (2.30b) and (2.30c) give

$$\langle \underline{T}^{\mu\nu}\rangle_{\rm gauge}+\langle \underline{T}^{\mu\nu}\rangle_{\rm ghost}=-\tfrac{1}{4}g^{\mu\nu}\lim_{x'\to x}G^{(1)}{}_{g}{}^{;\rho}{}_{\rho}$$
$$=-\tfrac{1}{4}m^2g^{\mu\nu}\lim_{x'\to x}G^{(1)}{}_{g}. \qquad (5.3)$$

This simplifies the calculation greatly.

Now define

$$G_{\mu\nu\alpha\beta}=g_{\nu}{}^{\nu'}g_{\beta}{}^{\beta'}G^{(1)}{}_{\mu\nu';\alpha\beta'},$$
$$G^{(1)}{}_{\alpha\beta}=g_{\beta}{}^{\beta'}G^{(1)}{}_{\alpha\beta'}.$$

This gives the parallel-transported $\langle \underline{T}^{\mu\nu}\rangle_{\rm matrix}$:

$$\langle \underline{T}^{\mu\nu}\rangle_{\rm Maxwell}=\lim_{x'\to x}\tfrac{1}{4}[G_{\lambda}{}^{\lambda\mu\nu}+G_{\lambda}{}^{\lambda\nu\mu}+G^{\mu\nu\lambda}{}_{\lambda}+G^{\nu\mu\lambda}{}_{\lambda}$$
$$-G^{\lambda\mu\nu}{}_{\lambda}-G^{\lambda\nu\mu}{}_{\lambda}-G^{\mu\lambda}{}_{\lambda}{}^{\nu}-G^{\nu\lambda}{}_{\lambda}{}^{\mu}$$
$$-(G_{\lambda}{}^{\lambda}{}_{\rho}{}^{\rho}-G_{\lambda\rho}{}^{\rho\lambda})g^{\mu\nu}], \qquad (5.4)$$

$$\langle \underline{T}^{\mu\nu}\rangle_{\rm gauge}+\langle \underline{T}^{\mu\nu}\rangle_{\rm ghost}=-\tfrac{1}{4}m^2g^{\mu\nu}\lim_{x'\to x}G^{(1)}{}_{g},$$

$$\langle \underline{T}^{\mu\nu}\rangle_{\rm mass}=\lim_{x'\to x}[\tfrac{1}{4}m^2(G^{(1)\mu\nu}+G^{(1)\nu\mu}-g^{\mu\nu}G^{(1)}{}_{\lambda}{}^{\lambda})$$
$$+\tfrac{1}{2}m^2g^{\mu\nu}G^{(1)}{}_{g}].$$

To determine the expansions of $G_{\mu\nu\alpha\beta}$ and $G^{(1)}{}_{\mu\nu}$ we need to use the recursion relations in Eqs. (3.6) and (3.7). For spin 1, the solution to Eq. (3.7) is $a_{0\mu\nu'}=g_{\mu\nu'}$ the bivector of parallel displacement. Now $\lim_{x'\to x}g_{\mu\nu'}=g_{\mu\nu}$, the metric. Equation (3.6) becomes

$$\sigma^{\rho}a_{k+1\mu\nu';\rho}+(k+1)a_{k+1\mu\nu'}=\Delta^{-1/2}(\Delta^{1/2}a_{k\mu\nu'})_{;\rho}{}^{\rho}$$
$$-R_{\mu}{}^{\lambda}a_{k\lambda\nu'}.$$

Using the usual commutation relations for covariant derivatives and the method of expansion in Ref. 9 gives the expansions for $g_{\mu\nu'}$, $a_{1\mu\nu'}$, $a_{2\mu\nu'}$ and their derivatives. These are put into Eqs. (5.4) and powers of σ^{ρ} are collected. The terms which arise from $L_{\rm div}$ are picked out and we again have

$$\langle \underline{T}^{\mu\nu}\rangle_{\rm div}\equiv\lim_{x'\to x}T^{\mu\nu}_{\rm div}[x,\sigma^{\rho}].$$

The results are in Eqs. (5.5)–(5.8).

In the calculations above the following relations are very useful:

$$R^{\mu}{}_{\alpha\beta\gamma}R^{\nu\gamma\beta\alpha}=\tfrac{1}{2}R^{\mu}{}_{\alpha\beta\gamma}R^{\nu\alpha\beta\gamma},$$
$$R^{\alpha\mu\beta}{}_{\rho;\beta\alpha}=R^{\alpha\mu\beta}{}_{\rho;\alpha\beta},$$
$$R^{\alpha\mu\beta}{}_{\rho;\alpha\beta}=\Box R^{\mu}{}_{\rho}-\tfrac{1}{2}R^{;\mu}{}_{\rho}+R^{\alpha\beta}R^{\mu}{}_{\alpha\rho\beta}-R^{\mu}{}_{\alpha}R^{\alpha}{}_{\rho},$$
$$R^{\mu}{}_{\lambda}{}^{\alpha}{}_{\xi;\alpha}{}^{\nu}\frac{\sigma^{\lambda}\sigma^{\xi}}{(\sigma^{\rho}\sigma_{\rho})}=(R_{\lambda\xi}{}^{;\mu\nu}-R^{\mu}{}_{\lambda}{}^{;\nu}{}_{\xi}-R^{\alpha}{}_{\lambda}R^{\mu}{}_{\alpha}{}^{\nu}{}_{\xi}$$
$$+R^{\mu}{}_{\alpha}R^{\alpha}{}_{\lambda}{}^{\nu}{}_{\xi})\frac{\sigma^{\lambda}\sigma^{\xi}}{(\sigma^{\rho}\sigma_{\rho})},$$
$$R^{\mu}{}_{\lambda}{}^{\nu}{}_{\alpha}{}^{;\alpha}{}_{\xi}\frac{\sigma^{\lambda}\sigma^{\xi}}{(\sigma^{\rho}\sigma_{\rho})}=(-R^{\nu}{}_{\lambda}{}^{;\mu}{}_{\xi}+R^{\mu\nu}{}_{;\lambda\xi})\frac{\sigma^{\lambda}\sigma^{\xi}}{(\sigma^{\rho}\sigma_{\rho})}.$$

After a great deal of algebra, we finally obtain the expressions for $\langle \underline{T}^{\mu\nu}\rangle_{\rm div}$. The notation is

$$\begin{bmatrix} \; \\ \; \\ \; \end{bmatrix}\begin{matrix}\leftarrow \text{coefficient for spin 0 } (\xi=\tfrac{1}{6}), \\ \leftarrow \text{coefficient for spin } \tfrac{1}{2}, \\ \leftarrow \text{coefficient for spin 1}\end{matrix}$$

and

$$H^{(1)\mu\nu}=g^{-1/2}\frac{\delta}{\delta g_{\mu\nu}}\int g^{1/2}R^2d^4x,$$
$$H^{(2)\mu\nu}=g^{-1/2}\frac{\delta}{\delta g_{\mu\nu}}\int g^{1/2}R^{\alpha\beta}R_{\alpha\beta}d^4x.$$

The results are

$$\langle \underline{T}^{\mu\nu}\rangle_{\rm div,quartic}=\lim_{x'\to x}\frac{1}{2\pi^2}\begin{bmatrix}1\\-2\\2\end{bmatrix}\frac{1}{(\sigma^{\rho}\sigma_{\rho})^2}\left(g^{\mu\nu}-4\frac{\sigma^{\mu}\sigma^{\nu}}{\sigma^{\rho}\sigma_{\rho}}\right), \qquad (5.5)$$

$$\langle \underline{T}^{\mu\nu}\rangle_{\text{div,quadratic}} = \lim_{x'\to x} \frac{1}{4\pi^2}\frac{1}{\sigma^\rho\sigma_\rho}\left\{ \begin{bmatrix} 0 \\ -\frac{1}{4} \\ 1 \end{bmatrix} \left[\tfrac{2}{3}\left(G^{\mu\nu} + R\frac{\sigma^\mu\sigma^\nu}{\sigma^\rho\sigma_\rho}\right) - 4R^{(\mu}{}_\lambda \frac{\sigma^{\nu)}\sigma^\lambda}{\sigma^\rho\sigma_\rho} + \tfrac{4}{3}R_{\lambda\xi}\frac{\sigma^\lambda\sigma^\xi}{\sigma^\rho\sigma_\rho}g^{\mu\nu} - \tfrac{4}{3}R^\mu{}_\lambda{}^\nu{}_\xi \frac{\sigma^\lambda\sigma^\xi}{\sigma^\rho\sigma_\rho}\right]\right.$$

