How to measure S-wave production in the reaction $\pi N \to \pi\pi\Delta$

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A method is proposed for performing a model-independent amplitude analysis of the reaction $\pi N \to \pi\pi\Delta$, with the $\pi\pi$ system a superposition of S and P waves and where all moments of the joint-decay-angular distribution are used. A set of necessary positivity conditions for the moments are presented, and their relation to the amplitude analysis is discussed. Given data satisfying these conditions, the separation of S- and P-wave cross sections can be carried out in a model-independent way, although some discrete ambiguities may occur. In addition, improved upper and lower bounds on the S-wave cross section are derived which employ partial knowledge of the Δ decay moments.

I. INTRODUCTION

The production of $\pi\pi$ systems of relatively low mass in the reactions

$$\pi N \to \pi\pi N \quad (\mathrm{A}),$$

$$\pi^+ p \to (\pi^+\pi^-)\Delta^{++} \quad (\mathrm{B}),$$

provides information on the $\pi\pi$ elastic scattering amplitudes. Recent high-statistics studies of reaction (A) have obtained precise measurements of the moments of the $(\pi\pi)$ decay-angular distribution as functions of momentum transfer and $\pi\pi$ mass.[1,2] These results have been employed to perform partial amplitude reconstruction, and to deduce $\pi\pi$ phase shifts.[3] However, an unfortunate aspect of the problem is that a certain amount of theoretical or phenomenological input is a necessary part of such analyses. In particular, the six experimental moments strictly provide only lower and upper bounds on the total S-wave production as a function of t and $m_{\pi\pi}$.[4,5] Without additional hypotheses the separation of $\pi\pi$ production into S and P waves is not possible. If reaction (A) is studied on a polarized nucleon target, however, the amount of S-wave production becomes unambiguous, and the analysis can be carried out without making such hypotheses.[6]

Reaction (B) has also been extensively used to study $\pi\pi$ (and $K\pi$) production,[7,8] but since it is generally observed in a bubble chamber, the number of events is typically an order of magnitude less than for reaction (A). If one compares the merits of the two reactions for studying π exchange, reaction (A) is evasive at $t=0$, while reaction (B) is not. On the other hand, t' effects associated with the large width of the Δ are a problem in reaction (B) absent in (A). Clearly, high statistics are more easily obtained in (A), although new generations of large rapid-cycling bubble chambers or electronic detectors with ~4π efficiency might be used to obtain large numbers of events in (B). An additional aspect which may be advanced in favor of (B) is that the correlations among the decay-angular distributions provide information on the amplitudes not available in (A) (on an unpolarized target). In principle, these correlation moments provide a means of measuring the amount of S-wave production under the ρ peak (we do not consider the problem of higher $\pi\pi$ partial waves). By measuring the amount of S wave, we intend to express the S-wave cross section in terms of the moments of the joint-decay-angular distribution without making any dynamical hypotheses. As we shall see, such an algorithm can be constructed, but its complicated form renders the application somewhat problematic. Nonetheless we think it useful to present such an analysis which also determines the amplitudes. At the very least the method proposed could be used in parallel with the more popular model-dependent analyses in order to verify the consistency of the hypotheses employed. Finally our discussion relies heavily on the positivity conditions for the $\pi\pi\Delta$ system, which have never been discussed in the literature, and whose enforcement at this stage of experimental analysis would certainly improve the estimates of the joint moments. A weak point of our method is that we must consider the Δ as a spin-$\frac{3}{2}$ state, whereas some small interfering background is probably present. While our analysis could be extended to consider the Δ as a superposition of S- and P- ($J=\frac{3}{2}$) wave states, formidable technical complications would be encountered.

This paper is organized as follows: In Sec. II, we discuss the observables, their expression in terms of amplitudes, and the positivity conditions. Section III gives an outline of the algorithm for expressing the amplitudes (modulo a known continuum and possible discrete ambiguities) in terms of the observables. In Sec. IV we present improved upper and lower bounds on the S-wave cross section. Certain numerical studies of our algorithm using random amplitudes are given in Sec. III and Sec. IV, bearing on questions of convexity and relative advantages of using the full correlation information rather than the $\pi\pi$ moments alone. Finally, our conclusions are presented in Sec. V. The Appendix contains details of the algorithm.

II. OBSERVABLES AND POSITIVITY CONDITIONS

A. Amplitudes and observables

The joint-decay-angular distribution of the system $\pi\pi\Delta$ may be written, assuming only $l=0$ and 1 partial waves for $\pi\pi$, as

$$W(\theta_1,\varphi_1,\theta_2,\varphi_2) = \sum_{L_1=0}^{2}\sum_{\substack{L_2=0\\ \text{even}}}^{2}\sum_{M_1=-L_1}^{L_1}\sum_{M_2=-L_2}^{L_2}\langle Y_{L_1}^{M_1}Y_{L_2}^{M_2}\rangle Y_{L_1}^{M_1*}(\theta_1,\varphi_1)\times Y_{L_2}^{M_2*}(\theta_2,\varphi_2),\quad(2.1)$$

where (θ_1,φ_1) and (θ_2,φ_2) are decay angles, in their rest frames, of the $\pi\pi$ and Δ, respectively. The quantities

$$\langle Y_{L_1}^{M_1}Y_{L_2}^{M_2}\rangle$$

are the expectation values of the spherical harmonics, and are functions of energy s, momentum transfer t, and the invariant masses of the $(\pi\pi)$ and (πN) final states. The problem we pose is to extract all possible dynamical information from these joint moments, in particular, the total amount of S-wave production. Even given arbitrarily precise moments there exists a four-parameter continuous ambiguity in the amplitudes,[9] but this ambiguity does not mix the relative amount of S- and P- wave $\pi\pi$ production. If the moments are averaged over some interval, either in mass or in momentum transfer, there is no guaranteed solution, and a set of discrete values for the average S-wave production will generally be found.

