Anisotropy damping through quantum effects in the early universe*

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We consider a quantized field in a Bianchi type-I anisotropically expanding universe. A suitable expectation value of the renormalized energy-momentum tensor acts as the source of the metric in the Einstein equations. The coupled set of differential equations is numerically integrated, with the help of several approximations, in the case when the quantized field is the massless conformal scalar field. Boundary conditions are imposed at an initial time t_0 of the order of the Planck time, with the initial expansion rates varying over a wide range consistent with the constraints. It is found that the expansion rates tend toward isotropy and approach a radiation-filled Friedmann expansion in an interval of less than 10^3 Planck times, for the full range of initial expansion rates considered.

I. INTRODUCTION

Quantum phenomena, such as particle creation resulting from the strong gravitational field near the cosmological singularity, may have a profound influence on the evolution of the metric at very early times. Here we investigate the question whether such a process could have brought about isotropization of an initially anisotropic expansion at a sufficiently early time.

Studies of helium formation[1-4] indicate that the expansion was effectively isotropic at $t \lesssim 10^{-1}$ sec. The isotropy of the observed cosmic blackbody radiation ($\Delta T/T \lesssim 10^{-3}$) has served as the basis for a number of papers implying limits on the time of isotropization.[5-10] Calculations in Refs. 8–10, based on evolutionary models involving classical fluids, have concluded that isotropization of the expansion must have occurred by as early a time as $t \lesssim 10^{-36}$ sec, a time which is close to the Planck time $t_P = (G\hbar/c^5)^{1/2} \sim 5\times10^{-44}$ sec. (Other studies of anisotropic models with fluids may be found in Refs. 11–19.)

The desire to avoid postulating special initial conditions, as well as the existence of particle horizons in isotropic models, suggested the study of inhomogeneous, anisotropic universes.[20] To bring about the observed isotropy in such models at a sufficiently early time requires a dynamical mechanism for damping inhomogeneity and anisotropy. One such dissipative mechanism is neutrino viscosity, which was investigated in homogeneous Bianchi type-I, -V, and -IX cosmologies, and was found not to be rapid enough to bring about isotropy at a sufficiently early stage.[20-23]

Another mechanism coming into play at much earlier times ($t \sim t_P$) is the production of elementary particles by the expansion of the universe.[24] Zeldovich[25] suggested that this process would bring about isotropy near the Planck time. Quantum aspects of particle production and renormalization of the energy-momentum tensor in Bianchi type-I and -IX universes were studied by various authors.[26-30] The reaction back on the metric of the created particles (treated as a classical relativistic gas) has been studied by Lukash and Starobinsky.[31] They assumed that the particle creation occurred at a time t_0 large with respect to t_P, so that the evolution of the metric at times near t_0 could be treated independently of the created particles [their models evolved as a Kasner $(\frac{2}{3}, \frac{2}{3}, -\frac{1}{3})$ expansion until the energy density of the created particles began to bring about isotropization at a later stage]. If their results are extrapolated back to $t_0 \sim t_P$, then they indicate that rapid isotropization should occur. In order to consider directly models in which $t_0 \sim t_P$, one would have to use a renormalized expression for the expectation value of the energy-momentum tensor of the quantized matter fields. (Eventually, one would also hope to include quantum gravity as more than a quantized perturbation on a classical background metric.)

In the present paper, we consider a Bianchi type-I universe in which the source of the gravitational field is the renormalized energy-momentum tensor of a quantized matter field. In this case, the conformally invariant massless scalar field is used, but similar results are expected for other fields. One wishes to solve self-consistently the coupled scalar wave equation and the Einstein equation. We solve the scalar wave equation in a low-frequency or early-time approximation and explicitly evaluate the mode sums appearing in the renormalized energy-momentum tensor. The resulting Einstein equations are then numerically integrated and the times t_F of effective isotropization are determined for various sets of initial conditions. There seems to be no natural unique choice of state at the initial time t_0. We choose the state vector such that if the initial expansion rates were chosen isotropically at t_0, then the energy density would vanish and remain zero (such a state is possible for the conformed field). We find for $t_0 \lesssim 3t_P$ that $t_F \lesssim 2\times10^3 t_P$, where t_F is the time of effective isotropization (as defined in Sec. IV). This result is largely independent of the initial values of the expansion rates and indicates that the particles creation mechanism can isotropize the expansion in these Bianchi type-I models at a sufficiently early time to be consistent with the observational limits.

In Sec. II, we give the gravitational field equations in a Bianchi type-I universe, with an expectation value of the renormalized energy-momentum tensor of quantized fields acting as the source. In Sec. III, we specialize to the conformal scalar field, and carry out the mode sums in the renormalized energy-momentum tensor, using a low-frequency or early-time approximation to the solutions of the scalar wave equation. In Sec. IV, we numerically integrate the Einstein equations for various sets of initial conditions, and present the results in the accompanying figures and tables.

II. GRAVITATIONAL FIELD EQUATIONS

In the semiclassical theory the Einstein equations take the form[32].

$$G_\mu{}^\nu = -8\pi\langle T_\mu{}^\nu\rangle \ , \tag{2.1}$$

where $G_\mu{}^\nu$ is the Einstein tensor formed from the classical metric and $\langle T_\mu{}^\nu\rangle$ denotes a suitable expectation value of the energy-momentum tensor of the quantized particle fields under consideration. Although the use of $\langle T_\mu{}^\nu\rangle$ as the source of the gravitational field in a semiclassical approximation may not be justified when the probable (in the quantum sense) matter distributions differ greatly from their average, its use appears to be correct in the present cosmological context. In general, the expectation values of the formal energy-momentum tensors are not well defined and must be renormalized. The considerations of the present section do not depend on the renormalization procedure, or the particular quantized fields present.

We are considering metrics of the form

$$ds^2 = dt^2 - a_1{}^2(t)\,(dx^1)^2 - a_2{}^2(t)(dx^2)^2 - a_3{}^2(t)(dx^3)^2 \ . \tag{2.2}$$

In a homogeneous state which has the symmetry of the metric, the nonvanishing components of $\langle T_\mu{}^\nu\rangle$ are

$$\langle T_0{}^0\rangle = \rho, \quad \langle T_i{}^j\rangle = -p_i\delta_i{}^j \ , \tag{2.3}$$

where ρ is the energy density and the p_i are the principal pressures [no sum on i in (2.3)]. The Einstein equations (2.1) take the form

$$-G_0{}^0 = V^{-2/3}(\alpha_1'\alpha_2' + \alpha_1'\alpha_3' + \alpha_2'\alpha_3') = 8\pi\rho \ , \tag{2.4}$$

$$-G_1{}^1 = V^{-2/3}\left[-\left(\frac{a_2''}{a_2}+\frac{a_3''}{a_3}\right) + \tfrac{1}{3}(\alpha_2')^2 + \tfrac{1}{3}(\alpha_3')^2 + \tfrac{1}{3}(\alpha_1'\alpha_2' + \alpha_1'\alpha_3' - \alpha_2'\alpha_3')\right] = 8\pi p_1 \ . \tag{2.5}$$