$$\left. + \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} \left[-\tfrac{1}{2}m^2\left(g^{\mu\nu} - 2\frac{\sigma^\mu\sigma^\nu}{\sigma^\rho\sigma_\rho}\right) + \tfrac{2}{3}R^{(\mu}{}_\lambda \frac{\sigma^{\nu)}\sigma^\lambda}{\sigma^\rho\sigma_\rho} - \tfrac{2}{3}R_{\lambda\xi}\frac{\sigma^\lambda\sigma^\xi\sigma^\mu\sigma^\nu}{(\sigma^\rho\sigma_\rho)^2}\right]\right\}, \tag{5.6}$$

$$\langle \underline{T}^{\mu\nu}\rangle_{\text{div,logarithmic}} = \lim_{x'\to x} \frac{1}{4\pi^2}\left\{ -\frac{1}{120}\begin{bmatrix} 1 \\ 3 \\ 12 \end{bmatrix}\left[H^{(2)\mu\nu} - \tfrac{1}{3}H^{(1)\mu\nu}\right] + \frac{1}{3}\begin{bmatrix} 0 \\ -\frac{1}{4} \\ 1 \end{bmatrix} m^2 G^{\mu\nu} - \frac{1}{8}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} m^4 g^{\mu\nu}\right\}\left(\gamma + \tfrac{1}{2}\left|\tfrac{1}{2}m^2\sigma\right|\right), \tag{5.7}$$

$$\langle \underline{T}^{\mu\nu}\rangle_{\text{div,finite}} = \lim_{x'\to x} \frac{1}{4\pi^2}\left\{ \frac{3}{32}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} m^4\left(g^{\mu\nu} - \frac{4}{3}\frac{\sigma^\mu\sigma^\nu}{\sigma^\rho\sigma_\rho}\right) - \frac{1}{6}\begin{bmatrix} 0 \\ -\frac{1}{4} \\ 1 \end{bmatrix} m^2\left(G^{\mu\nu} + R\frac{\sigma^\mu\sigma^\nu}{\sigma^\rho\sigma_\rho}\right)\right.$$

$$- \frac{1}{720}\left[\begin{bmatrix} 1 \\ \frac{7}{4} \\ -13 \end{bmatrix} R^{\lambda\xi\kappa\iota}R_{\lambda\xi\kappa\iota} + \begin{bmatrix} -1 \\ 2 \\ 88 \end{bmatrix} R^{\lambda\xi}R_{\lambda\xi} + \begin{bmatrix} 0 \\ -\frac{5}{4} \\ -25 \end{bmatrix} R^2 + \begin{bmatrix} 1 \\ 3 \\ -18 \end{bmatrix} \Box R\right]\frac{\sigma^\mu\sigma^\nu}{\sigma^\rho\sigma_\rho}$$

$$+ \left[\frac{1}{30}\begin{bmatrix} 3 \\ -1 \\ -14 \end{bmatrix} R^{(\mu}{}_{\lambda;\xi\kappa} + \frac{1}{60}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} R_{\lambda\xi}{}^{;(\mu}{}_\kappa + \frac{1}{36}\begin{bmatrix} 2 \\ -1 \\ -8 \end{bmatrix} R^{(\mu}{}_\lambda R_{\xi\kappa}\right.$$

$$\left. + \frac{1}{180}\begin{bmatrix} 8 \\ -1 \\ -44 \end{bmatrix} R^{(\mu\alpha}{}_\lambda{}^\beta R_{\xi\alpha\kappa\beta} + \frac{1}{90}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} R_\lambda{}^\alpha R^{(\mu}{}_{\xi\alpha\kappa}\right]\frac{\sigma^{\nu)}\sigma^\lambda\sigma^\xi\sigma^\kappa}{(\sigma^\rho\sigma_\rho)^2}$$

$$+ \left[-\frac{1}{180}\begin{bmatrix} 1 \\ -12 \\ 42 \end{bmatrix} R^\mu{}_\lambda{}^\nu{}_{\xi;\kappa\iota} - \frac{1}{36}\begin{bmatrix} 0 \\ -1 \\ 4 \end{bmatrix} R_{\lambda\xi}R^\mu{}_\kappa{}^\nu{}_\iota + \frac{1}{180}\begin{bmatrix} 2 \\ 1 \\ -16 \end{bmatrix} R^\mu{}_\lambda{}^\alpha{}_\xi R^\nu{}_{\kappa\alpha\iota}\right]\frac{\sigma^\lambda\sigma^\xi\sigma^\kappa\sigma^\iota}{(\sigma^\rho\sigma_\rho)^2}$$

$$+ \frac{1}{2}\begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}\left(\tfrac{1}{20}R_{\lambda\xi;\kappa\iota} + \tfrac{1}{72}R_{\lambda\xi}R_{\kappa\iota} + \tfrac{1}{90}R^\alpha{}_\lambda{}^\beta{}_\xi R_{\alpha\kappa\beta\iota}\right)\frac{\sigma^\lambda\sigma^\xi\sigma^\kappa\sigma^\iota}{(\sigma^\rho\sigma_\rho)^2}\left(g^{\mu\nu} - 4\frac{\sigma^\mu\sigma^\nu}{\sigma^\rho\sigma_\rho}\right)$$

$$- \frac{1}{72}\begin{bmatrix} 1 \\ 0 \\ -6 \end{bmatrix}\left(2R_{\lambda\xi;\kappa\iota} + R_{\lambda\xi}R_{\kappa\iota} + R^\alpha{}_\lambda{}^\beta{}_\xi R_{\alpha\kappa\beta\iota}\right)\frac{\sigma^\lambda\sigma^\xi\sigma^\kappa\sigma^\iota}{(\sigma^\rho\sigma_\rho)^2}g^{\mu\nu}$$