The joint moments may be obtained using any coordinate system, but two popular choices are as follows:

1. *Helicitylike systems.*[10,11] The z axes lie in the production plane, and the y axes are perpendicular to the plane. For the Jacob-Wick convention, the y axis in the Δ rest frame is antiparallel to that in the $\pi\pi$ rest frame.

2. *Transversity system.*[12,13] The z axes in both frames are parallel and perpendicular to the production plane (using the convention of Doncel *et al.*[13]).

In a helicitylike system the joint moments are real, and satisfy

$$\langle Y_{L_1}^{M_1}Y_{L_2}^{M_2}\rangle=(-1)^{M_1+M_2}\langle Y_{L_1}^{-M_1}Y_{L_2}^{-M_2}\rangle.\quad(2.2)$$

There are 30 independent real moments, and their expression in terms of the joint-density matrix and the helicity amplitudes is given by Irving,[14] to which which we refer the reader for details. Our discussion of the problem is based on transversity amplitudes, for which the task of expressing the amplitudes in terms of observables (modulo the four-parameter ambiguity) is more easily executed. A standard rotation to the helicity amplitudes may be performed to obtain the latter. In the transversity basis, the $\langle Y_{L_1}^{M_1}Y_{L_2}^{M_2}\rangle$ are generally complex, satisfy the relation

$$\langle Y_{L_1}^{M_1}Y_{L_2}^{M_2}\rangle^*=(-1)^{M_1+M_2}\langle Y_{L_1}^{-M_1}Y_{L_2}^{-M_2}\rangle,\quad(2.3)$$

and are nonvanishing only when $L_1+L_2+M_1+M_2$ is even. (This behavior follows from symmetry under reflection in the production plane; all odd moments of the normal must be zero, since it is a pseudovector.) For our purposes, it is useful to introduce certain linear combinations of these transversity observables, and to group them in such a way as to facilitate the amplitude analysis.

The combinations of transversity moments we use are

$$\Lambda_{nn'}^{mm'}=\sum_{\substack{L_1L_2\\ \text{even}}}\sum_{M_1M_2}f(L_1,L_2)C_{m,M_1,m'}^{1,L_1,1}C_{n,M_2,n'}^{3/2,L_2,3/2}\langle Y_{L_1}^{M_1}Y_{L_2}^{M_2}\rangle,\quad(2.4\text{a})$$

where

$$f(L_1,L_2)=\pi(2L_1+1)(2L_2+1)(6\sqrt{3}\,C_{0,0,0}^{1,1,L_1}C_{1/2,-1/2,0}^{3/2,3/2,L_2})^{-1},\quad(2.4\text{b})$$

$$\Lambda_{nn'}^{sm'}=\sum_{\substack{L_2\\ \text{even}}}\sum_{M_2}\tilde f(L_2)C_{n,M_2,n'}^{3/2,L_2,3/2}\langle Y_1^{m'}Y_{L_2}^{M_2}\rangle,\quad(2.5\text{a})$$

where

$$\tilde f(L_2)=-\pi(2L_2+1)(4C_{1/2,-1/2,0}^{3/2,3/2,L_2})^{-1}.\quad(2.5\text{b})$$

The quantities Λ are simply related to the joint-density matrix in a transversity basis, namely

$$\Lambda_{nn'}^{mm'}=\tfrac{1}{4}[\rho_{nn'}^{mm'}+(-1)^{m-m'}\rho_{nn'}^{-m',-m}+(-1)^{n-n'}\rho_{-n',-n}^{mm'}+(-1)^{m-m'+n-n'}\rho_{-n',-n}^{-m',-m}]+\tfrac{1}{6}\delta_{mm'}[\rho_{nn'}^{ss}+(-1)^{n-n'}\rho_{-n',-n}^{ss}],\quad(2.6)$$

$$\Lambda_{nn'}^{sm'}=\tfrac{1}{4}[\rho_{nn'}^{sm'}+(-1)^{m'}\rho_{nn'}^{-m's}+(-1)^{n-n'}\rho_{-n',-n}^{sm'}+(-1)^{m+n-n'}\rho_{-n',-n}^{-m's}].\quad(2.7)$$