[The equations involving p_2 and p_3 are obtained from (2.5) by cyclic interchange of indices 1, 2, 3.] Here a prime denotes derivative with respect to

$$\eta = \int^t V^{-1/3} dt' \tag{2.6}$$

and

$$\alpha_j = \ln a_j \ , \tag{2.7}$$

$$V = a_1 a_2 a_3 \ . \tag{2.8}$$

One may regard Eq. (2.4) as a constraint to be satisfied at an initial time t_0. The dynamical equations (2.5) may be rewritten in a convenient form (for numerical integration) containing only one second time derivative, as

$$V^{2/3}R_1{}^1 = \alpha_1'' + \tfrac{2}{3}\alpha_1'(\alpha_1' + \alpha_2' + \alpha_3') = 8\pi(p_1 + \tfrac{1}{2}T)V^{2/3} \ , \tag{2.9}$$

and cyclic permutations of indices 1, 2, 3, where T is the renormalized trace of the energy-momentum tensor discussed below. The constraint Eq. (2.4) is consistently propagated by the dynamical equation, provided that

$$\langle T_\mu{}^\nu\rangle_{;\nu} = 0 \tag{2.10}$$

or

$$V^{-1}(V\rho)' + \sum_i \alpha_i' p_i = 0 \ . \tag{2.11}$$

For simplicity, we will assume that initially

$$a_1 = a_2, \quad p_1 = p_2 \ . \tag{2.12}$$

This axial symmetry will continue to hold at all times. The renormalized trace of the energy-momentum tensor is

$$\langle T_\mu{}^\mu\rangle = \rho - \sum_i p_i = T(\eta) \ . \tag{2.13}$$

In a theory in which $T_\mu{}^\mu$ vanishes on a classical level, a nonvanishing trace anomaly $T(\eta)$ may arise in the quantum theory as a result of renormalization. In the axially symmetric case, Eqs. (2.11) and (2.13) can be used to express p_1 and p_3 in terms of ρ, ρ', and T. Thus,

$$2(\alpha_3' - \alpha_1')p_1 = \rho' + 2(\alpha_1' + \alpha_3')\rho - \alpha_3' T \tag{2.14}$$

and

$$(\alpha_1' - \alpha_3')p_3 = \rho' + (3\alpha_1' + \alpha_3')\rho - \alpha_1' T . \tag{2.15}$$

With these equations, one can generate the regularized expressions for p_1 and p_3 from those for ρ and T in the anisotropically expanding universe ($\alpha_1' \neq \alpha_3'$). The form of the trace anomaly T for conformally invariant theories has been calculated by a number of authors.[33-42]

In anisotropically expanding universes we expect that for $t \gtrsim t_P$ the process of anisotropy damping is dominated by the created particles, rather than by other vacuum contributions to the energy-momentum tensor, which make up the trace anomaly. As we take $t_0 \lesssim t_P$ in these calculations, we have ignored the trace anomaly and set $T = 0$ throughout. However, it would be of interest to investigate the effect of the trace anomaly by direct calculation, especially for smaller values of t_0 (the quantum nature of the background metric would also be important for $t \lesssim t_P$). Our present results do not depend significantly on the particular form of the subtractions involved in the renormalization of $\langle T_\mu{}^\nu\rangle$, as described in later sections. (The possibility that including higher than second derivatives of the a_i in the subtractions may alter the fundamental character of the equations is not considered here.)

An explicit expression for the regularized energy density ρ of the quantized conformal scalar field in the metric of Eq. (2.2) has been obtained by means of adiabatic regularization (Ref. 29) and shown to agree with that obtained by "n-wave" regularization (Ref. 26). We use that expression here. Bunch[43] has shown for a Robertson-Walker metric that adiabatic regularization yields the same trace anomalies as the manifestly covariant techniques. Explicit expressions for the energy density in anisotropically expanding universes have not yet been obtained by other methods.

III. ENERGY-MOMENTUM TENSOR

We take the quantized massless conformal scalar field as the source of the gravitational field. The Lagrangian of the scalar field is

$$\mathcal{L} = \tfrac{1}{2}(-g)^{1/2}(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{6}R\Phi^2) . \tag{3.1}$$

The energy-momentum tensor is

$$T_{\mu\nu} = (\partial_\mu\phi)(\partial_\nu\phi) - \tfrac{1}{2}g_{\mu\nu}g^{\lambda\sigma}(\partial_\lambda\phi)(\partial_\sigma\phi) - \tfrac{1}{6}\nabla_\mu\partial_\nu(\phi^2) + \tfrac{1}{6}g_{\mu\nu}g^{\lambda\sigma}\nabla_\lambda\partial_\sigma(\phi^2) - \tfrac{1}{6}\phi^2 G_{\mu\nu} , \tag{3.2}$$

where ∇_μ denotes covariant derivative. One can write in the present metric

$$\phi = V^{-1/3}\chi , \tag{3.3}$$

with

$$\chi = (2\pi)^{-3/2}\int d^3k[A_{\vec{k}}\chi_{\vec{k}}(\eta)e^{i\vec{k}\cdot\vec{x}} + A_{\vec{k}}^\dagger\chi_{\vec{k}}^*(\eta)e^{-i\vec{k}\cdot\vec{x}}] . \tag{3.4}$$

The function $\chi_{\vec{k}}(\eta)$ satisfies

$$\chi_{\vec{k}}'' + (\Omega_{\vec{k}}^2 + Q)\chi_{\vec{k}} = 0 , \tag{3.5}$$

where

$$\Omega_{\vec{k}} = V^{1/3}\left(\sum k_i^2/a_i^2\right)^{1/2} \tag{3.6}$$

and

$$Q = \tfrac{1}{18}[(\alpha_1' - \alpha_2')^2 + (\alpha_1' - \alpha_3')^2 + (\alpha_2' - \alpha_3')^2] . \tag{3.7}$$

The operators $A_{\vec{k}}$, $A_{\vec{k}}^\dagger$ obey the usual commutation relations for annihilation and creation operators as a consequence of the canonical commutators of the field and its conjugate momentum, provided that the conserved Wronskian of Eq. (3.5) has the value

$$\chi_{\vec{k}}'^*\chi_{\vec{k}} - \chi_{\vec{k}}^*\chi_{\vec{k}}' = i . \tag{3.8}$$

Let $|0_A\rangle$ be the state annihilated by all the $A_{\vec{k}}$. The complete specification of $|0_A\rangle$ depends on the boundary conditions imposed on the functions $\chi_{\vec{k}}(\eta)$. All expectation values below will refer to the state $|0_A\rangle$. Boundary conditions will be imposed on the $\chi_{\vec{k}}$ later, by means of a physical argument.

The regularized energy density given in Eqs. (2.26) and (2.45) of Ref. 29 is

$$\rho = (16\pi^3)^{-1}V^{-4/3}\int d^3k\left\{|\chi_{\vec{k}}'|^2 + (\Omega_{\vec{k}}^2 - Q)|\chi_{\vec{k}}|^2 - \Omega_k - (2\Omega_{\vec{k}})^{-1}\left[\frac{1}{4}\left(\frac{\Omega'}{\Omega}\right)^2 - Q\right] + (2\Omega_{\vec{k}})^{-1}\left[\frac{1}{8}\left(\frac{\Omega'}{\Omega}\right)^2\epsilon_{2(2)} - \frac{1}{4}\frac{\Omega'}{\Omega}\epsilon_{2(3)}' - \tfrac{1}{4}\Omega^2\epsilon_{2(2)}{}^2 - \tfrac{1}{2}Q\epsilon_{2(2)}\right]\right\} , \tag{3.9}$$

where the expressions for $\epsilon_{2(j)}$ are defined and evaluated in Eqs. (B1) and (B2) of Ref. 29. The first two terms in Eq. (3.9) come from the formal expression for $\langle T_0{}^0\rangle$. The term $(-\Omega_{\vec{k}})$ removes the quartic divergence, while the next term removes the quadratic divergence, and the final term eliminates the logarithmic high frequency divergence. In the case when $m=0$, the adiabatic expansion upon which the final term was based is not valid for small $\vec{k}$ and it in fact introduces an evidently spurious logarithmic infrared divergence. That term will therefore not be subtracted in the low-frequency modes in our work below so that the infrared behavior is determined by the original formal contribution to ρ. This procedure has the additional feature of not introducing higher than second derivatives of the a_j into the Einstein equations. The quartically divergent term $\Omega_{\vec{k}}$ is subtracted at low frequencies even in flat-spacetime renormalization. To explore the question whether the quadratically divergent terms in Eq. (3.9) should be subtracted at low frequencies, we have performed our calculations with and without those subtractions and find that the orders of magnitude of the anisotropy damping times are not influenced by those terms. For convenience of exposition we will include the quadratically divergent subtractions in all equations below, rather than explicitly consider each possibility.