$$
\begin{aligned}
&+\Bigg[\frac{1}{12}\begin{bmatrix}-2\\1\\8\end{bmatrix} m^2R^{(\mu}{}_{\lambda}+\frac{1}{144}\begin{bmatrix}0\\1\\-40\end{bmatrix} RR^{(\mu}{}_{\lambda}+\frac{1}{720}\begin{bmatrix}-4\\3\\12\end{bmatrix} R^{;(\mu}{}_{\lambda}\\
&\qquad+\frac{1}{120}\begin{bmatrix}2\\1\\-16\end{bmatrix} \Box R^{(\mu}{}_{\lambda}+\frac{1}{360}\begin{bmatrix}4\\7\\128\end{bmatrix} R^{\alpha\beta}R_{\alpha}{}^{(\mu}{}_{\beta\lambda}+\frac{1}{360}\begin{bmatrix}4\\7\\-52\end{bmatrix} R^{\alpha\beta\gamma(\mu}R_{\alpha\beta\gamma\lambda}\\
&\qquad+\frac{1}{360}\begin{bmatrix}-8\\1\\224\end{bmatrix} R^{(\mu}{}_{\alpha}R^{\alpha}{}_{\lambda}\Bigg]\frac{\sigma^{\nu)}\sigma^{\lambda}}{\sigma^{\rho}\sigma_{\rho}}\\
&+\Bigg[\frac{1}{12}\begin{bmatrix}0\\-1\\4\end{bmatrix} m^2R^{\mu}{}_{\lambda}{}^{\nu}{}_{\xi}-\frac{1}{144}\begin{bmatrix}0\\1\\20\end{bmatrix} RR^{\mu}{}_{\lambda}{}^{\nu}{}_{\xi}+\frac{1}{360}\begin{bmatrix}1\\3\\-18\end{bmatrix} R_{\lambda\xi}{}^{;(\mu\nu)}\\
&\qquad-\frac{1}{120}\begin{bmatrix}2\\1\\-16\end{bmatrix} R^{(\mu}{}_{\lambda}{}^{;\nu)}{}_{\xi}+\frac{1}{180}\begin{bmatrix}-1\\2\\88\end{bmatrix} R^{\mu}{}_{\alpha}R^{\alpha}{}_{\lambda}{}^{\nu}{}_{\xi}-\frac{1}{72}\begin{bmatrix}0\\1\\14\end{bmatrix} R^{\mu\nu}R_{\lambda\xi}\\
&\qquad-\frac{1}{120}\begin{bmatrix}3\\4\\-4\end{bmatrix} R^{\mu\nu}{}_{;\lambda\xi}-\frac{1}{36}\begin{bmatrix}1\\1\\-10\end{bmatrix} R^{\mu}{}_{\lambda}R^{\nu}{}_{\xi}-\frac{1}{360}\begin{bmatrix}4\\7\\-52\end{bmatrix} R^{\mu\alpha\nu\beta}R_{\alpha\lambda\beta\xi}\\
&\qquad-\frac{1}{360}\begin{bmatrix}2\\1\\164\end{bmatrix} R_{\lambda\alpha}R^{\alpha(\mu\nu)}{}_{\xi}-\frac{1}{360}\begin{bmatrix}4\\7\\-52\end{bmatrix} R^{\mu\alpha\beta}{}_{\lambda}R^{\nu}{}_{\beta\alpha\xi}\\
&\qquad-\frac{1}{360}\begin{bmatrix}4\\7\\128\end{bmatrix} R^{\mu\alpha\beta}{}_{\lambda}R^{\nu}{}_{\alpha\beta\xi}\Bigg]\frac{\sigma^{\lambda}\sigma^{\xi}}{\sigma^{\rho}\sigma_{\rho}}\\
&+\Bigg[-\frac{1}{24}\begin{bmatrix}1\\-2\\2\end{bmatrix} m^2R_{\lambda\xi}-\frac{1}{144}\begin{bmatrix}0\\-1\\4\end{bmatrix} RR_{\lambda\xi}-\frac{1}{720}\begin{bmatrix}1\\-12\\42\end{bmatrix} R_{;\lambda\xi}\\
&\qquad+\frac{1}{240}\begin{bmatrix}1\\-2\\2\end{bmatrix} \Box R_{\lambda\xi}-\frac{1}{180}\begin{bmatrix}1\\-2\\2\end{bmatrix} R_{\lambda\alpha}R^{\alpha}{}_{\xi}+\frac{1}{360}\begin{bmatrix}1\\-2\\2\end{bmatrix} R^{\alpha\beta}R_{\alpha\lambda\beta\xi}\\
&\qquad+\frac{1}{1440}\begin{bmatrix}4\\7\\-52\end{bmatrix} R^{\alpha\beta\gamma}{}_{\lambda}R_{\alpha\beta\gamma\xi}\Bigg]\frac{\sigma^{\lambda}\sigma^{\xi}}{\sigma^{\rho}\sigma_{\rho}}\left(g^{\mu\nu}-2\frac{\sigma^{\mu}\sigma^{\nu}}{\sigma^{\rho}\sigma_{\rho}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&+\Bigg[-\frac{1}{4}\begin{bmatrix}0\\0\\1\end{bmatrix} m^2R_{\lambda\xi}+\frac{1}{6}\begin{bmatrix}0\\0\\1\end{bmatrix} RR_{\lambda\xi}+\frac{1}{96}\begin{bmatrix}1\\0\\0\end{bmatrix} R_{;\lambda\xi}\\
&\quad-\frac{1}{144}\begin{bmatrix}1\\0\\-6\end{bmatrix}\Box R_{\lambda\xi}+\frac{1}{72}\begin{bmatrix}1\\0\\-24\end{bmatrix} R_{\lambda\alpha}R^{\alpha}{}_{\xi}-\frac{1}{4}\begin{bmatrix}0\\0\\1\end{bmatrix} R^{\alpha\beta}R_{\alpha\lambda\beta\xi}\\
&\quad+\frac{1}{8}\begin{bmatrix}0\\0\\1\end{bmatrix} R^{\alpha\beta\gamma}{}_{\lambda}R_{\alpha\beta\gamma\xi}\Bigg]\frac{\sigma^{\lambda}\sigma^{\xi}}{\sigma^{\rho}\sigma_{\rho}}g^{\mu\nu}\Bigg\}. \qquad (5.8)
\end{aligned}
$$

Note that as in Ref. 9, we have eliminated odd powers of σ^{ρ} by forming

$$\langle \underline{T}^{\mu\nu}\rangle_{\rm div}=\tfrac{1}{2}\lim_{x'\to x}\left(T^{\mu\nu}_{\rm div}[x,\sigma^{\rho}]+T^{\mu\nu}_{\rm div}[x,-\sigma^{\rho}]\right).$$

The notation $\langle T^{\mu\nu}\rangle_{\rm div,finite}$ seems a bit confusing perhaps. The "div, finite" means that these are finite terms in the stress tensor which arise from the divergent part of $W_{\rm eff}$, namely $W_{\rm div}$.

Many of the terms in $\langle \underline{T}^{\mu\nu}\rangle_{\rm div}$ have coefficients

$$\begin{bmatrix}1\\-2\\2\end{bmatrix}\ \text{or}\ \begin{bmatrix}0\\-\frac{1}{4}\\1\end{bmatrix}.$$

The first is $(-1)^{2\bar{s}}$ times the number of helicity states for each spin while the second is $(-1)^{2\bar{s}}\bar{s}^2$. Why these particular dependences on spin occur is not completely understood as yet.

VI. THE MASSLESS CASE AND ANOMALIES

As we can easily see the massless field presents a special problem. In the expressions for $\langle \underline{T}^{\mu\nu}\rangle_{\rm div}$ we find that all the divergences and finite subtraction terms have smooth $m\to 0$ limits *except* the logarithmic divergence. The $H^{(2)\mu\nu}-\frac{1}{3}H^{(1)\mu\nu}$ term blows up. If we go back to $L_{\rm div}$ in Eq. (4.14) we see that this term comes from the functional derivative of the a_2 term which is divergent as $m\to 0$. This term appears in Eq. (4.14) because the integral

$$\mathrm{Tr}\int_0^{\infty}\frac{ds}{s^3}\,e^{i\sigma/2s}a_2(is)^2 \qquad (6.1)$$

in Eq. (4.13) not only diverges at the upper ($s=0$) limit when $x'\to x$ but also diverges at the upper ($s=\infty$) limit even when x and x' are separated. This is an infrared divergence. We cannot include this divergence in $L_{\rm div}$ since it would also make $L_{\rm eff,ren}$ infrared-divergent. So we put an upper cutoff on the integral by introducing a factor $e^{-i\mu^2 s}$. This makes the integral finite and allows us to write Eq. (4.14) as

$$
\begin{aligned}
L_{\rm div}=\lim_{x'\to x}\mathrm{Tr}\,\frac{D^{1/2}}{8\pi^2}\big[&a_0\sigma^{-2}+\tfrac{1}{2}a_1\sigma^{-1}\\
&-\tfrac{1}{2}a_2(\gamma+\tfrac{1}{2}\ln|\tfrac{1}{2}\mu^2|)\big]. \qquad (6.2)
\end{aligned}
$$

The cutoff factor μ, which has units of mass, is completely arbitrary. Suppose we write $\mu=\mu'(\mu/\mu')$, then

$$\ln|\tfrac{1}{2}\mu^2\sigma|=\ln|\tfrac{1}{2}\mu'^2\sigma|+\ln|(\mu/\mu')^2|. \qquad (6.3)$$

The finite $a_2\ln|(\mu/\mu')^2|$ term only contributes to the renormalization of α_0 in $S_{\rm grav,ren}$. No extra terms are produced in $\langle \underline{T}^{\mu\nu}\rangle_{\rm div}$ or $\langle \underline{T}^{\mu\nu}\rangle_{\rm vac,ren}$.

The $\langle \underline{T}^{\mu\nu}\rangle_{\rm div}$ for a massless field is the same as in Sec. V with m set equal to zero everywhere except in the $\ln|\frac{1}{2}m^2\sigma|$ factor in the logarithmic divergence. There m becomes μ. Note that exactly the same sort of cutoff factor appears in the order regularization methods where the symbol for it is κ,[5] L,[6] T,[1] or μ.[11,29]

For massless fields of spin 0 ($\xi=\frac{1}{6}$), spin $\frac{1}{2}$, and spin 1, the classical theory is invariant under conformal transformations of the metric, i.e., $g_{\mu\nu}\to\Omega^2(x)g_{\mu\nu}$. This symmetry carries over to the stress tensor making it trace-free. In the quantum theory, the renormalization process causes the trace of the stress tensor to be nonzero. This trace is called an anomaly.[31]

We can see the form that the anomaly will have. The trace of $\langle \underline{T}^{\mu\nu}\rangle_{\rm div}$ is

$$\langle \underline{T}^{\mu}{}_{\mu}\rangle_{\rm div} = \frac{-1}{2880\pi^2}\left[\begin{bmatrix} 1 \\ \frac{7}{4} \\ -13 \end{bmatrix} C^2 + \begin{bmatrix} 1 \\ \frac{11}{2} \\ 62 \end{bmatrix} (R^{\alpha\beta}R_{\alpha\beta} - \tfrac{1}{3}R^2) + \begin{bmatrix} 1 \\ 3 \\ -18 \end{bmatrix} \Box R \right]. \tag{6.4}$$

Thus, since the trace of $\langle T^{\mu\nu}\rangle_{\rm vac}$ is zero by construction, the trace of $\langle T^{\mu\nu}\rangle_{\rm vac,ren}$ must be the negative of $\langle \underline{T}^{\mu}{}_{\mu}\rangle_{\rm div}$ in Eq. (6.4).