We then group these quantities, weighted by the differential cross section according to

$$H=2\frac{d\sigma}{dt}\begin{pmatrix}\Lambda_{1/2,1/2}^{0,0} & -\Lambda_{1/2,-3/2}^{0,0}\\ -\Lambda_{1/2,-3/2}^{0,0*} & \Lambda_{3/2,3/2}^{0,0}\end{pmatrix},\quad(2.8\text{a})$$

$$A=4\frac{d\sigma}{dt}\begin{pmatrix}\Lambda_{3/2,3/2}^{1,1} & \Lambda_{3/2,-1/2}^{1,1}\\ \Lambda_{3/2,-1/2}^{1,1*} & \Lambda_{1/2,1/2}^{1,1}\end{pmatrix},\quad(2.8\text{b})$$

$$B=2\frac{d\sigma}{dt}\begin{pmatrix}\Lambda_{3/2,3/2}^{1,-1} & \Lambda_{3/2,-1/2}^{1,-1}\\ \Lambda_{-1/2,3/2}^{1,-1} & \Lambda_{1/2,1/2}^{1,-1}\end{pmatrix},\quad(2.8\text{c})$$

$$C=4\frac{d\sigma}{dt}\begin{pmatrix}\Lambda^{s,-1}_{3/2,3/2} & \Lambda^{s,-1}_{3/2,-1/2}\\ \Lambda^{s,-1}_{-1/2,3/2} & \Lambda^{s,-1}_{1/2,1/2}\end{pmatrix}, \tag{2.8d}$$

$$z_1=4\frac{d\sigma}{dt}\Lambda^{1,0}_{3/2,1/2}, \tag{2.8e}$$

$$z_2=-4\frac{d\sigma}{dt}\Lambda^{0,1}_{3/2,1/2}, \tag{2.8f}$$

$$z_3=4\frac{d\sigma}{dt}\Lambda^{s,0}_{3/2,1/2}. \tag{2.8g}$$

The transversity amplitudes for the P wave are denoted by $T_{m,n,\mu}$, where m denotes the spin-1 meson, n denotes the Δ, and μ denotes the initial nucleon transversity, with $T_{m,n,\mu}=0$ whenever $m+n-\mu$ is odd. Let us group them into

$$P=\begin{pmatrix}T_{0,-1/2,-1/2} & -T_{0,3/2,-1/2}\\ T^*_{0,1/2,1/2} & -T^*_{0,-3/2,1/2}\end{pmatrix}, \tag{2.9a}$$

$$Q=\begin{pmatrix}T_{-1,-3/2,-1/2} & T_{-1,+1/2,-1/2}\\ T^*_{1,3/2,1/2} & T^*_{1,-1/2,1/2}\end{pmatrix}, \tag{2.9b}$$

$$R=\begin{pmatrix}T_{1,-3/2,-1/2} & T_{1,1/2,-1/2}\\ T^*_{-1,3/2,1/2} & T^*_{-1,-1/2,1/2}\end{pmatrix}. \tag{2.9c}$$

The transversity amplitudes $T_{n,\mu}$ for the S-wave $\pi\pi$ system are zero unless $n-\mu$ is odd, and may be grouped into

$$S=\begin{pmatrix}-T_{-3/2,-1/2} & -T_{1/2,-1/2}\\ T^*_{3/2,1/2} & T^*_{-1/2,1/2}\end{pmatrix}, \tag{2.10}$$

and we shall introduce also the 2×2 matrix, depending on S,

$$M=[\det(S^\dagger S)](S^\dagger S)^{-1}. \tag{2.11}$$

The transversity observables may now be written in terms of the transversity amplitudes as

$$H=P^\dagger P+\tfrac{1}{3}M, \tag{2.12a}$$

$$A=Q^\dagger Q+R^\dagger R+\tfrac{2}{3}S^\dagger S, \tag{2.12b}$$

$$B=Q^\dagger R, \tag{2.12c}$$

$$C=S^\dagger R-Q^\dagger S, \tag{2.12d}$$

$$z_1=\mathrm{Tr}(Q^\dagger P), \tag{2.12e}$$

$$z_2=\mathrm{Tr}(R^\dagger P), \tag{2.12f}$$

$$z_3=\mathrm{Tr}(S^\dagger P). \tag{2.12g}$$

Having written the observables in this way, we see that if the transversity amplitudes represented by P, Q, R, and S are a solution, then so are those represented by UP, UQ, UR, and US, with U any 2×2 unitary matrix. This four-parameter ambiguity is an inescapable part of a model-independent amplitude analysis.[9] However, this ambiguity does not mix S and P waves, hence one may attempt to separate them. We remark that while there are 30 real observables, there are only 28 real parameters among the amplitudes, hence two constraints must exist. As these constraints are certainly not linear, any data averaged over some mass or t interval need not satisfy them, and hence need not be analyzable.

B. Positivity constraints and the equal-phase hypothesis

Given the experimental moments, one may ask whether they are consistent with the general requirements of positivity of the combined $\pi\pi\Delta$ spin-density matrix. In a previous work it was found that data on the reactions $\pi^+p\to\rho^0\Delta^{++}$, $\pi^+p\to\omega\Delta^{++}$, and $KN\to K^*\Delta^{++}$ were at slight variance with certain necessary conditions of positivity for the vector-meson Δ system.[15] It seems thus advisable to extend these conditions to the case of $\pi\pi\Delta$ (or $K\pi\Delta$) production. The key ingredients are the Eberhard-Good theorem[16] on the rank of the density matrix and the fact that the experimental joint moments in a helicity basis do not depend on the initial nucleon polarization if they are all in phase.