To facilitate integration of the coupled Einstein-particle field equations, we will make use of a low-frequency or early-time approximation for $\chi_{\vec{k}}$ given in Ref. 26. We will then be able to obtain an expression for ρ in terms of the unspecified expansion functions $a_j(\eta)$. First one reduces Eq. (3.5) to two first-order differential equations by defining the functions $\alpha_{\vec{k}}(\eta)$ and $\beta_{\vec{k}}(\eta)$ through

$$\chi_{\vec{k}}=(2\Omega_{\vec{k}})^{-1/2}\left[\alpha_{\vec{k}}\exp\left(-i\int^{\eta}\Omega_{\vec{k}}d\eta'\right)+\beta_{\vec{k}}\exp\left(i\int^{\eta}\Omega_{\vec{k}}d\eta'\right)\right] \tag{3.10}$$

and

$$\chi'_{\vec{k}}=-i(\Omega_{\vec{k}}/2)^{1/2}\left[\alpha_k\exp\left(-i\int^{\eta}\Omega_{\vec{k}}d\eta'\right)-\beta_{\vec{k}}\exp\left(i\int^{\eta}\Omega_{\vec{k}}d\eta'\right)\right]. \tag{3.11}$$

Then Eq. (3.8) becomes

$$|\alpha_{\vec{k}}(\eta)|^2-|\beta_{\vec{k}}(\eta)|^2=1, \tag{3.12}$$

and Eq. (3.5), together with the equation obtained by comparing (3.10) and (3.11), leads to

$$\alpha'=\frac{1}{2}\left(\frac{\Omega'}{\Omega}-i\frac{Q}{\Omega}\right)\beta\exp\left(2i\int_{\eta_0}^{\eta}\Omega d\eta'\right)-i\frac{Q}{2\Omega}\alpha \tag{3.13}$$

and

$$\beta'=\frac{1}{2}\left(\frac{\Omega'}{\Omega}+i\frac{Q}{\Omega}\right)\alpha\exp\left(-2i\int_{\eta_0}^{\eta}\Omega d\eta'\right)+i\frac{Q}{2\Omega}\beta, \tag{3.14}$$

where the subscript $\vec{k}$ has been dropped. At early times (η near η_0) or low frequencies, such that $\exp(2i\int_{\eta_0}^{\eta}\Omega d\eta')\approx 1$, one finds that the general solution of Eqs. (3.13) and (3.14) is

$$\alpha=c_1\left(\Omega^{1/2}-i\Omega^{-1/2}\int_{\eta_0}^{\eta}Qd\eta'\right)+c_2\Omega^{-1/2}, \tag{3.15}$$

$$\beta=c_1\left(\Omega^{1/2}+i\Omega^{-1/2}\int_{\eta_0}^{\eta}Qd\eta'\right)-c_2\Omega^{-1/2}, \tag{3.16}$$

where $c_1(\vec{k})$ and $c_2(\vec{k})$ are complex numbers, which by virtue of Eq. (3.12) satisfy

$$c_1c_2^*+c_2c_1^*=\tfrac{1}{2}. \tag{3.17}$$

In this approximation, one finds that

$$|\chi'|^2+(\Omega^2-Q)|\chi|^2=2|c_1|^2\left[(\Omega^2-Q)+\left(\int_{\eta_0}^{\eta}Qd\eta'\right)^2\right]+2|c_2|^2+2i(c_1^*c_2-c_1c_2^*)\left(\int_{\eta_0}^{\eta}Qd\eta'\right). \tag{3.18}$$

Thus,

$$\rho(t)=(16\pi^3)^{-1}V^{-4/3}\int_{q(t)}d^3k\left\{2|c_1|^2\left[(\Omega^2-Q)+\left(\int_{\eta_0}^{\eta}Qd\eta'\right)^2\right]+2|c_2|^2+2i(c_1^*c_2-c_1c_2^*)\left(\int_{\eta_0}^{\eta}Qd\eta'\right)-\Omega-(2\Omega)^{-1}\left[\frac{1}{4}\left(\frac{\Omega'}{\Omega}\right)^2-Q\right]\right\}+\rho_{c(t)}, \tag{3.19}$$

where $q(t)$ is the domain in $\vec{k}$ space over which the approximate form of $\chi_{\vec{k}}$ is valid at time t, and ρ_c refers to the integral in Eq. (3.9) over the rest of $\vec{k}$ space.

In the domain $q(t)$ of $\vec{k}$ space in which the low-frequency approximation is valid, quantum effects such as particle creation and vacuum polarization are dominant. For higher frequencies such that the WKB approximation is valid, the matter can be treated classically. In the absence of an approximation for the intermediate range of frequencies, and to make explicit integration over $\vec{k}$ possible, we take the classical domain $c(t)$, over which the WKB approximation can be used, to consist of all $\vec{k}$ such that $\omega > t^{-1}$; and we take the quantum domain $q(t)$ to consist[44] of all $\vec{k}$ such that $\omega \leq t^{-1}$. For $\vec{k} \in c(t)$, $\chi_{\vec{k}}$ can be written as a linear combination of positive- and negative-frequency extended WKB solutions (Ref. 29) of Eq. (3.5), and the value of ρ_c will depend on the coefficient of the negative-frequency part of $\chi_{\vec{k}}$. This coefficient is determined by the behavior of $\chi_{\vec{k}}$ at earlier times when $\vec{k}$ was in q [for the initial conditions we consider, V increases with time and one generally has $q(t) \subset q(t')$ if $t' < t$]. At such times positive and negative frequencies are not well defined (i.e., particle creation is occurring). Thus ρ_c consists of the remnant of particles created at earlier times. As modes $\vec{k}$ pass out of q into c, their contribution can be treated as a classical relativistic ($m = 0$) gas of already created particles.

It is still necessary to complete the specification of the $\chi_{\vec{k}}$ (or equivalently of the state vector $|0_A\rangle$) for modes $\vec{k} \in q$. This corresponds to choosing $c_1(\vec{k})$ and $c_2(\vec{k})$ in Eq. (3.19). That choice is dictated by the following considerations. If the expansion were isotropic then there would be a natural choice of vacuum state for the conformal field.[45] In that case $\Omega_{\vec{k}} = k$ is constant and Eq. (3.10) with $\alpha_{\vec{k}}$ and $\beta_{\vec{k}}$ constants is the solution of Eq. (3.5). Positive and negative frequencies are well defined, which coincide with the usual ones in Minkowski space during any interval when the expansion ceases and which do not mix during the expansion. The state $|0_A\rangle$ annihilated by the $A_{\vec{k}}$ of Eq. (3.4) corresponds to the vacuum state if $\chi_{\vec{k}}$ is a positive-frequency solution, that is, if $\alpha_k = 1$ (to within an arbitrary phase) and $\beta_k = 0$. When the expansion is not isotropic we would like to choose a state which corresponds to the absence of particles at t_0 (so that one can find the anisotropy damping produced by particles created after t_0). However, there is no clear definition of positive frequency (or of particle) in the early anisotropic expansion. Nevertheless, if one assumes that the natural choice of quantum state at t_0 is independent of the initial values of α_1' and α_3', then one is led to choosing the state vector which would correspond to the vacuum if the initial conditions were isotropic (i.e., if α_1' were equal to α_3'). Thus, when the expansion is not isotropic, we complete the definition of the state vector $|0_A\rangle$ by imposing the initial condition

$$\alpha_{\vec{k}}(\eta_0) = 1, \quad \beta_{\vec{k}}(\eta_0) = 0 \; , \tag{3.20}$$

where η_0 is the η time corresponding to t_0. This requires that the constants $c_1(\vec{k})$ and $c_2(\vec{k})$ in the low-frequency approximation of Eqs. (3.15) and (3.16) have the values

$$c_1 = \tfrac{1}{2}\Omega_0^{-1/2} \; , \quad c_2 = \tfrac{1}{2}\Omega_0^{1/2} \; , \tag{3.21}$$

where Ω_0 is the value of Ω at time t_0. Then the average energy density of Eq. (3.19) is

$$\rho = \rho_{q(t)} + \rho_{c(t)} \; , \tag{3.22}$$

with

$$\rho_{q(t)} = (16\pi^3)^{-1} V^{-4/3} \int_{q(t)} d^3k \left\{ (2\Omega_0)^{-1} \left[(\Omega^2 - Q) + \left(\int_{\eta_0}^{\eta} Q\, d\eta' \right)^2 \right] + 2^{-1}\Omega_0 - \Omega - (2\Omega)^{-1} \left[\frac{1}{4} \left(\frac{\Omega'}{\Omega} \right)^2 - Q \right] \right\} . \tag{3.23}$$

In the integrand of Eq. (3.23) it appears that the term involving $(\int_{\eta_0}^{\eta} Q\, d\eta')^2$, which depends on the past history of the system, correponds to the accumulated particles or energy produced in the low-frequency modes by the expansion between t_0 and t, while the other terms evidently correspond to vacuum polarization or vacuum energy effects.