We might ask why we cannot keep conformal invariance by adding more finite counterterms to $S_{\rm grav,ren}$ in such a way that $\langle T^{\mu}{}_{\mu}\rangle_{\rm vac,ren}$ is zero. The reason is that such counterterms must be nonlocal (see Ref. 10). There are no local action functionals whose functional derivatives with respect to $g_{\mu\nu}$ will give stress tensors whose traces are C^2 or $R^{\alpha\beta}R_{\alpha\beta} - \frac{1}{3}R^2$. The $\Box R$ term's status as an anomaly is still vague. There is a local action,

$$\int g^{1/2}R^2 d^4x,$$

whose functional derivative has a trace proportional to $\Box R$. So by adding an R^2 counterterm we can eliminate the $\Box R$ trace. To further confuse the issue we find that in the spin-1 case, dimensional regularization gives $12\Box R$ in the anomaly instead of $-18\Box R$ which both zeta function and point separation give. All other anomaly coefficients are the same in all methods. We know why dimensional regularization gives a different answer. It is the fact that unlike the spin-0 and spin-$\frac{1}{2}$ cases the spin-1 theory is not conformally invariant for arbitrary dimensions. The n-dimensional manipulations of the effective action force extra R^2-type counterterms and therefore an extra $30\Box R$ anomaly term to appear when continuing back to four dimensions. (See Refs. 5, 6, and 11 for more details.) Only with new physical arguments, such as those presented by Horowitz and Wald,[32] will we know more about what the $\Box R$ anomaly should be. At present we will accept the value predicted by our method.

It is an interesting and relatively simple exercise to find the form of the anomaly for arbitrary dimensions. We will confine our discussion to theories which are classically conformally invariant in all dimensions.

The anomaly arises because of the logarithm term in $L_{\rm div}$. We can find the form for this term in any dimension. Looking to how we found $L_{\rm div}$ in Eq. (4.16) we see that the logarithm term arises from the $a_2\sigma\ln|\frac{1}{2}m^2\sigma|$ (or $a_2\sigma\ln|\frac{1}{2}\mu^2\sigma|$ for massless fields) term in $G^{(1)}$ when we differentiate with respect to σ [Eq. (4.9)].

Consider Eq. (3.10) which gives G in arbitrary dimensions. Suppose n is odd, then

$$H^{(2)}_{n/2-1}(z) = (2z/\pi)^{1/2}h^{(2)}_{(n-3)/2},$$

for $n \geq 3$, where $h^{(2)}_{(n-3)/2}$, is a spherical Hankel function of the second kind of order $\frac{1}{2}(n-3)$. The asymptotic expansion of $h^{(2)}_{(n-3)/2}$ is

$$h^{(2)}_{(n-3)/2} = i^{(n+1)/2}z^{-1}e^{iz} \times \sum_{l=0}^{(n-3)/2} \frac{[\frac{1}{2}(n-3)+l]!}{l!\,\Gamma(\frac{1}{2}(n-3)-l+1)}(2iz)^{-l}.$$

We see immediately that the $H^{(2)}_{n/2-1}(z)$ has no logarithmic terms and therefore there is no possibility for an anomaly. Hence we have[33]

$$\langle \underline{T}^{\mu}{}_{\mu}\rangle_{\rm vac,ren} = 0,$$

for all odd dimensions.

In even dimensions, the situation is totally different. Using the definition of $H^{(2)}_{n/2-1}(z)$,

$$H^{(2)}_{n/2-1}(z) = J_{n/2-1}(z) - iY_{n/2-1}(z), \tag{6.5}$$

where $J_{n/2-1}$ and $Y_{n/2-1}$ are Bessel functions of the first and second kind, respectively, and $z^2 \equiv -2m^2\sigma$, we get

$$G^{(1)}(x,x') = \frac{2\pi\Delta^{1/2}}{(4\pi)^{n/2}}\sigma^{1-n/2}\sum_{k=0}^{\infty}a_k(2\sigma)^k\left(\frac{\partial}{\partial z^2}\right)^k \times (z^2)^{(n/2-1)/2}Y_{n/2-1}(z) \tag{6.6}$$

from Eqs. (3.10) and (3.1). Now we write the expansion of $Y_{n/2-1}(z)$,

$$Y_{n/2-1}(z) = (-\tfrac{1}{2}z)^{-(n/2-1)}\pi^{-1}\sum_{l=0}^{n/2}\frac{(n/2-l)!}{l!}2^{-2l}(z^2)^l + 2^{1-n/2}\pi^{-1}\ln|\tfrac{1}{4}z^2|(z^2)^{(n/2-1)/2}\sum_{l=0}^{\infty}\frac{(-1)^l 2^{-2l}(z^2)^l}{l!\,\Gamma(n/2+l)}$$
$$- 2^{1-n/2}\pi^{-1}(z^2)^{(n/2-1)/2}\sum_{l=0}^{\infty}[\psi(l+1)+\psi(n/2+l)]\frac{(-1)^l 2^{-2l}(z^2)^l}{l!(n/2+l-1)!}, \tag{6.7}$$

and note the log term in the second term on the right-hand side. In Eq. (6.6) this takes the form

$$G^{(1)}(x,x')=\frac{2\pi\Delta^{1/2}}{(4\pi)^{n/2}}\sigma^{1-n/2}\sum_{k=0}^{\infty}a_k(2\sigma)^k\left(\frac{\partial}{\partial z^2}\right)^k 2^{1-n/2}\pi^{-1}\ln\left|\tfrac{1}{4}z^2\right|(z^2)^{n/2-1}$$
$$\times\sum_{l=0}^{\infty}\frac{(-1)^l 2^{-2l}(z^2)^l}{l!\,\Gamma(n/2+l)}+\text{nonlog terms.} \tag{6.8}$$

If we do the differentiations in Eq. (6.8) we obtain

$$G^{(1)}(x,x')=\frac{2\pi\Delta^{1/2}}{(4\pi)^{n/2}}2^{1-n/2}\pi^{-1}\sigma^{1-n/2}\sum_{k=0}^{\infty}a_k(2\sigma)^k\sum_{l=0}^{\infty}\frac{(-1)^l 2^{-2l}}{l!\,\Gamma(n/2+l)}\frac{(l+n/2-1)!}{(l+n/2-k-1)!}(-2m^2\sigma)^{l-k+n/2-1}\ln\left|\tfrac{1}{2}m^2\sigma\right|$$
$$+\text{nonlog terms.}$$

We now find the term which is $\sigma\ln|\frac{1}{2}m^2\sigma|$. Counting powers of σ we find that only when $l=1$ will we find such a term. We must also have no m's in the term so we require $l-k+n/2-1=0$. With $l=1$, this gives $k=n/2$. Finally,

$$G^{(1)}(x,x')=-\frac{\Delta^{1/2}(x,x')}{(4\pi)^{n/2}}a_{n/2}(x,x')\sigma\ln\left|\tfrac{1}{2}m^2\sigma\right|$$
$$+\text{non-}\sigma\ln\left|\tfrac{1}{2}m^2\sigma\right|\text{terms,}$$

which gives the log term in L_{div}

$$L_{\text{div,log}}=\lim_{x'\to x}\text{Tr}\left[-\frac{D^{1/2}(x,x')}{2(4\pi)^{n/2}}a_{n/2}(x,x')\right.$$
$$\left.\times\ln\left|\tfrac{1}{2}m^2\sigma(x,x')\right|\right]. \tag{6.9}$$

This leads to the anomaly (for $m\to\mu$)[34]

$$\langle \underline{T}^{\mu}{}_{\mu}\rangle_{\text{vac,ren}}=\frac{1}{(4\pi)^{n/2}}\text{Tr}a_{n/2}(x,x). \tag{6.10}$$

As a check, for $n=2$

$$\langle \underline{T}^{\mu}{}_{\mu}\rangle_{\text{vac,ren}}=\frac{1}{4\pi}\text{Tr}a_1(x,x),$$

and for $n=4$

$$\langle \underline{T}^{\mu}{}_{\mu}\rangle_{\text{vac,ren}}=\frac{1}{16\pi^2}\text{Tr}a_2(x,x),$$

in agreement with previous calculations. Note that

$$a_k(x,x)=\begin{cases}a_k(x,x), & \text{for spin } 0,\\ \text{Tr}\mathcal{A}_k(x,x) & \text{for spin } \tfrac{1}{2}.\end{cases}$$

Spin 1 is not conformally invariant in all dimensions.[35]

VII. DISCUSSION

In this section we will look at three subjects. First, we will outline briefly how to use $\langle \underline{T}^{\mu\nu}\rangle_{\text{div}}$ in a practical calculation. Second, we will show how we could functionally differentiate W_{div} to obtain $\langle \underline{T}^{\mu\nu}\rangle_{\text{div}}$. Finally, we will discuss calculations in other dimensions.