Let us consider the reaction $\pi N\to(\pi\pi)\Delta^{++}$, and suppose that in its rest frame the $\pi\pi$ system decays along a direction (θ_1,φ_1). The spin-density matrix for the Δ when this happens may be denoted by

$${}^{\theta_1\varphi_1}\rho_{nn'}$$

and its corresponding multipole parameters be

$${}^{\theta_1\varphi_1}t^{M_2}_{L_2}.$$

The joint-decay-angular distribution may then by written as

$$W(\theta_1,\varphi_1,\theta_2,\varphi_2)=\sum_{\substack{L_2=0\\ \text{even}}}^{2} g(L_2)\sum_{M_2=-L_2}^{L_2}{}^{\theta_1\varphi_1}t^{M_2}_{L_2}Y^{M_2*}_{L_2}(\theta_2,\varphi_2), \tag{2.13}$$

with

$$g(L_2)=(4\pi)^{-1/2}(-1)^{L_2/2}. \tag{2.14}$$

From Eq. (2.1) it then follows that

$${}^{\theta_1\varphi_1}t^{M_2}_{L_2}=\frac{1}{g(L_2)}\sum_{L_1=0}^{2}\sum_{M_1=-L_1}^{L_1}\langle Y^{M_1}_{L_1}Y^{M_2}_{L_2}\rangle Y^{M_1*}_{L_1}(\theta_1,\varphi_1). \tag{2.15}$$

Positivity then implies

$${}^{\theta_1\varphi_1}t^0_0\geq 0, \tag{2.16a}$$

$$({}^{\theta_1\varphi_1}t^0_0)^2-5\sum_{M_2=-2}^{2}|{}^{\theta_1\varphi_1}t^{M_2}_2|^2\geq 0, \tag{2.16b}$$

for all values of (θ_1,φ_1).

If we replace ${}^{\theta_1\varphi_1}t^{M_2}_{L_2}$ by its definition, we may express the requirements of positivity as

$$\sum_{L_1=0}^{2}\sum_{M_1=-L_1}^{L_1}\langle Y^{M_1}_{L_1}Y^0_0\rangle Y^{M_1*}_{L_1}(\theta_1,\varphi_1)\geq 0, \tag{2.17}$$

$$\sum_{S=0}^{4}\sum_{\sigma=-S}^{S}D^{S*}_{\sigma 0}(\varphi_1,\theta_1,0)\tilde{C}^S_\sigma\geq 0, \tag{2.18}$$

for all (θ_1,φ_1), where we have defined

$$C_\sigma^S = \sum_{L_1 L_1'} [(2L_1+1)(2L_1'+1)]^{1/2} C_{0,0,0}^{L_1', L_1', S} \sum_{M_1 M_1'} C_{M_1 - M_1' \sigma}^{L_1 L_1' S} (-1)^{M_1'} \left(\langle Y_{L_1}^{M_1} Y_0^0 \rangle^* \langle Y_{L_1'}^{M_1'} Y_0^0 \rangle - 5 \sum_M \langle Y_{L_1}^{M_1} Y_2^M \rangle^* \langle Y_{L_1'}^{M_1'} Y_2^M \rangle \right), \quad (2.19)$$

and where $D_{\sigma 0}^S$ is the usual Wigner rotation function. These inequalities involve only the raw experimental moments, and may well prove quite constraining in practice. Since there are odd as well as even spherical harmonics, we cannot eliminate the (θ_1, φ_1) dependence in an algebraic manner. We further note that the Eberhard-Good theorem implies that the left-hand side of (2.18) must be proportional to $(1-P^2)$ where $\vec{P}$ is the initial nucleon polarization vector. If all helicity amplitudes were in phase, the moments would not depend on P, and hence the $\tilde{C}_\sigma^S$ would all be zero. This provides an extension to the $\pi\pi$ system of the tests of the equal-phase hypothesis proposed in Ref. 17. We note that if one uses a helicity system, then the value of the expression (2.18) at $\theta = 0$, π is sensitive only to phase difference in the helicity zero P-wave and S-wave amplitudes. However, such tests are sensitive to averaging, and as one expects phase differences to arise from the final state phase shifts, it would be necessary to have copious data in small $m_{\pi\pi}$ and t intervals in order to perform the tests.

III. DETERMINATION OF THE AMPLITUDES

Given a set of observables

$$\langle Y_{L_1}^{M_1} Y_{L_2}^{M_2} \rangle$$

which satisfy the positivity conditions (2.17) and (2.18), one may attempt to determine the amplitudes. The observables must be grouped into the matrices H, A, B, C and the complex numbers z_1, z_2, z_3 of (2.8). The problem is then to solve the set of equations (2.12) for the transversity amplitudes. Our method relies on certain techniques for solving algebraic matrix equations.[18] Since the equations are fourth order in the unknown matrix, this implies the solution of an eighth-order ordinary algebraic equation, which can only be done numerically. In consequence, we are not able to prove rigorously some results which are suggested by our numerical studies.