The integration over $\vec{k}$ in Eq. (3.23) can be carried out explicitly, giving $\rho_{q(t)}$ as a function of the a_j. We are considering the axisymmetric case, $a_1 = a_2$, so that k_1 and k_2 always appear in the combination

$$k_{12} = (k_1^{\,2} + k_2^{\,2})^{1/2} \; . \tag{3.24}$$

The domain $q(t)$ for which $\omega \leq t^{-1}$ is an ellipse in $\vec{k}$ space,

$$(k_{12}/k_{12m})^2 + (k_3/k_{3m})^2 = 1 \; , \tag{3.25}$$

with k_{12m} and k_{3m} being functions of time given by

$$k_{12m} = a_1(t)/t \; , \quad k_{3m} = a_3(t)/t \; . \tag{3.26}$$

We are free to scale the coordinates x^j such that at t_0,

$$a_1(t_0) = a_3(t_0) \ . \qquad (3.27)$$

One finds

$$\rho_q = (4\pi t)^{-2}\Big\{\tfrac{1}{2}V^{-2/3}a_1{}^2 I_0 t^{-2} + \tfrac{1}{4}V^{-4/3}a_1{}^2(a_3{}^2 + a_1{}^2 I_0)t^{-2} - t^{-2} + V^{-4/3}a_1{}^2 I_0\Big[\Big(\int_{\eta_0}^{\eta} Q d\eta'\Big)^2 - Q\Big] + V^{-2/3}Q - \tfrac{1}{45}V^{-2/3}(\alpha_1' - \alpha_3')^2\Big\} , \qquad (3.28)$$

where I_0 is defined in Eq. (A5). At t_0 the energy density does not vanish, and is given by

$$\rho_q(t_0) = -(720\pi^2 a_0{}^2 t_0{}^2)^{-1}(\alpha_1' - \alpha_3')^2_{t_0} \ . \qquad (3.29)$$

This negative-energy density arises from the subtraction term involving $(\Omega'/\Omega)^2$ in Eq. (3.23). As noted earlier, we find that when the calculations are repeated with that subtraction absent there is no significant change in the anisotropy damping time. It is nevertheless of qualitative interest to note that negative-energy density causes the anisotropy to increase, as we have found using a model with fluid source. In the present case, the energy density soon becomes positive and brings about anisotropy damping.

To find the part of ρ_q which goes over into ρ_c with the passage of time as the domain q becomes smaller, let $\rho_q(k_m, \eta)$ denote the expression in Eq. (3.23), where the domain q is given by Eq. (3.25) with $k_m = (k_{12m}, k_{3m})$ regarded as independent variables related to the limits of integration. Then the part of the energy density $\rho_q(\eta)$ which becomes classical in the interval from η to $\eta + \Delta\eta$ is given at time η by

$$\delta\rho_q(\eta) = \rho_q(k_m(\eta), \eta) - \rho_q(k_m(\eta + \Delta\eta), \eta) = -\Big(\frac{\partial\rho_q}{\partial k_{12m}}\frac{dk_{12m}}{d\eta} + \frac{\partial\rho_q}{\partial k_{3m}}\frac{dk_{3m}}{d\eta}\Big)\Delta\eta \ , \qquad (3.30)$$

where $k_m(\eta)$ is given by Eq. (3.26) with $\eta = \eta(t)$. One can obtain the effective equation of state of the remnant of created particles by using the WKB approximation in the classical domain $c(t)$, as discussed in the paragraph following Eq. (3.19) [that procedure yields $\rho_c \approx (2\pi V)^{-3}\int_{c(t)} d^3k\, \omega_{\vec{k}}^{\mp} |\beta_{\vec{k}}^{\mp}|^2$]. The complicated dependence on the $a_i(t)$ makes it difficult in that case to use ρ_q to generate ρ_c as the domain $q(t)$ decreases with time. For simplicity, we treat the classical remnant as a relativistic fluid with energy density ρ_c proportional to $V^{-4/3}$, and with the p_i generated by the conservation law, i.e., by Eqs. (2.14) and (2.15). Because of the presence of ρ_q, the pressures p_i will not be isotropic. However, as ρ_c becomes large with respect to ρ_q the p_i will tend toward isotropy. One might at first expect that anisotropic responses of the p_i to the different expansion rates were necessary to bring about isotropy of the expansion. However, we find that equalization of the expansion rates does take place, so that the anisotropy parameter ΔH decreases below unity.

In the approximation that $\rho_c \propto V^{-4/3}$, the classical contributions to the energy density are red-shifted during any interval, say from η_1 to η_2, by the factor $[V(\eta_1)/V(\eta_2)]^{4/3}$. Then we have from Eqs. (3.22) and (3.30)

$$\begin{aligned}\rho(\eta+\Delta\eta) &= [V(\eta)/V(\eta+\Delta\eta)]^{4/3}[\rho_c(\eta) + \delta\rho_q(\eta)] + \rho_q(k_m(\eta+\Delta\eta), \eta+\Delta\eta) \\ &= [V(\eta)/V(\eta+\Delta\eta)]^{4/3}[\rho(\eta) - \rho_q(k_m(\eta+\Delta\eta), \eta)] + \rho_q(k_m(\eta+\Delta\eta), \eta+\Delta\eta) \\ &= [V(\eta)/V(\eta+\Delta\eta)]^{4/3}\rho(\eta) + \Delta\eta\Big[\frac{\partial}{\partial\eta}\rho_q(k_m(\eta), \eta) + \frac{4}{3}\frac{V'(\eta)}{V(\eta)}\rho_q(k_m(\eta), \eta)\Big] \ , \end{aligned} \qquad (3.31)$$

where $k_m(\eta)$ is held fixed in the partial derivative in Eq. (3.31). This was the expression used to generate $\rho(\eta)$ in the numerical integration of the Einstein equations, with $\Delta\eta$ being the step size. The quantity $\rho_q(k_m(\eta), \eta)$ appearing in Eq. (3.31) is given in Eq. (3.28). To find $\partial\rho_q(k_m(\eta), \eta)/\partial\eta$ we perform the integration over k in Eq. (3.23) with k_m, which specifies the domain q of integration, left arbitrary (so that η can be varied independently). The integrals involved and their partial derivatives with respect to η are listed in the Appendix. The result is that Eq. (3.31) becomes

$$\begin{aligned}\rho(\eta+\Delta\eta) =& [V(\eta)/V(\eta+\Delta\eta)]^{4/3}\rho(\eta) \\ &+ \Delta\eta(2\pi)^{-2}V^{-4/3}\Big\{\frac{\alpha_1{}^2 V^{2/3}}{8t^4}(\alpha_1' - \alpha_3')\Big[\tfrac{2}{3}I_0 + e_0{}^{-1}\Big(1 - \frac{a_1{}^2}{a_3{}^2}I_0\Big)\Big] + \frac{1}{18t^2}(\alpha_1' - \alpha_3')(\alpha_1'' - \alpha_3'')(V^{2/3} - a_1{}^2 I_0) \\ &+ \frac{a_1{}^2}{2t^2}I_0 Q\Big(\int_{\eta_0}^{\eta} Q d\eta'\Big) - \frac{V^{2/3}}{90t^2}(\alpha_1' - \alpha_3')[\tfrac{4}{21}(\alpha_1' - \alpha_3')^2 + (\alpha_1'' - \alpha_3'')]\Big\} \ , \end{aligned} \qquad (3.32)$$