In several recent papers, the method for finding $\langle \underline{T}^{\mu\nu}\rangle_{\text{vac,ren}}$ in various cosmological models has been given.[36] The method described in those papers will work in all cases if certain pieces of information are given. The first bit of information is the definition of the vacuum state $|\text{in, vac}\rangle$. As we said in the Introduction, this may not be easy. Let us suppose that through perseverance we discover a complete set of mode functions $u_i(x)$ which we find give a physically reasonable $|\text{in, vac}\rangle$ state. It is then easy to show that if we expand the field operator in terms of creation ($\underline{a}_i^*$) and annihilation operators ($\underline{a}_i$) as

$$\underline{\phi}(x)=\sum_i\left[\underline{a}_i u_i(x)+\underline{a}_i^* u_i^*(x)\right],$$

then

$$\langle \underline{T}^{\mu\nu}\rangle_{\text{vac}}=\sum_i T^{\mu\nu}[u_i(x),u_i^*(x)]. \tag{7.1}$$

The $\sum_i$ in Eq. (7.1) represents a sum (when i is a discrete index) or an integral (when i is continuous) over all modes. $T^{\mu\nu}[u_i(x),u_i^*(x)]$ is constructed by substituting u_i and u_i^* in for the $\phi(x)$'s in Eqs. (2.4), (2.15), or (2.26).

We already know that

$$\langle \underline{T}^{\mu\nu}\rangle_{\text{vac,ren}}=\langle \underline{T}^{\mu\nu}\rangle_{\text{vac}}-\langle \underline{T}^{\mu\nu}\rangle_{\text{div}}, \tag{7.2}$$

when we eliminate the infinite $\langle \underline{T}^{\mu\nu}\rangle_{\text{div}}$ with infinite counterterms. We also know the forms for

$$\langle \underline{T}^{\mu\nu}\rangle_{\text{div}}=\lim_{x'\to x}T^{\mu\nu}_{\text{div}}[x,\sigma^{\rho}], \tag{7.3}$$

from Sec. V. We use this information in the following way. Take $\langle \underline{T}^{\mu\nu}\rangle_{\text{vac}}$ in Eq. (7.1) and form

$$\langle \underline{T}^{\mu\nu}\rangle_{\text{vac}}=\tfrac{1}{2}\lim_{x'\to x}\sum_i\{T^{\mu\nu}[u_i(x),u_i^*(x')]$$
$$+T^{\mu\nu}[u_i(x'),u_i^*(x)]\}$$
$$\equiv\lim_{x'\to x}T^{\mu\nu}_{\text{vac}}[x,\sigma^{\rho}], \tag{7.4}$$

being very careful to make the point separation in *exactly* the same symmetrical manner as Sec. II. Calculate the mode sums (usually a gruesome task) if possible. Expand the sums in powers of σ^ρ. Now because $\langle \underline{T}^{\mu\nu}\rangle_{\rm div}$ contains the divergences and finite subtraction terms which are renormalized away, if we form

$$\langle \underline{T}^{\mu\nu}\rangle_{\rm vac,ren} = \lim_{x'\to x}(T^{\mu\nu}_{\rm vac}[x,\sigma^\rho] - T^{\mu\nu}_{\rm div}[x,\sigma^\rho]) \tag{7.5}$$

we will have the finite result of Eq. (4.5). The $\lim_{x'\to x}$ in Eqs. (7.5) becomes superfluous since we will have no terms left which depend on σ^ρ and which also survive in the coincidence limit. This procedure is quite general and will work for any quantum field or background field so long as we can define an $|{\rm in,vac}\rangle$ state and do the mode sums.

Next we consider the possibility of finding $\langle \underline{T}^{\mu\nu}\rangle_{\rm div}$ by functionally differentiating. DeWitt[37] has shown that

$$\frac{\delta\sigma(x,x')}{\delta g_{\mu\nu}} = \tfrac{1}{2}\sigma^\mu\sigma^\nu + \text{terms of order } (\sigma^\rho)^3 \text{ and higher}. \tag{7.6}$$

Consider the scalar field where $a_0 = 1$. Remember that $D^{1/2}(x,x') = g^{1/2} + $ terms of order $(\sigma^\rho)^2$ and higher. Let us look at the direction-independent terms of line one in Eq. (4.14) which are

$$\frac{1}{8\pi^2}\int g^{1/2}[\sigma^{-2} - \tfrac{1}{2}m^2\sigma^{-1} - \tfrac{1}{4}m^4(\gamma + \tfrac{1}{2}\ln|\tfrac{1}{2}m^2\sigma|) + \tfrac{3}{16}m^4]d^4x\,. \tag{7.7}$$

Making a variation of $g_{\mu\nu}$ in Eq. (7.7) gives

$$\frac{1}{8\pi^2}\int\{\delta g^{1/2}[\sigma^{-2} - \tfrac{1}{2}m^2\sigma^{-1} - \tfrac{1}{4}m^4(\gamma + \tfrac{1}{2}\ln|\tfrac{1}{2}m^2\sigma|) + \tfrac{3}{16}m^4] + g^{1/2}[-2\sigma^{-3}\delta\sigma + \tfrac{1}{2}m^2\sigma^{-2}\delta\sigma - \tfrac{1}{8}m^4\sigma^{-1}\delta\sigma]\}d^4x\,.$$

Using $\delta g^{1/2} = \frac{1}{2}g^{1/2}g^{\mu\nu}\delta g_{\mu\nu}$, Eq. (7.6), and the definition of $\langle \underline{T}^{\mu\nu}\rangle_{\rm div}$ gives

$$\frac{1}{2\pi^2}\frac{1}{(\sigma^\rho\sigma_\rho)^2}\left(g^{\mu\nu} - 4\frac{\sigma^\mu\sigma^\nu}{\sigma^\rho\sigma_\rho}\right) - \frac{1}{8\pi^2}\frac{1}{(\sigma^\rho\sigma_\rho)}m^2\left(g^{\mu\nu} - 2\frac{\sigma^\mu\sigma^\nu}{\sigma^\rho\sigma_\rho}\right)$$
$$- \frac{1}{32\pi^2}m^4g^{\mu\nu}(\gamma + \tfrac{1}{2}\ln|\tfrac{1}{2}m^2\sigma) + \frac{3}{128\pi^2}m^4\left(g^{\mu\nu} - \frac{4}{3}\frac{\sigma^\mu\sigma^\nu}{\sigma^\rho\sigma_\rho}\right) + \text{higher-order terms in } \sigma^\rho\,,$$

which are exactly the same as the terms which appear in Sec. V. The anomaly comes from

$$-\frac{1}{32\pi^2}\lim_{x'\to x}\int g^{1/2}a_2\ln|\tfrac{1}{2}m^2\sigma|d^4x\,.$$

Varying $g_{\mu\nu}$ in this term gives one finite term from

$$-\frac{1}{32\pi^2}\lim_{x'\to x}\int g^{1/2}a_2\delta\ln|\tfrac{1}{2}m^2\sigma|d^4x$$
$$= -\frac{1}{32\pi^2}\lim_{x'\to x}\int g^{1/2}a_2(x,x')\frac{\sigma^\mu\sigma^\nu}{\sigma^\rho\sigma_\rho}\delta g_{\mu\nu}d^4x\,.$$

So the trace term in the finite part of $\langle T^{\mu\nu}\rangle_{\rm div}$ is

$$-\frac{1}{16\pi^2}\lim_{x'\to x}a_2(x,x')\frac{\sigma^\mu\sigma^\nu}{\sigma^\rho\sigma_\rho}\,,$$

exactly as in Sec. V. In the massless case, this is the *only* finite non-trace-free term which arises from the functional derivative of $L_{\rm div}$. The rest of the terms in Sec. V come from the expansions of $D^{1/2}$, a_1, a_2, and $\delta\sigma$ of higher powers in σ^ρ. As far as we know, the expansion for $\delta\sigma$ has not been calculated. We hope to carry this out in later research just to prove that it can be done and find $\langle \underline{T}^{\mu\nu}\rangle_{\rm div}$ in another way.

Finally, a few comments on $\langle \underline{T}^{\mu\nu}\rangle_{\rm div}$ in other dimensions. In Eq. (6.6) we give the expression for $G^{(1)}$ in arbitrary dimension. To find $\langle \underline{T}^{\mu\nu}\rangle_{\rm div}$ we simply follow the same procedure as we did in four dimensions. We need only be careful to take into account terms such as $\xi(n)$ and remember that $g^{\mu\nu}g_{\mu\nu} = n$. In dimensions higher than four, the calculation would be extremely long and probably would have to be done on a computer.