The first step is to study the unnatural-parity-exchange sector, represented by the amplitudes Q, R of Eqs. (2.9b) and (2.9c), S of Eq. (2.10), and by the observables A, B, and C of Eqs. (2.8b), (2.8c), and (2.8d). The relevant equations are (2.12b), (2.12c), and (2.12d), and while we show in the appendix how and under which conditions these equations may be solved, we give here a brief discussion of our results. The basic conclusion is that if the 2×2 matrix $N(\varphi)$, where

$$N(\varphi) = A + (\tfrac{2}{3})^{1/2} C e^{i\varphi} + (\tfrac{2}{3})^{1/2} C^\dagger e^{-i\varphi} - B e^{2i\varphi} - B^\dagger e^{-2i\varphi}, \quad (3.1)$$

is positive for all values of φ, then there exist 16 sets of matrices Q, R, and S such that Eqs. (2.12b), (2.12c), and (2.12d) are satisfied. We cannot prove rigorously this result; it would appear to be a generalization of the Fejer-Riesz theorem on positive trigonometric polyomials (as discussed for example in Ref. 19) to the case of 2×2 Hermitian matrices. However, the positivity of $N(\varphi)$ turns out to be just the positivity of the Δ density matrix when the $\pi\pi$ system decay occurs in the production plane with some azimuthal angle φ. Thus a subset of the positivity conditions (2.18) suffices to ensure 16 discrete solutions for Q, R, and S.

Next the determination of the natural-parity amplitudes grouped in the matrix P may be carried out using Eq. (2.12a). For each of the 16 candidates found in the preceding step, one forms the 2×2 matrix M of Eq. (2.11), and writes

$$P^\dagger P = H - M. \quad (3.2)$$

Clearly the right-hand side of this equality must be a positive matrix, but there is no reason for this to be true for all 16 values of M. What is suggested by numerical studies using randomly generated amplitudes and observables is that if the positivity condition (2.18) is satisfied for all (θ, φ), then at least one of the 16 possible matrices M will be such that $H - M$ is positive. Thus positivity implies that there exists at least one solution. At this point, if P, Q, R, and S form a solution, then so also do UP, $U'Q$, $U'R$, and $U'S$, where U and U' are two arbitrary unitary matrices.

The final step is to use Eqs. (2.12e), (2.12f), and (2.12g) to find the relative unitary matrix between the natural-parity amplitude matrix P, and the unnatural parity Q, R, and S. For each of the candidates P, Q, R, and S surviving the previous selection, we can write

$$z_1 = \mathrm{Tr}(UPQ^\dagger), \quad (3.3a)$$

$$z_2 = \mathrm{Tr}(UPR^\dagger), \quad (3.3b)$$

$$z_3 = \mathrm{Tr}(UPS^\dagger). \quad (3.3c)$$

Choosing any two of z_1, z_2, z_3, one can solve for the unitary matrix U, as was shown in Ref. 18. In order for a solution to exist, a rather complicated inequality among the observables given in Ref. 18 must be satisfied. Finally the constraint that the third z_i be given by the matrix U found from the other two z_j's can be used to make a final selection among the candidates. On the basis of numerical studies we conjecture that given perfect data, there is only one solution for the amplitudes. In support of this hypothesis, we present the following evidence: Using randomly generated amplitudes, we have calculated the observables, and then fed these observables into a computer program which carries out the amplitude analysis. The program always found (in 1000 attempts) the input solution, and in no cases found that any of the 15 other candidates were acceptable; the final constraint always eliminated them. However, this result depends essentially on the error-free nature of our generated observables. When errors are included the discrete solutions become bands, and hence will tend to overlap. Furthermore, the constraints will not be exactly satisfied by any of the solutions, and it would be hard to choose the "best" solution.

A curious result is that if one had a set of experimental moments such that the 10 S-P interference terms were zero, there would exist discrete solutions for which the S-wave amplitudes would not be zero (but which in general would not satisfy the constraints at the final stage of the analysis).

From the analysis outlined here, we conclude that it is feasible to determine Q, R, and S up to one arbitrary unitary matrix, with P determined up to another matrix, since positivity conditions allow one to proceed thus far. The amplitudes in P correspond to natural-parity exchange such as A_2, while those in Q, R, and S correspond to unnatural-parity exchange, π, B, A_1, etc. There will in general be some discrete ambiguities, the number not exceeding 16.