where I_0 and e_0 are defined in Eqs. (A5) and (A16), respectively. In the computation one also needs this expression when e_0 is small (i.e., when a_1 and a_3 are nearly equal). In that case,

$$\rho(\eta+\Delta\eta)\approx [V(\eta+\Delta\eta)/V(\eta)]^{-4/3}\rho(\eta) + \Delta\eta(2\pi)^{-2}V^{-4/3}\Bigg\{\frac{a_1^2V^{2/3}}{8t^4}(\alpha_1'-\alpha_3')(\tfrac{4}{3}+\tfrac{16}{45}e_0)+\frac{V^{2/3}}{18t^2}(\alpha_1'-\alpha_3')(\alpha_1''-\alpha_3'') + \frac{a_1^2}{t^2}(1+\tfrac{1}{3}e_0)\left[\tfrac{1}{2}Q\int_{\eta_0}^{\eta}Qd\eta'-\tfrac{1}{18}(\alpha_1'-\alpha_3')(\alpha_1''-\alpha_3'')\right] - \frac{V^{2/3}}{90t^2}(\alpha_1'-\alpha_3')[\tfrac{4}{21}(\alpha_1'-\alpha_3')^2+(\alpha_1''-\alpha_3'')]\Bigg\}. \tag{3.33}$$

We now turn to the numerical integration of Einstein's equations and describe the results.

IV. NUMERICAL SOLUTIONS AND RESULTS

In this section we solve the Einstein equations with the energy density and pressure derived above. Before we embark upon the numerical integration procedure, let us first summarize the system of equations that we deal with.

The Einstein equations. The Einstein equations for an axisymmetric type-I metric are given by Eqs. (2.4) and (2.9) with $\alpha_1=\alpha_2$:

$$\alpha_1'^2+2\alpha_1'\alpha_3'=8\pi\rho V^{2/3}, \tag{4.1a}$$

$$\alpha_1''+\tfrac{2}{3}\alpha_1'(2\alpha_1'+\alpha_3')=8\pi(p_1+\tfrac{1}{2}T)V^{2/3}, \tag{4.1b}$$

$$\alpha_3''+\tfrac{2}{3}\alpha_3'(2\alpha_1'+\alpha_3')=8\pi(p_3+\tfrac{1}{2}T)V^{2/3}. \tag{4.1c}$$

The energy density ρ is generated in the numerical integration by means of Eq. (3.32). The pressures p_1 and p_3 are related to ρ and T, the trace of $\langle T_{\mu\nu}\rangle$ by Eqs. (2.14) and (2.15), in which ρ' is determined numerically from ρ. As explained earlier, we expect the anisotropy damping effect for $t>t_P$ to be dominated by the created particles, so that we have used the approximation that $T\simeq 0$ in these calculations. The unknown functions α_i appear on both sides of the system of equations (4.1), (2.14), and (2.15) which has to be solved self-consistently.

Initial conditions. The dynamical equations (4.1b), and (4.1c) are second-order equations, so that at some initial time t_0 one must specify both the values $(a_1,a_3)_0$ and their first derivatives $(a_1',a_3')_0$. In a spatially flat type-I universe, as the only dynamical variables appearing in the Einstein equations are the expansion rates α_i', one can freely scale the axes a_i. For convenience, we set $a_1=a_2=a_3\equiv a_0$ at t_0. The first-order constraint equation (4.1a) relates the expansion rates α_1' and α_3' to the energy density ρ. The initial energy density ρ_0 is given by Eq. (3.29). This nonvanishing initial energy density arises from the vacuum subtraction involving $(\Omega'/\Omega)^2$ in Eq. (3.19). To compare the effect on our results of the various subtraction terms in the energy density, we have performed the numerical integration for three different cases. In *case 1,* the complete expression in Eq. (3.19) is used for $\rho(t)$. In *case 2,* the term $\frac{1}{4}(\Omega'/\Omega)^2$ is omitted from Eq. (3.19), and in *case 3,* the full term $[\frac{1}{4}(\Omega'/\Omega)^2-Q]$ is omitted from Eq. (3.19). These terms arise from the second-order vacuum subtraction in the renormalization process. The initial energy density ρ_0 is zero in cases 2 and 3. From the $G_0{}^0$ equation [Eq. (4.1a)], one derives a relation between α_1' and α_3' at t_0. In case 1,

$$(\alpha_3')_0=(\alpha_1')_0\{1+[-1+(1-3\sigma)^{1/2}]/\sigma\},$$

where

$$\sigma=\frac{1}{90\pi t_0^2}, \tag{4.2}$$

whereas in cases 2 and 3, $(\alpha_3')_0=-(\alpha_1')_0/2$. To calculate p_1 and p_3 at t_0 from Eqs. (2.14) and (2.15) one also needs the inital values of ρ_0'. In case 1, Eq. (3.32) yields

$$\rho_0'=\frac{(\alpha_1'-\alpha_3')_0}{24\pi^2a_0^4}\Bigg\{\left(\frac{a_0}{t_0}\right)^4-\frac{1}{15}\left(\frac{a_0}{t_0}\right)^2 \times[\tfrac{4}{21}(\alpha_1'-\alpha_3')_0^2+(\alpha_1''-\alpha_3'')_0 - \tfrac{2}{3}(2\alpha_1'+\alpha_3')_0(\alpha_1'-\alpha_3')_0]\Bigg\}. \tag{4.3}$$

The quantity $\alpha_1''-\alpha_3''$ in Eq. (4.3) can be expressed in terms of ρ, ρ', and α_i' by means of the Einstein equations [Eq. (4.1)], the divergence equation, and the trace condition [Eqs. (2.14) and (2.15)]. Thus from Eq. (4.1) one gets

$$\alpha_1''-\alpha_3''=-8\pi V^{2/3}(p_1-p_3) -\tfrac{2}{3}(\alpha_1'+\alpha_2'+\alpha_3')(\alpha_1'-\alpha_3')$$

and from Eqs. (2.14) and (2.15),

$$p_1-p_3=-\frac{1}{2(\alpha_1'-\alpha_3')}[3\rho'+4\rho(2\alpha_1'+\alpha_3')].$$

Hence

$$\alpha_1'' - \alpha_3'' = \frac{4\pi V^{2/3}}{(\alpha_1' - \alpha_3')}[3\rho' + 4\rho(2\alpha_1' + \alpha_3')] - \tfrac{2}{3}(2\alpha_1' + \alpha_3')(\alpha_1' - \alpha_3') .$$

Substituting this into Eq. (4.3) and solving for ρ_0', one gets

$$\rho_0' = \frac{(\alpha_1' - \alpha_3')_0}{(1 + 1/30\pi t_0^2)} \frac{1}{24\pi^2 t_0^4} \left\{ 1 + \tfrac{1}{45}(\alpha_1' - \alpha_3')_0 \frac{t_0^2}{a_0^2} \left[(2\alpha_1' + \alpha_3')_0 \left(4 + \frac{1}{15\pi t_0^2} \right) - \tfrac{4}{7}(\alpha_1' - \alpha_3')_0 \right] \right\} . \quad (4.4)$$

The expressions for p_1 and p_3 follow from Eqs. (2.14) and (2.15),

$$(p_1)_0 = -\frac{1}{2(\alpha_1' - \alpha_3')_0}[\rho_0' + 2\rho_0(\alpha_1' + \alpha_3')_0] ,$$
$$(p_3)_0 = \rho_0 - 2(p_1)_0 , \quad (4.5)$$

with ρ_0' given by Eq. (4.4).

In summary, as initial conditions, one has to specify only t_0, a_0, and $(\alpha_1')_0$. The quantity $(\alpha_3')_0$ is given by (4.2). The initial energy density ρ_0 and pressures $(p_i)_0$ are given by (3.29) and (4.5), respectively.