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APPENDIX

Here we present the expansions of the various bitensors used in the text which did not appear in Ref. 9. The methods used to find these expansions can be found in Refs. 9 and 29:

$$g_\mu{}^{\lambda'}\Delta^{1/2}{}_{;\lambda'} = -\tfrac{1}{6}R_{\mu\rho}\sigma^\rho + \tfrac{1}{24}(2R_{\mu\rho;\tau}+R_{\rho\tau;\mu})\sigma^\rho\sigma^\tau - (\tfrac{1}{40}R_{\mu\rho;\tau\kappa}+\tfrac{1}{40}R_{\rho\tau;\mu\kappa}+\tfrac{1}{90}R_\mu{}^\lambda{}_\rho{}^\xi R_{\lambda\tau\xi\kappa}$$
$$+\tfrac{1}{72}R_{\mu\rho}R_{\tau\kappa}+\tfrac{1}{360}R_\rho{}^\lambda R_{\mu\tau\lambda\kappa})\sigma^\rho\sigma^\tau\sigma^\kappa+\cdots,$$

$$g_\nu{}^{\lambda'}\Delta^{1/2}{}_{;\mu\lambda'} = -\tfrac{1}{6}R_{\mu\nu}+\tfrac{1}{12}(R_{\mu\nu;\rho}+R_{\mu\rho;\nu}-R_{\nu\rho;\mu})\sigma^\rho$$
$$+(-\tfrac{1}{40}R_{\mu\nu;\rho\tau}+\tfrac{7}{240}R_{\rho\tau;\mu\nu}-\tfrac{1}{80}R_{\rho\tau;\mu\nu}-\tfrac{1}{20}R_{\mu\rho;\nu\tau}+\tfrac{1}{30}R_{\nu\rho;\mu\tau}-\tfrac{1}{72}R_{\mu\nu}R_{\rho\tau}$$
$$-\tfrac{1}{36}R_{\mu\rho}R_{\nu\tau}-\tfrac{1}{360}R_\mu{}^\lambda R_{\nu\rho\lambda\tau}-\tfrac{1}{360}R_\nu{}^\lambda R_{\mu\rho\lambda\tau}-\tfrac{1}{90}R_\mu{}^\lambda{}_\nu{}^\xi R_{\lambda\rho\xi\tau}$$
$$-\tfrac{1}{90}R_\mu{}^{\lambda\xi}{}_\rho R_{\nu\lambda\xi\tau}-\tfrac{1}{90}R_\mu{}^{\lambda\xi}{}_\rho R_{\nu\xi\lambda\tau}+\tfrac{19}{360}R_{\rho\lambda}R^\lambda{}_{\mu\nu\tau}-\tfrac{11}{360}R_{\rho\lambda}R^\lambda{}_{\nu\mu\tau})\sigma^\rho\sigma^\tau+\cdots,$$

$$g_\beta{}^{\lambda'}g_{\alpha\lambda'}=g_{\alpha\beta},$$

$$g_\beta{}^{\lambda'}g_{\alpha\lambda';\mu} = -\tfrac{1}{2}R_{\alpha\beta\mu\rho}\sigma^\rho+\tfrac{1}{6}R_{\alpha\beta\mu\rho;\tau}\sigma^\rho\sigma^\tau-\tfrac{1}{24}(R_{\alpha\beta\mu\rho;\tau\kappa}+R_{\alpha\beta\lambda\rho}R^\lambda{}_{\tau\mu\kappa})\sigma^\rho\sigma^\tau\sigma^\kappa+\cdots,$$

$$g_\beta{}^{\lambda'}g_\mu{}^{\xi'}g_{\alpha\lambda';\xi'} = -\tfrac{1}{2}R_{\alpha\beta\mu\rho}\sigma^\rho+\tfrac{1}{3}R_{\alpha\beta\mu\rho;\tau}\sigma^\rho\sigma^\tau-\tfrac{1}{24}(3R_{\alpha\beta\mu\rho;\tau\kappa}+R_{\alpha\beta\lambda\rho}R^\lambda{}_{\tau\mu\kappa})\sigma^\rho\sigma^\tau\sigma^\kappa+\cdots,$$

$$g_\beta{}^{\lambda'}g_\nu{}^{\xi'}g_{\alpha\lambda';\mu\xi'} = \tfrac{1}{2}R_{\alpha\beta\mu\nu}-\tfrac{1}{6}(R_{\alpha\beta\mu\rho;\nu}+R_{\alpha\beta\mu\nu;\rho})\sigma^\rho+(\tfrac{1}{12}R_{\alpha\beta\mu\rho;\nu\tau}+\tfrac{1}{24}R_{\alpha\beta\mu\nu;\rho\tau}+\tfrac{5}{24}R_{\alpha\lambda\mu\rho}R^\lambda{}_{\beta\nu\tau}-\tfrac{1}{24}R_{\lambda\beta\mu\rho}R^\lambda{}_{\alpha\nu\tau}$$
$$+\tfrac{1}{24}R_{\alpha\beta\mu\lambda}R^\lambda{}_{\rho\nu\tau}-\tfrac{1}{24}R_{\alpha\beta\nu\lambda}R^\lambda{}_{\rho\mu\tau}+\tfrac{1}{12}R_{\alpha\beta\lambda\rho}R^\lambda{}_{\tau\mu\nu})\sigma^\rho\sigma^\tau+\cdots,$$

$$g_\beta{}^{\lambda'}a_{1\alpha\lambda'} = (\tfrac{1}{6}Rg_{\alpha\beta}-R_{\alpha\beta})+(-\tfrac{1}{12}R_{;\rho}g_{\alpha\beta}+\tfrac{1}{6}R_{\alpha\beta\rho}{}^\lambda{}_{;\lambda}+\tfrac{1}{2}R_{\alpha\beta;\rho})\sigma^\rho$$
$$+[-\tfrac{1}{6}R_{\alpha\beta;\rho\tau}-\tfrac{1}{12}R_{\alpha\beta}{}^\lambda{}_{\rho;\lambda\tau}-\tfrac{1}{12}R_{\lambda\alpha\xi\rho}R^\lambda{}_\beta{}^\xi{}_\tau+\tfrac{1}{6}(\tfrac{3}{20}R_{;\rho\tau}+\tfrac{1}{20}\Box R_{\rho\tau}-\tfrac{1}{15}R_{\rho\lambda}R^\lambda{}_\tau+\tfrac{1}{30}R^{\lambda\xi}R_{\lambda\rho\xi\tau}$$
$$+\tfrac{1}{30}R^{\lambda\xi\kappa}{}_\rho R_{\lambda\xi\kappa\tau})g_{\alpha\beta}]\sigma^\rho\sigma^\tau+\cdots,$$

$$g_\beta{}^{\lambda'}a_{1\alpha\lambda';\mu} = (\tfrac{1}{12}R_{;\mu}g_{\alpha\beta}-\tfrac{1}{2}R_{\alpha\beta;\mu}+\tfrac{1}{6}R_{\alpha\beta\mu}{}^\lambda{}_{;\lambda})$$
$$+[\tfrac{1}{6}R_{\alpha\beta;\mu\rho}-\tfrac{1}{12}RR_{\alpha\beta\mu\rho}+\tfrac{1}{6}R_\alpha{}^\lambda R_{\lambda\beta\mu\rho}-\tfrac{1}{3}R_\beta{}^\lambda R_{\lambda\alpha\mu\rho}+\tfrac{1}{12}(R_{\alpha\beta}{}^\lambda{}_{\mu;\lambda\rho}-R_{\alpha\beta}{}^\lambda{}_{\rho;\lambda\mu})$$
$$-\tfrac{1}{12}(R_{\lambda\beta\xi\mu}R^\lambda{}_\alpha{}^\xi{}_\rho+R_{\lambda\beta\xi\rho}R^\lambda{}_\alpha{}^\xi{}_\mu)+\tfrac{1}{3}(-\tfrac{1}{10}R_{;\mu\rho}+\tfrac{1}{20}\Box R_{\mu\rho}-\tfrac{1}{15}R_{\mu\lambda}R^\lambda{}_\rho$$
$$+\tfrac{1}{30}R^{\lambda\xi}R_{\lambda\mu\xi\rho}+\tfrac{1}{30}R^{\lambda\xi\kappa}{}_\mu R_{\lambda\xi\kappa\rho})g_{\alpha\beta}]\sigma^\rho+\cdots,$$