IV. BOUNDS ON S-WAVE PRODUCTION

The amplitude analysis outlined in the preceding section is quite complicated, involving the numerical solution of an eighth-order ordinary equation, and the testing of a large number of candidates. If the positivity conditions are (2.18) even slightly violated the procedure will break down, and if some averaging over either $\pi\pi$ or πN masses or t is carried out there is but little chance that a unique solution will emerge. Since there exist no published data on the full set of joint moments, we cannot put our method of analysis to a practical test. In order to show the interest it may present, we discuss the problem of bounds on the amount of S-wave production. The reactions $\pi^+p \to \pi^+\pi^-\Delta^{++}$ and $K^+p \to K^+\pi^-\Delta^{++}$ have been studied by various experimental groups, and diverse methods of analysis have been used to estimate the $\pi\pi$ and $K\pi$ phase shifts.[7,8] In these analyses, however, only those moments with $L_2=0$ are used, which amounts to summing over all spin states of the Δ. The analysis then reduces, essentially, to that of $\pi^-p \to \pi^+\pi^-n$ on an unpolarized target. In this case the positivity requirements on the $(\pi\pi)$ moments are known[4,5,13]; they imply that a cubic equation, whose coefficients are functions of the experimental moments, should have two positive roots r_1, r_2 less than a certain upper limit furnished by the roots of a certain quadratic equation. When this happens, the S wave may be assigned an arbitrary value in the interval (r_1, r_2) and a consistent amplitude analysis will result. Thus if one sums over the Δ spin, upper and lower estimates of the amount of S-wave production result. Clearly the use of our analysis, even if the final step, the linking of natural- and unnatural-parity exchange amplitudes, cannot be carried out, would provide tighter estimates of the S-wave production. In our opinion, our method provides a means to critically examine the results of those analyses which do not use the Δ decay moments. Our method is not in itself a replacement for model-dependent analyses, since the latter yield results which, although they depend on various hypotheses, are more readily interpreted in terms of dynamics. Ideally, our method furnishes a means to judge the validity of the hypotheses employed, since any acceptable model-dependent analysis should be roughly consistent with at least one solution found by the model-independent means.

It is interesting to note that even if our full analysis is not employed, better estimates of the S-wave production can still be obtained by using the moments

$$\langle Y_{L_1}^{M_1} Y_2^0 \rangle$$

as well as the

$$\langle Y_{L_1}^{M_1} Y_0^0 \rangle$$

Using these additional moments, one may form the quantities

$$\Lambda_{nn}^{00}, \quad \Lambda_{nn}^{11}, \quad \Lambda_{nn}^{1-1}, \quad \Lambda_{nn}^{s-1},$$

where n is $\frac{1}{2}$ or $\frac{3}{2}$. Using these observable quantities, one can derive from positivity requirements upper and lower bounds on the amount of S-wave production when the Δ is either $\pm\frac{1}{2}$ or $\pm\frac{3}{2}$ transversity states. The prescription for calculating these bounds is given in the Appendix; the important result is that tighter bounds on the total amount of S-wave production are obtained than those which one finds by using only moments with $L_2=0$. Noting for example that the hypotheses used in Ref. 3 to analyze the reaction $\pi N \to \pi\pi N$ require the S-wave production to saturate its upper or lower bounds, it would be interesting to see whether $\pi\pi\Delta$ would be consistent with such hypotheses.

The family of discrete solutions obtained by our full analysis, minus the last step, is assured to have S-wave cross sections lying between these bounds. Since no published data exist, we have attempted to see how the interval provided by the S-wave bounds compares to the smallest and largest S-wave cross sections found by our detailed analysis. Using randomly generated amplitudes, we have generated observables, found the various candidates for the S-wave cross section, and computed the bounds. In most cases the acceptable solutions were grouped in an interval smaller, typically by a factor of 2, than the interval given by the bounds. Of course there is no reason that our random amplitudes should mimic the true situation; but we nonetheless feel that the complications inherent in our method may well be worth the trouble.

Finally we have studied, again numerically, what happens when the observables are found by superposing two sets of observables each generated by amplitudes. The nonlinear constraints are in general not satisfied, hence no exact amplitudes exist. However, omitting the final step, one obtains discrete candidates, the number varying from 1 to 16. In no case did our approach fail to yield acceptable candidates, which is consistent with our conjecture that our positivity conditions are sufficient to ensure an analysis, up to the final step.

Let us mention that in the limit of production in the forward direction, a restricted amplitude analysis can be carried out using those helicity amplitudes which are nonzero there. This has been done in Ref. 20. Our general method employs transversity amplitudes, for which the limiting behavior as $t' \to 0$ is rather complicated. Among the solutions generated by our procedure, some may not have the correct t' behavior, and could in principle be ruled out. In addition, the full four-parameter ambiguity reduces to one overall phase. We have studied this aspect of the problem using observables generated from helicity amplitudes having the correct t' behavior. Indeed certain solutions did have unreasonable behavior as $t' \to 0$, but even at quite small t' it would be difficult, in practice, to rule them out. In our opinion exceedingly precise data in very small t' intervals would be required to eliminate certain solutions on grounds of incorrect t' behavior (Of course, all the observables furnished by the false solutions *do* have the correct t' behavior.)

V. SUMMARY AND CONCLUSIONS

In this paper we have presented a method for performing a model-independent analysis of the reaction

$$\pi(K) + p \to \pi\pi(K\pi) + \Delta,$$

in which the $\pi\pi$ system is assumed to be in S and P waves. A set of necessary positivity conditions on the moments of the joint-decay-angular distribution have been presented, and their relation to tests of the equal-phase hypothesis have been discussed. The proposed method yields transversity amplitudes up to a four-parameter ambiguity and given perfect data, there are no discrete ambiguities. If any averaging of experiment quantities is done (as is inevitable) there will no longer be a unique solution, but rather separate four-parameter ambiguities for both natural- and unnatural-parity exchange amplitudes. In addition, between one and 16 discrete solutions may be found. One quantity which is not affected by the continuous ambiguities is the total S-wave cross section; only the discrete ambiguities occur.