For the simpler cases 2 and 3, one finds

$$\rho_0' = \frac{(\alpha_1' - \alpha_3')_0}{24\pi^2 t_0^4}$$

and from Eqs. (2.14) and (2.15),

$$(p_1)_0 = -\frac{1}{48\pi^2 t_0^4} , \quad (p_3)_0 = \frac{1}{24\pi^2 t_0^4} . \quad (4.6)$$

Numerical integration and results. We have numerically integrated the Einstein equations (4.1) with a source energy density ρ given by Eq. (3.32) and the related pressures p_1, p_3 given by Eqs. (2.14) and (2.15). The expressions for $\rho'(\eta)$, $\alpha_1''(\eta)$, and $\int_{\eta_0}^{\eta} Q(\eta')d\eta'$ are computed numerically as the integration proceeds. We used a fourth-order Runge-Kutta subroutine for the solution of ordinary linear differential equations on an IBM 360/95 computer. We have performed computations for different initial values of t_0 and α_1' for all three cases. The results are given in the accompanying graphs and tables.

We define $\Delta H \equiv (\alpha_1' - \alpha_3')/2$ as the anisotropy parameter and $H \equiv \frac{1}{3}(\sum_i \alpha_i')$ as the average Hubble parameter. The quantity $\Delta H/H$ is a measure of the variance in the expansion rates in different directions and hence gives the degree of anisotropy. For an isotropic expansion, $\Delta H/H = 0$. In case 1 for $t_0 \sim 1$, one sees from Eq. (4.2) that $\Delta H/H \approx 1.5$, whereas in cases 2 and 3 as $\alpha_3' = -\alpha_1'/2$ at t_0, $\Delta H/H = 1.5$. Although $\Delta H/H$ is approximately the same as for a Kasner solution (where $a_i \sim t^{p_i}$ and $\sum_i p_i = \sum_i p_i^2 = 1$), the initial values of α_1' and α_3' are *not* restricted to the Kasner values. As the universe evolves, $\Delta H/H$ decreases; and we define t_F as the earliest time after which $\Delta H/H$ remains less than unity. This definition of t_F agrees with that of Ref. 31 and is in ratio $(\sqrt{6} : 3/2)$ with that of Ref. 20 (1968) in the axisymmetric case. It serves as a measure of the time of effective isotropization. In the following we give a discussion of the results of our computations (see Figs. 1 to 4 and Tables I and II).

Figures 1(a), 1(b), and 1(c) show $\Delta H/H$ as a func-

TABLE I. t_F as a function of t_0 for $(\alpha_1')_0 = \frac{4}{3}$ $(a_0 = 1)$ for cases 1, 2, and 3 $(a_0 = 1)$.

t_0	Case	t_F	$\rho V^{4/3}$ $(t \gg t_F)$
0.5	1	1.0	0.208×10^{-1}
	2	1.0	0.232×10^{-1}
	3	0.76	0.297×10^{-1}
1.0	1	21.0	0.148×10^{-2}
	2	21.0	0.161×10^{-2}
	3	14.6	0.199×10^{-2}
2.0	1	385	0.955×10^{-4}
	2	366	0.103×10^{-3}
	3	275	0.125×10^{-3}
3.0	1	2078	0.190×10^{-4}
	2	1970	0.202×10^{-4}
	3	1450	0.246×10^{-4}

TABLE II. t_F as a function of $(\alpha_1')_0$ for $t_0 = 1$ for cases 1, 2 and 3 $(a_0 = 1)$.

$(\alpha_1')_0$	Case	t_F	$\rho V^{4/3}$ $(t \gg t_F)$
10	1	23.5	0.558×10^{-1}
	2	23.24	0.623×10^{-1}
	3	2.60	0.317
5	1	32.5	0.181×10^{-1}
	2	29.85	0.207×10^{-1}
	3	4.97	0.703×10^{-1}
1	1	27.0	0.234×10^{-2}
	2	24.76	0.258×10^{-2}
	3	14.26	0.361×10^{-2}
$\frac{1}{3}$	1	13.2	0.677×10^{-3}
	2	13.9	0.732×10^{-3}
	3	11.5	0.80×10^{-3}

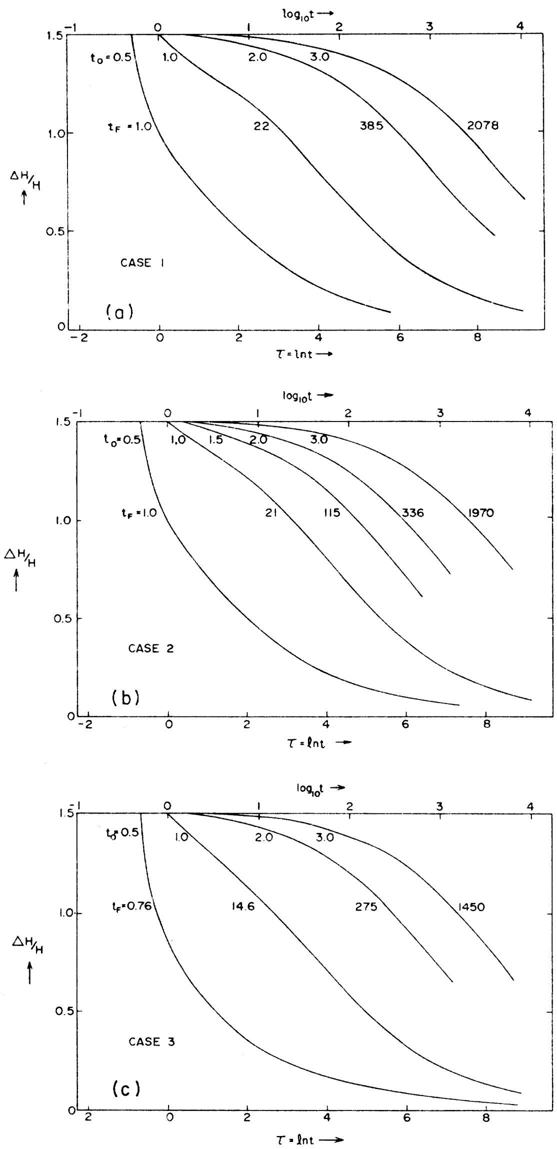

FIG. 1. Plot of $\Delta H/H$ as a function of $\tau \equiv \ln t$ for initial time $t_0 = 0.5$, 1.0, 2.0, 3.0, $(\alpha_1')_0 = 2a_0/3t_0$, $a_0 = 1$. (a), (b), and (c) correspond to cases 1, 2, and 3, respectively.

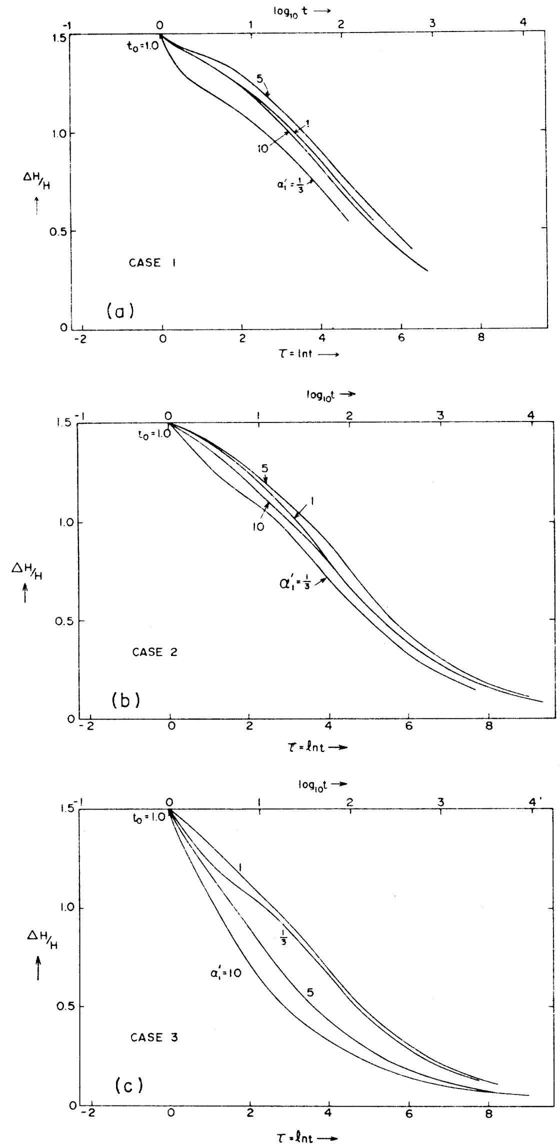

FIG. 2. Plot of $\Delta H/H$ as a function of τ for $t_0 = 1$ and $(\alpha_1')_0 = \frac{1}{3}$, 1, 5, 10 ($a_0 = 1$). (a), (b), and (c) correspond to cases 1, 2, and 3, respectively.