$$g_\beta{}^{\lambda'}g_\mu{}^{\xi'}a_{1\alpha\lambda';\xi'} = (\tfrac{1}{12}R_{;\mu}g_{\alpha\beta}-\tfrac{1}{12}R_{\alpha\beta;\mu}-\tfrac{1}{6}R_{\alpha\beta\mu}{}^\lambda{}_{;\lambda})$$
$$+[\tfrac{1}{3}R_{\alpha\beta;\mu\rho}-\tfrac{1}{12}RR_{\alpha\beta\mu\rho}+\tfrac{1}{3}R_{\alpha\lambda}R^\lambda{}_{\beta\mu\rho}-\tfrac{1}{6}R_{\beta\lambda}R^\lambda{}_{\alpha\mu\rho}-\tfrac{1}{12}(R_{\alpha\beta}{}^\lambda{}_{\mu;\lambda\rho}+R_{\alpha\beta}{}^\lambda{}_{\rho;\lambda\mu})$$
$$+\tfrac{1}{12}(R_{\lambda\beta\xi\mu}R^\lambda{}_\alpha{}^\xi{}_\rho+R_{\lambda\beta\xi\rho}R^\lambda{}_\alpha{}^\xi{}_\nu)-\tfrac{1}{3}(\tfrac{3}{20}R_{;\mu\rho}+\tfrac{1}{20}\Box R_{\mu\rho}-\tfrac{1}{15}R_{\mu\lambda}R^\lambda{}_\rho$$
$$+\tfrac{1}{30}R^{\lambda\xi}R_{\lambda\mu\xi\rho}+\tfrac{1}{30}R^{\lambda\xi\kappa}{}_\mu R_{\lambda\xi\kappa\rho})g_{\alpha\beta}]\sigma^\rho+\cdots,$$

$$g_\beta{}^{\lambda'}g_\nu{}^{\xi'}a_{1\alpha\lambda';\mu\xi'} = [-\tfrac{1}{6}R_{\alpha\beta;\mu\nu}+\tfrac{1}{12}RR_{\alpha\beta\mu\nu}-\tfrac{1}{6}R_{\alpha\lambda}R^\lambda{}_{\beta\mu\nu}+\tfrac{1}{3}R_{\beta\lambda}R^\lambda{}_{\alpha\mu\nu}$$
$$+\tfrac{1}{12}(R_{\alpha\beta}{}^\lambda{}_{\nu;\lambda\mu}-R_{\alpha\beta}{}^\lambda{}_{\mu;\lambda\nu})+\tfrac{1}{12}(R_{\lambda\beta\xi\mu}R^\lambda{}_\alpha{}^\xi{}_\nu+R_{\lambda\beta\xi\nu}R^\lambda{}_\alpha{}^\xi{}_\mu)$$
$$-\tfrac{1}{3}(-\tfrac{1}{10}R_{;\mu\nu}+\tfrac{1}{20}\Box R_{\mu\nu}-\tfrac{1}{15}R_{\mu\lambda}R^\lambda{}_\nu+\tfrac{1}{30}R^{\lambda\xi}R_{\lambda\mu\xi\nu}+\tfrac{1}{30}R^{\lambda\xi\kappa}{}_\mu R_{\lambda\xi\kappa\nu})g_{\alpha\beta}]+\cdots,$$

$$g_\beta{}^{\lambda'}a_{2\alpha\lambda'} = [-\tfrac{1}{6}RR_{\alpha\beta}-\tfrac{1}{6}\Box R_{\alpha\beta}+\tfrac{1}{2}R_{\alpha\lambda}R^\lambda{}_\beta-\tfrac{1}{12}R^{\lambda\xi\kappa}{}_\alpha R_{\lambda\xi\kappa\beta}+(\tfrac{1}{72}R^2+\tfrac{1}{30}\Box R-\tfrac{1}{180}R^{\lambda\xi}R_{\lambda\xi}+\tfrac{1}{180}R^{\lambda\xi\kappa\iota}R_{\lambda\xi\kappa\iota})g_{\alpha\beta}]+\cdots,$$

$$\mathcal{G}^2=\underline{1},$$

$$\mathcal{G}\mathcal{G}_{;\mu} = -\tfrac{1}{4}G_{[\alpha\beta]}R^{\alpha\beta}{}_{\mu\rho}\sigma^\rho+\tfrac{1}{12}G_{[\alpha\beta]}R^{\alpha\beta}{}_{\mu\rho;\tau}\sigma^\rho\sigma^\tau-\tfrac{1}{48}G_{[\alpha\beta]}(R^{\alpha\beta}{}_{\mu\rho;\tau\kappa}+R^{\alpha\beta}{}_{\lambda\rho}R^\lambda{}_{\tau\mu\kappa})\sigma^\rho\sigma^\tau\sigma^\kappa+\cdots,$$

$$g_\mu{}^{\lambda'}\mathcal{G}\mathcal{G}_{;\lambda'} = -\tfrac{1}{4}G_{[\alpha\beta]}R^{\alpha\beta}{}_{\mu\rho}\sigma^\rho+\tfrac{1}{6}G_{[\alpha\beta]}R^{\alpha\beta}{}_{\mu\rho;\tau}\sigma^\rho\sigma^\tau-\tfrac{1}{48}G_{[\alpha\beta]}(3R^{\alpha\beta}{}_{\mu\rho;\tau\kappa}+R^{\alpha\beta}{}_{\lambda\rho}R^\lambda{}_{\tau\mu\kappa})\sigma^\rho\sigma^\tau\sigma^\kappa+\cdots,$$

$$\mathcal{G}\mathcal{G}_{;\mu\nu} = -\tfrac{1}{4}G_{[\alpha\beta]}R^{\alpha\beta}{}_{\mu\nu}+\tfrac{1}{12}G_{[\alpha\beta]}(R^{\alpha\beta}{}_{\mu\nu;\rho}-2R^{\alpha\beta}{}_{\mu\rho;\nu})\sigma^\rho+[-\tfrac{1}{48}G_{[\alpha\beta]}(R^{\alpha\beta}{}_{\mu\nu;\rho\tau}-2R^{\alpha\beta}{}_{\mu\rho;\nu\tau}+R^{\alpha\beta}{}_{\lambda\mu}R^\lambda{}_{\rho\nu\tau}+R^{\alpha\beta}{}_{\lambda\nu}R_{\rho\mu\tau}$$
$$+2R^{\alpha\beta}{}_{\lambda\rho}R^\lambda{}_{\nu\mu\tau}+2R^{\alpha\beta}{}_{\lambda\rho}R^\lambda{}_{\mu\nu\tau})$$
$$+\tfrac{1}{32}G_{[\alpha\beta]}G_{[\gamma\delta]}(R^{\alpha\beta}{}_{\mu\rho}R^{\gamma\delta}{}_{\nu\tau}+R^{\alpha\beta}{}_{\nu\rho}R^{\gamma\delta}{}_{\mu\tau})]\sigma^\rho\sigma^\tau+\cdots,$$

$$\mathcal{G}g_\nu{}^{\lambda'}\mathcal{G}_{;\mu\lambda'} = \tfrac{1}{4}G_{[\alpha\beta]}R^{\alpha\beta}{}_{\mu\nu}-\tfrac{1}{12}G_{[\alpha\beta]}(R^{\alpha\beta}{}_{\mu\nu;\rho}+R^{\alpha\beta}{}_{\mu\rho;\nu})\sigma^\rho$$
$$+[\tfrac{1}{48}G_{[\alpha\beta]}(R^{\alpha\beta}{}_{\mu\nu;\rho\tau}+10R^{\alpha\beta}{}_{\mu\rho;\nu\tau}-8R^{\alpha\beta}{}_{\mu\rho;\tau\nu}+R^{\alpha\beta}{}_{\lambda\mu}R^\lambda{}_{\rho\nu\tau}+R^{\alpha\beta}{}_{\lambda\nu}R^\lambda{}_{\rho\mu\tau}+2R^{\alpha\beta}{}_{\lambda\rho}R^\lambda{}_{\nu\mu\tau}+2R^{\alpha\beta}{}_{\lambda\rho}R^\lambda{}_{\mu\nu\tau})$$
$$-\tfrac{1}{32}G_{[\alpha\beta]}G_{[\gamma\delta]}(R^{\alpha\beta}{}_{\mu\rho}R^{\gamma\delta}{}_{\nu\tau}-3R^{\alpha\beta}{}_{\nu\rho}R^{\gamma\delta}{}_{\mu\tau})]\sigma^\rho\sigma^\tau+\cdots,$$