A set of refined bounds on the total S-wave production have also been discussed which employ partial knowledge of the Δ decay moments. Their use may provide a means of testing hypotheses used in analyses which neglect the Δ moments.

We remark that the positivity conditions we derive are valid even if there is some S-wave πN background under the Δ peak. If there is some P-wave $J=\frac{1}{2}$ background, however, the conditions could be violated by correct data.

Finally we note that our method can be extended to other spin-0–spin-1 superposition problems (such as $\pi\rho$ in both 0^- and 1^+ states) provided the correlation moments are available. Similarly, the restriction to S- and P-wave states could be lifted, although the method would involve considerably greater numerical complexity.

Let us close with the plea that experimental groups analyzing $\pi\pi\Delta$ and $K\pi\Delta$ final states publish the full joint-moment analysis, as it contains considerable dynamical information.

APPENDIX

1. Amplitude analysis

The problem is: Given (2×2) matrices, A, B, and C, find matrices Q, R, and S such that

$$Q^\dagger Q + R^\dagger R + \tfrac{2}{3} S^\dagger S = A\,, \tag{A1}$$

$$Q^\dagger R = B\,, \tag{A2}$$

$$S^\dagger R - Q^\dagger S = C\,. \tag{A3}$$

Let X be a matrix such that

$$-QX + RX^{-1} + (\tfrac{2}{3})^{1/2} S = 0\,. \tag{A4}$$

(This quadratic matrix equation always has a solution, given Q, R, and S.) In order to obtain an equation for X in which only the known matrices A, B, and C occur, we may carry out the following steps:

(a) Multiply Eq. (A4) on the left by $-Q^\dagger$ and on the right by X^{-1};

(b) Multiply Eq. (A4) on the left by $R^\dagger$ and on the right by X;

(c) Multiply Eq. (A4) by $(\frac{2}{3})^{1/2}S^\dagger$ on the left side and add to this the results of steps (a) and (b).

This yields

$$A - BX^{-2} - B^\dagger X^2 + (\tfrac{2}{3})^{1/2}CX^{-1} + (\tfrac{2}{3})^{1/2}C^\dagger X = 0 . \quad \text{(A5)}$$

This fourth-order matrix equation, following the discussion of Gantmacher,[21] may be converted into an eighth-order ordinary equation, which requires numerical solution. From the roots of this algebraic equation, one can construct all solutions of Eq. (A5). For each possible value of X, one must carry out the following steps in order to determine Q, R, and S. Assuming that X is known, it follows from Eq. (A4) that

$$S = (\tfrac{2}{3})^{1/2}(QX - RX^{-1}) . \quad \text{(A6)}$$

Left-multiplying this equation by Q, and adding to it the adjoint of Eq. (A6) right-multiplied by R yields

$$(\tfrac{2}{3})^{1/2}C = X^\dagger B + BX^{-1} - X^{-1\dagger}R^\dagger R - Q^\dagger QX . \quad \text{(A7)}$$

By appropriate multiplications Eq. (A7) may be recast into two forms,

$$Q^\dagger Q + (RX^{-1})^\dagger(RX^{-1}) = X^\dagger BX^{-1} + BX^{-2} - (\tfrac{2}{3})^{1/2}CX^{-1} , \quad \text{(A8)}$$

or

$$(QX)^\dagger(QX) + R^\dagger R = (X^\dagger)^2 B + X^\dagger BX^{-1} - (\tfrac{2}{3})^{1/2}X^\dagger C . \quad \text{(A9)}$$

Clearly the left-hand sides of Eqs. (A8) and (A9) are positive Hermitian matrices, whereas the right-hand sides are not manifestly so. For those values of X such that the right-hand sides are positive and Hermitian, one may continue the analysis. In Ref. 18, it was shown that the system of (2×2) matrix equations

$$E^\dagger E + F^\dagger F = A , \quad \text{(A10)}$$

$$E^\dagger F = B \quad \text{(A11)}$$

can be solved for E and F provided that $A + e^{i\varphi}B + e^{-i\varphi}B^\dagger$ is a positive matrix for all φ. It is then possible to rewrite Eq. (A2) in the form

$$(QX)^\dagger R = X^\dagger B \quad \text{(A12)}$$

so that Eqs. (A9) and (A12) can be solved for QX and R up to a unitary transformation, and four discrete solutions.

Given Q, R, and S, we may write

$$N(\varphi) = [-Qe^{i\varphi} + Re^{-i\varphi} + (\tfrac{2}{3})^{1/2}S]^\dagger \times [-Qe^{i\varphi} + Re^{-i\varphi} + (\tfrac{2}{3})^{1/2}S] , \quad \text{(A13)}$$

where N is given by Eq. (3.1). Clearly, it is a necessary condition for an amplitude analysis that $N(\varphi)$ be positive, but our numerical studies suggest that it is also sufficient. If $N(\varphi)$ is positive we find that the procedure yields 16 different sets of matrices Q, R, and S.