tion of $\tau \equiv \ln t$ with initial conditions $(\alpha_1')_0 = 2a_0/(3t_0)$, $(a_0 = 1)$ and initial times $t_0 = 0.5, 1.0, 2.0, 3.0$ (in units of Planck time) for the three cases. First, one notices that the results in all three cases are quite close and indeed cases 1 and 2 are almost indistinguishable. This means that the effect of the $(\Omega'/\Omega)^2$ term is small compared with the $-Q$ term in the second-order subtraction. Comparison of Fig. 1(c) with Fig. 1(a) or 1(b) shows that the entire second-order term has a relatively small effect on t_F. In all three cases, the time of isotropization t_F depends on the choice of t_0 rather strongly. For $t_0 < 1$, the particle energy density builds up rapidly and the universe isotropizes almost instantly. For $t_0 > 1$, the particle density builds up more gradually, and the time of isotropization increases rapidly with increasing t_0.

For larger t_0 (e.g., $t_0 = 3$), $\Delta H/H$ remains nearly ≈ 1.5 for a longer time ($\sim 10^3$) while the universe stays as an anisotropic near-vacuum Kasner solution, after which it isotropizes rather quickly. Table I gives t_F for different values of t_0. The constant values of $\rho V^{4/3}$ for $t \gg t_F$ are also listed, showing that more matter is created when t_0 is

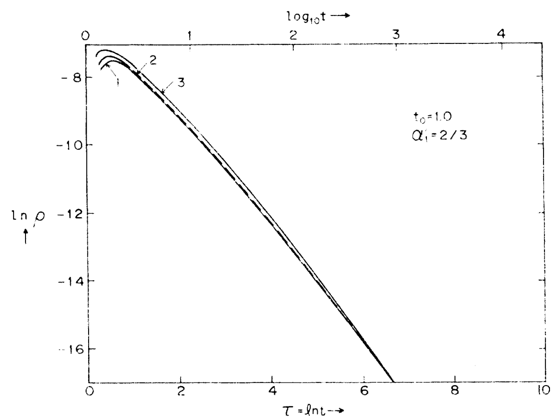

FIG. 3. Plot of energy density ρ as a function of τ for $t_0 = 1$ and $(\alpha_1')_0 = \frac{2}{3}$ for cases 1, 2, and 3 (as indicated by arrowhead).

smaller. For $t_0 \gtrsim 1$, one sees from the table that $\rho V^{4/3}$ is proportional to t_0^{-4}, as might be expected on dimensional grounds ($V_0 = 1$ here).

In Figs. 2(a), 2(b), and 2(c), we show the dependence of $\Delta H/H$ on the initial rate of expansion $(\alpha_1')_0 = (\frac{1}{3}, 1, 5, 10)(a_0/t_0)$ for a fixed starting time, $t_0 = 1$, $(a_0 = 1)$ for the three cases. Again, cases 1 and 2 are almost identical and differ only slightly from case 3. All three cases show that the time of isotropization depends little on the initial values of α_1'. Table II gives t_F and $\rho V^{4/3}$ (for $t \gg t_F$) for different values of α_1' with $t_0 = 1$.

Figure 3 shows the energy density of particles, as a function of time for $t_0 = 1$, $(\alpha_1')_0 = \frac{2}{3}$ for all three cases. There is no significant difference between the three cases. Particles are created abundantly in a short interval after t_0, during which ρ_q grows rapidly for a brief period ($\sim 1 t_P$). Afterwards, the classical energy density ρ_c remains as the dominant contribution to ρ, which decreases as $V^{-4/3}$.

In Figures 4(a) and 4(b), we show the radii functions $\ln a_1$ and $\ln a_3$ as a function of time $\tau = \ln t$, for $t_0 = 1$, $(\alpha_1')_0 = \frac{2}{3}$ [cases 1 and 2 are indistinguishable and are presented in Fig. 4(a)]. At the start a_1 expands as $t^{2/3}$ and a_3 contracts as $t^{-1/3}$. At about t_F, the energy density from particle creation becomes comparable to the anisotropy energy of the background and causes a_3 to reverse to an expansion while the expansion rate in the a_1 direction gradually slows down. After $t \sim 10^3 t_P$, a_1 and a_3 both approach $t^{1/2}$ behavior, and the universe approximates a radiation-filled Friedmann solution. In Fig. 4(c), the radii functions are plotted for $t_0 = 1$, but with an initial expansion rate of $(\alpha_1')_0 = 10$. After $t \sim 10^3 t_P$, a_1 and a_3 again both approach the $t^{1/2}$ behavior of a radiation-filled Friedmann universe.

Our numerical results indicate that the anisotro-

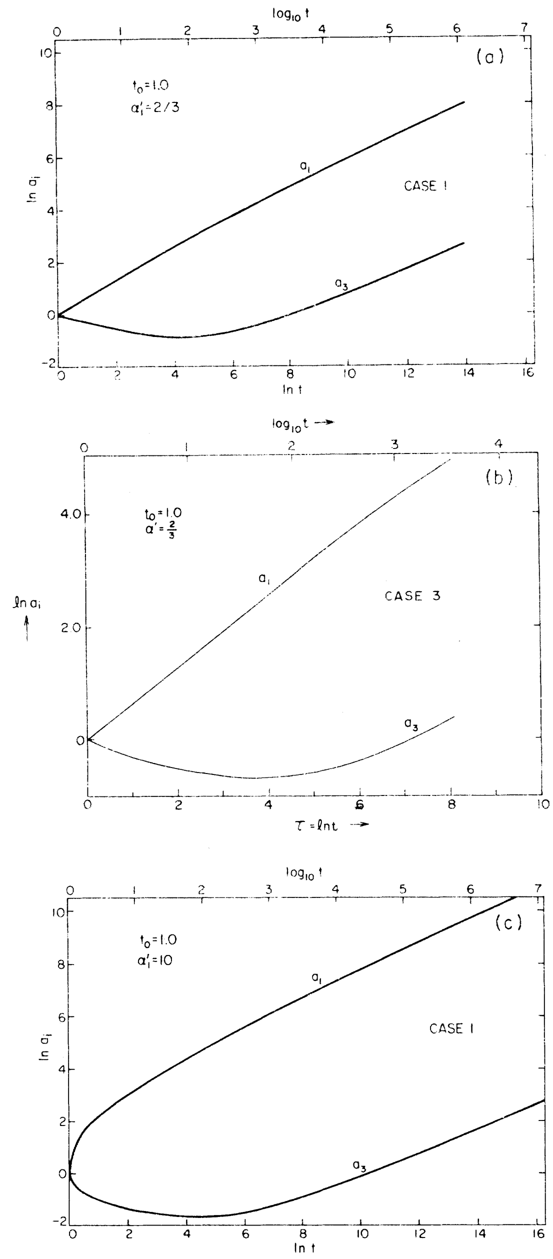

FIG. 4. Plot of radii function a_i as a function of τ for $t_0 = 1$. (a) corresponds to cases 1 and 2 with $(\alpha_1')_0 = \frac{2}{3}$ while (b) corresponds to case 3 with $(\alpha_1')_0 = \frac{2}{3}$, and (c) corresponds to cases 1 and 2 with $(\alpha_1')_0 = 10$.

py damping resulting from particle creation in a Bianchi type-I expansion is strong enough when $t_0 \sim t_P$ to bring about effective isotropization at a sufficiently early time t_F to be consistent with the various upper bounds on t_F arrived at from considerations of the observed cosmic blackbody radiation, helium abundance, and deuterium abundance. These results also indicate that the value of t_F for a given t_0 is only weakly dependent on the initial values of the expansion rates (consistent with the Einstein constraint equation and the axial symmetry). This is, of course, not a proof that particle creation occurring near the Planck time will bring about isotropy at a sufficiently early time regardless of the initial expansion rates, but it is encouraging that isotropization did take place for the full range of initial values considered.