$$\mathcal{G}\mathcal{A}_1 = -\tfrac{1}{12}R\underline{1} + (\tfrac{1}{24}R_{;\rho}\underline{1} + \tfrac{1}{12}G_{[\alpha\beta]}R^{\alpha\beta}{}_{\rho}{}^{\lambda}{}_{;\lambda})\sigma^{\rho} + [\tfrac{1}{24}G_{[\alpha\beta]}R^{\alpha\beta\lambda}{}_{\rho;\lambda\tau} + \tfrac{1}{48}G_{[\alpha\beta]}G_{[\gamma\delta]}R^{\alpha\beta\lambda}{}_{\rho}R^{\gamma\delta}{}_{\lambda\tau}$$
$$+ (-\tfrac{1}{60}R_{;\rho\tau} + \tfrac{1}{120}\Box R_{\rho\tau} - \tfrac{1}{90}R_{\rho\lambda}R^{\lambda}{}_{\tau} + \tfrac{1}{180}R^{\lambda\xi}R_{\lambda\rho\xi\tau} + \tfrac{1}{180}R^{\lambda\xi\kappa}{}_{\rho}R_{\lambda\xi\kappa\tau})\underline{1}]\sigma^{\rho}\sigma^{\tau}$$
$$+ \cdots,$$

$$\mathcal{G}\mathcal{A}_{1;\mu} = (-\tfrac{1}{24}R_{;\mu}\underline{1} + \tfrac{1}{12}G_{[\alpha\beta]}R^{\alpha\beta}{}_{\mu}{}^{\lambda}{}_{;\lambda}) + [\tfrac{1}{48}G_{[\alpha\beta]}(RR^{\alpha\beta}{}_{\mu\rho} + 2R^{\alpha\beta\lambda}{}_{\mu;\lambda\rho} - 2R^{\alpha\beta\lambda}{}_{\rho;\lambda\mu})$$
$$+ \tfrac{1}{48}G_{[\alpha\beta]}G_{[\gamma\delta]}(R^{\alpha\beta\lambda}{}_{\mu}R^{\gamma\delta}{}_{\lambda\rho} + R^{\alpha\beta\lambda}{}_{\rho}R^{\gamma\delta}{}_{\lambda\mu})$$
$$+ (\tfrac{1}{120}R_{;\mu\rho} + \tfrac{1}{60}\Box R_{\mu\rho} - \tfrac{1}{45}R_{\mu\lambda}R^{\lambda}{}_{\rho} + \tfrac{1}{90}R^{\lambda\xi}R_{\lambda\mu\xi\rho} + \tfrac{1}{90}R^{\lambda\xi\kappa}{}_{\mu}R_{\lambda\xi\kappa\rho})\underline{1}]\sigma^{\rho} + \cdots,$$

$$g_{\mu}{}^{\lambda'}\mathcal{G}\mathcal{A}_{1;\lambda'} = (-\tfrac{1}{24}R_{;\mu}\underline{1} - \tfrac{1}{12}G_{[\alpha\beta]}R^{\alpha\beta}{}_{\mu}{}^{\lambda}{}_{;\lambda}) + [\tfrac{1}{48}G_{[\alpha\beta]}(RR^{\alpha\beta}{}_{\mu\rho} - 2R^{\alpha\beta\lambda}{}_{\mu;\lambda\rho} - 2R^{\alpha\beta\lambda}{}_{\rho;\lambda\mu})$$
$$- \tfrac{1}{48}G_{[\alpha\beta]}G_{[\gamma\delta]}(R^{\alpha\beta\lambda}{}_{\mu}R^{\gamma\delta}{}_{\lambda\rho} + R^{\alpha\beta\lambda}{}_{\rho}R^{\gamma\delta}{}_{\lambda\mu})$$
$$- (-\tfrac{1}{30}R_{;\mu\rho} + \tfrac{1}{60}\Box R_{\mu\rho} - \tfrac{1}{45}R_{\mu\lambda}R^{\lambda}{}_{\rho} + \tfrac{1}{90}R^{\lambda\xi}R_{\lambda\mu\xi\rho} + \tfrac{1}{90}R^{\lambda\xi\kappa}{}_{\mu}R_{\lambda\xi\kappa\rho})\underline{1}]\sigma^{\rho}$$
$$+ \cdots,$$

$$g_{\nu}{}^{\lambda'}\mathcal{G}\mathcal{A}_{1;\mu\lambda'} = [-\tfrac{1}{48}G_{[\alpha\beta]}(RR^{\alpha\beta}{}_{\nu\mu} + 2R^{\alpha\beta\lambda}{}_{\mu;\lambda\nu} + 2R^{\alpha\beta\lambda}{}_{\nu;\lambda\mu}) - \tfrac{1}{48}G_{[\alpha\beta]}G_{[\gamma\delta]}(R^{\alpha\beta\lambda}{}_{\mu}R^{\gamma\delta}{}_{\lambda\nu} + R^{\alpha\beta\lambda}{}_{\nu}R^{\gamma\delta}{}_{\lambda\mu})$$
$$- (\tfrac{1}{120}R_{;\mu\nu} + \tfrac{1}{60}\Box R_{\mu\nu} - \tfrac{1}{45}R_{\mu\lambda}R^{\lambda}{}_{\nu} + \tfrac{1}{90}R^{\lambda\xi}R_{\lambda\mu\xi\nu} + \tfrac{1}{90}R^{\lambda\xi\kappa}{}_{\mu}R_{\lambda\xi\kappa\nu})\underline{1}] + \cdots,$$

$$\mathcal{G}\mathcal{A}_{1;\mu\nu} = [\tfrac{1}{48}G_{[\alpha\beta]}(RR^{\alpha\beta}{}_{\mu\nu} - 2R^{\alpha\beta\lambda}{}_{\mu;\lambda\nu} - 2R^{\alpha\beta\lambda}{}_{\nu;\lambda\mu}) + \tfrac{1}{48}G_{[\alpha\beta]}G_{[\gamma\delta]}(R^{\alpha\beta\lambda}{}_{\mu}R^{\gamma\delta}{}_{\lambda\nu} + R^{\alpha\beta\lambda}{}_{\nu}R^{\gamma\delta}{}_{\lambda\mu})$$
$$+ (-\tfrac{1}{30}R_{;\mu\nu} + \tfrac{1}{60}\Box R_{\mu\nu} - \tfrac{1}{45}R_{\mu\lambda}R^{\lambda}{}_{\nu} + \tfrac{1}{90}R^{\lambda\xi}R_{\lambda\mu\xi\nu} + \tfrac{1}{90}R^{\lambda\xi\kappa}{}_{\mu}R_{\lambda\xi\kappa\nu})\underline{1}] + \cdots,$$

$$\mathcal{G}\mathcal{A}_2 = [(\tfrac{1}{288}R^2 - \tfrac{1}{120}\Box R - \tfrac{1}{180}R^{\lambda\xi}R_{\lambda\xi} + \tfrac{1}{180}R^{\lambda\xi\kappa\iota}R_{\lambda\xi\kappa\iota})\underline{1} + \tfrac{1}{48}G_{[\alpha\beta]}G_{[\gamma\delta]}R^{\alpha\beta\lambda\xi}R^{\gamma\delta}{}_{\lambda\xi}] + \cdots.$$

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[17]Reference 14, pp. 121, 122, 139, 140, 158, 232, and 233.

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[20]Recently some questions have arisen as to thè mathematical rigor of the steps leading to Eq. (3.3). Work is in progress, and it is hoped that it will provide all of the distribution theory details involved. Some preliminary discussion of point separation as distribution theory will be presented in T. S. Bunch, S. M. Christensen, and S. A. Fulling (unpublished).

[21]More discussion of the bitensors used here may be found in Refs. 1, 5, 9, and 13.

[22]G. N. Watson, *Theory of Bessels Functions* (Cambridge Univ. Press, New York, 1944). See pp. 178 and 179.

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[24]Script symbols are always bispinors.

[25]Occasionally $\langle \underline{T}^{\mu\nu}\rangle_{\text{matrix}}$, properly renormalized, is used as a source in Einstein's equations. Since $\langle \underline{T}^{\mu\nu}\rangle_{\text{matrix}}$ is in general a complex quantity it can lead to complex metrics whose physical interpretation is unclear. However, the vacuum expectation value of $\underline{T}^{\mu\nu}$ in some physically meaningful state is always real and has a well-defined physical meaning in terms of densities and fluxes.

[26]Reference 1, pp. 336-340.

[27]Reference 1, pp. 340-356.

[28]Reference 13, DeWitt has shown that the $L_{\mathrm{eff,ren}}$ obtained by Brown is identical to the one derived from Eqs. (4.11) - (4.13).

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[38]There are many other useful papers, letters, etc. which are not listed here. The references listed are those that were referred to over and over in the course of this research. For other relevant works, see the references in the papers listed here.