The possibility of determining the matrix P via Eq. (2.12a) provides a selection among the 16 candidates for solutions. Using each candidate S, one forms M, Eq. (2.11), and asks whether

$$H - M$$

is positive-semidefinite. If not, the candidate is eliminated. We have no proof that the full positivity conditions (2.18) imply that at least one successful candidate exists, but we believe it to be true, on the basis of our numerical studies.

2. Bounds on S-Wave production when the Δ is in a definite transversity state

Let the $(S\text{-}P)$ Δ joint-density matrix be denoted by

$$\rho^{\mu\mu'}_{nn'}, \quad \mu = s, 0, \pm 1, \quad n = \pm\tfrac{3}{2}, \pm\tfrac{1}{2} .$$

In order to obtain bounds directly from the observables, it is necessary to choose certain linear combinations of the joint-density matrix. For this we introduce the 4×4 unitary matrix Γ whose only nonvanishing elements are

$$\Gamma^{s,s} = \Gamma^{0,0} = -\Gamma^{1,-1} = -\Gamma^{-1,1} = 1 , \quad \text{(A14)}$$

and we consider, for n either $\frac{1}{2}$ or $\frac{3}{2}$,

$$\bar{\rho}_{nn} = \tfrac{1}{4}(\rho_{nn} + \rho_{-n-n} + \Gamma\rho_{nn}\Gamma + \Gamma\rho_{-n-n}\Gamma) . \quad \text{(A15)}$$

This 4×4 matrix may be expressed in terms of the observables $\Lambda^{\mu\mu'}_{nn'}$, together with the total amount of S-wave production when the Δ has transversity n, which we call S_{nn}. Explicitly,

$$\bar{\rho}_{nn} = \begin{pmatrix} \Lambda^{0,0}_{nn} - \frac{1}{3}S_{nn} & 0 & 0 & 0 \\ 0 & S_{nn} & -\Lambda^{s,-1*}_{nn} & \Lambda^{s,-1}_{nn} \\ 0 & -\Lambda^{s,-1}_{nn} & \Lambda^{1,1}_{nn} - \frac{1}{3}S_{nn} & \Lambda^{1,-1}_{nn} \\ 0 & \Lambda^{s,-1*}_{nn} & \Lambda^{1,-1*}_{nn} & \Lambda^{1,1}_{nn} - \frac{1}{3}S_{nn} \end{pmatrix} . \quad \text{(A16)}$$

The positivity condition is that there must exist at least one value of S_{nn} such that $\bar{\rho}_{nn}$ is a non-negative matrix. Obviously one must have

$$0 \leqslant S_{nn} \leqslant 3 \min\{\Lambda_{nn}^{0,0}, \Lambda_{nn}^{1,1}\}. \tag{A17}$$

Next, by considering 2×2 minors, one improves this to

$$\tfrac{3}{2}(\Lambda_{nn}^{1,1} - [(\Lambda_{nn}^{1,1})^2 - \tfrac{4}{3}|\Lambda_{nn}^{s,-1}|^2]^{1/2}) \leqslant S_{nn} \leqslant \min\{3(\Lambda_{nn}^{1,1} - |\Lambda_{nn}^{1,-1}|), 3\Lambda_{nn}^{0,0}, \tfrac{3}{2}(\Lambda_{nn}^{1,1} + [(\Lambda_{nn}^{1,1})^2 - \tfrac{4}{3}|\Lambda_{nn}^{s,-1}|^2]^{1/2})\}. \tag{A18}$$

The strongest condition is that the determinant be positive,

$$S_{nn}{}^3 + pS_{nn}{}^2 + qS_{nn} + r \geqslant 0, \tag{A19}$$

where

$$p = -6\Lambda_{nn}^{1,1}, \tag{A20}$$

$$q = 9[(\Lambda_{nn}^{1,1})^2 - |\Lambda_{nn}^{1,-1}|^2] + 6|\Lambda_{nn}^{s,-1}|^2, \tag{A21}$$

$$r = -18\{\Lambda_{nn}^{1,1}|\Lambda_{nn}^{s,-1}|^2 + \mathrm{Re}[\Lambda_{nn}^{1,1*}(\Lambda_{nn}^{s,-1})^2]\}. \tag{A22}$$

The requirement of positivity is that the roots of Eq. (A19), considered as an equation, be real and positive, and that the two smallest roots lie inside the interval in S_{nn} given by Eq. (A18). Then any value of S_{nn} between these two roots is a possible value for the S-wave cross section with the Δ in the transversity state $\pm n$. We have shown that if the experimental moments satisfy the positivity conditions (2.18), then Eq. (A19) does have three positive roots, two occurring in the correct interval.

The same procedure can be performed using $\bar{\rho}_{1/2,1/2} + \bar{\rho}_{3/2,3/2}$, but the bounds on the sum $S_{1/2,1/2} + S_{3/2,3/2}$ are looser than individual bounds on $S_{1/2\,1/2}$ and $S_{3/2,3/2}$ separately.

A similar procedure can be carried out if instead of transversity states for the Δ, one considers helicitylike states for the Δ for an arbitrary choice of the z axis in the production plane. One then obtains bounds on $S_{1/2,1/2}$ and $S_{3/2,3/2}$ as functions of the azimuthal angle.

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