In this first attempt at directly calculating $\langle T_{\mu\nu}\rangle$ by solving the equation governing the quantized field ϕ in suitable approximation and integrating the resulting Einstein equations in the Bianchi type-I metric, the following simplifications were necessarily made: (1) The expectation value of $T_{\mu\nu}$ was taken in a quantum state corresponding as nearly as possible to the absence of matter at t_0. Other state vectors were not considered, as our objective was limited to studying the effect of particle creation on the anisotropy of the expansion for $t > t_0$. (2) A low-frequency approximation to the solutions of the scalar field equation was used, and the high-frequency modes were treated as a classical gas. (3) The assumption that the energy density ρ_c of the classical remnant was proportional to $V^{-4/3}$ implied that at late times, when the total energy density became mainly classical, the pressures p_1 and p_3 became equal and the equation of state approached $p=\rho/3$. (4) The initial time t_0 was taken to be of order t_P. In a complete theory one would hope that t_0 could be eliminated or would enter in a more fundamental way. (5) The trace anomaly was not fully incorporated into the numerical calculation. This would be of increasing significance as t_0 became small with respect to t_P. (6) The metric was treated classically. Quantum gravity would be very important for $t_0 < t_P$. (7) Nongravitational interactions were not included. (8) Consideration was limited to Bianchi type-I metrics (with $a_1 = a_2$). It is hoped that these points can be more fully treated in future work.

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APPENDIX

The integrals and derivatives involved in Eqs. (3.23) and (3.31), with the domain of integration given by Eq. (3.25), are as follows:

$$\int \Omega d^3k = -\frac{\pi a_1^2 V^{1/3}}{3a_3^3} k_{3m}^4 + \frac{\pi V^{1/3}}{6a_3}\left[k_{3m}^2(5k_{12m}^2 + 2k_{3m}^2 f) + 3\frac{a_3}{a_1}k_{12m}^4 I\right], \quad (A1)$$

where

$$f = (a_1/a_3)^2 - (k_{12m}/k_{3m})^2 \quad (A2)$$

and

$$I = \begin{cases} (f)^{-1/2}\ln\left\{\dfrac{k_{3m}}{k_{12m}}\left[\dfrac{a_1}{a_3} + (f)^{1/2}\right]\right\}, & \text{if } f > 0 \\ a_3/a_1, & \text{if } f = 0 \\ (-f)^{-1/2}\sin^{-1}\left[\dfrac{k_{3m}}{k_{12m}}(-f)^{1/2}\right], & \text{if } f < 0 \end{cases} \quad (A3)$$

$$\int \Omega_0 d^3k = (\pi/2)k_{12m}^2(k_{3m}^2 + k_{12m}^2 I_0), \quad (A4)$$

where

$$I_0 = \begin{cases} (f_0)^{-1/2}\ln\left\{\dfrac{k_{3m}}{k_{12m}}[1 + (f_0)^{1/2}]\right\}, & \text{if } f_0 > 0 \\ 1, & \text{if } f_0 = 0 \\ (-f_0)^{-1/2}\sin^{-1}\left[\dfrac{k_{3m}}{k_{12m}}(-f_0)^{1/2}\right], & \text{if } f_0 < 0 \end{cases} \quad (A5)$$

and

$$f_0 = 1 - (k_{12m}/k_{3m})^2, \quad (A6)$$

$$\int \Omega^{-1} d^3k = 2\pi a_1 V^{-1/3} k_{12m} I, \quad (A7)$$

$$\int \Omega_0^{-1} d^3k = 2\pi k_{12m} I_0, \quad (A8)$$

$$\int \Omega^2 \Omega_0^{-1} d^3k = \frac{\pi V^{2/3}}{3a_1^2}\left\{3\left(\frac{a_1^2}{a_3^2} - 1\right)\left(\frac{k_{3m}^2}{k_{12m}^2} - 1\right)^{-1} k_{3m}^4 + \tfrac{3}{2}k_{12m}^2\left[1 + \left(1 - \frac{k_{12m}^2}{k_{3m}^2}\right)^{-1}\left(1 - \frac{a_1^2}{a_3^2}\right)\right][k_{3m}^2 + k_{12m}^2 I_0]\right\}. \quad (A9)$$

$$\int \Omega'^2\Omega^{-3}d^3k = 2\pi a_1 V^{-1/3}k_{12m}{}^2(\alpha_1' - \alpha_3')^2 \left[\left(\tfrac{1}{3} - f^{-1}\frac{a_1{}^2}{a_3{}^2}\right)^2 I + f^{-1}\left(\tfrac{1}{3} - f^{-1}\frac{a_1{}^2}{a_3{}^2}\right)\frac{a_1}{a_3}\right] . \quad \text{(A10)}$$

To obtain Eq. (3.28) one sets $k_{12m} = a_1(t)/t$, $k_{3m} = a_3(t)/t$, and makes use of the expansion of I for small f:

$$I = (a_3/a_1)(1 + \tfrac{1}{3}e + \tfrac{1}{5}e^2 + \tfrac{1}{7}e^3 + \cdots) , \quad \text{(A11)}$$

where

$$e = (a_3{}^2/a_1{}^2)f = 1 - (k_{12m}/a_1)^2(k_{3m}/a_3)^{-2} . \quad \text{(A12)}$$

To evaluate Eq. (3.31) one must take the derivative of the above expressions with respect to η, keeping k_{12m} and k_{3m} fixed, and then set $k_{12m} = a_1(t)/t$ and $k_{3m} = a_3(t)/t$. Performing those operations one finds after some calculation that

$$\frac{\partial}{\partial\eta}\int \Omega d^3k = 0 , \quad \frac{\partial}{\partial\eta}\int \Omega^{-1}d^3k = 0 . \quad \text{(A13)}$$

One also has

$$\frac{\partial}{\partial\eta}\int \Omega_0 d^3k = 0 , \quad \frac{\partial}{\partial\eta}\int \Omega_0{}^{-1}d^3k = 0 . \quad \text{(A14)}$$

Furthermore, one finds that

$$\frac{\partial}{\partial\eta}\int \Omega^2\Omega_0{}^{-1}d^3k = \pi V^{2/3}a_1{}^2t^{-4}(\alpha_1' - \alpha_3') \times\left[\tfrac{2}{3}I_0 + \left(1 - \frac{a_1{}^2}{a_3{}^2}\right)^{-1}\left(1 - \frac{a_1{}^2}{a_3{}^2}I_0\right)\right] . \quad \text{(A15)}$$

When a_1 and a_3 are nearly equal it is useful to expand this last expression to first order in

$$e_0 = 1 - a_1{}^2a_3{}^{-2} , \quad \text{(A16)}$$

using the expansion

$$I_0 = 1 + \tfrac{1}{3}e_0 + \tfrac{1}{5}e_0{}^2 + \cdots , \quad \text{(A17)}$$

with the result

$$\frac{\partial}{\partial\eta}\int \Omega^2\Omega_0{}^{-1}d^3k \cong \pi V^{2/3}a_1{}^2t^{-4}(\alpha_1' - \alpha_3') \times(\tfrac{4}{3} + \tfrac{16}{45}e_0) . \quad \text{(A18)}$$

Finally, one finds

$$\frac{\partial}{\partial\eta}\int \Omega'^2\Omega^{-3}d^3k = \tfrac{16}{45}\pi V^{2/3}t^{-2}(\alpha_1' - \alpha_3') \times[\tfrac{4}{21}(\alpha_1' - \alpha_3')^2 + (\alpha_1'' - \alpha_3'')] . \quad \text{(A19)}$$

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