Comments and Addenda

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Comment on the three-body unitary formalism

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We present a simplified version of the three-body unitary (K-matrix) formalisms proposed by Cahill, by Kowalski, and by Sasakawa.

Kowalski[1] has shown that the alternative three-body unitary formalisms proposed by Cahill[2] and by Kowalski[3] are equivalent, the difference coming essentially from the order in which the various singularities are removed from the three-body transition amplitudes. The demonstration of this fact involved a simplification of Cahill's formalism, in that one intermediate set of equations was eliminated, leaving one set of equations for the three-body K matrices and a hierarchy of two sets of Heitler-type integral equations for the three-body transition amplitudes in terms of the K matrices.

The simple rederivation of the formalisms to be presented in this comment shows that the two sets of Heitler-type equations can be replaced by one single set, corresponding to a simultaneous (rather than stepwise) removal of the two- and three-body singularities from the three-body transition amplitude.

Using the notation of Ref. 1, we recall that all three-body transition amplitudes can be obtained from one single (matrix of) operator(s) $F(\pm)$ which satisfies the matrix equations

$$\begin{aligned} F(\pm) &= \bar{\delta} G_0(\pm) + \bar{\delta} G_0(\pm)\, t(\pm)\, F(\pm) \\ &= \bar{\delta} \boldsymbol{G}_0(\pm) + F(\pm)\, t(\pm)\, \bar{\delta} G_0(\pm). \end{aligned} \tag{1}$$

Here

$$\begin{aligned} t(\pm) &= V + VG_0(\pm)\, t(\pm) \\ &= V + t(\pm) G_0(\pm) V \end{aligned}$$

is a diagonal 3×3 matrix with elements t_α, $\alpha = 1, 2, 3$, $G_0(\pm) = (E \pm i0 - H_0)^{-1}$, and

$$\bar{\delta} = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

More specifically, the physical amplitudes are obtained as on-shell matrix elements of the operators

$$\begin{aligned} M(\pm) &= t(\pm)\, F(\pm)\, t(\pm), \\ M^R(\pm) &= t(\pm)\, F(\pm)\, VP, \\ M^L(\pm) &= P\, V\, F(\pm)\, t(\pm), \\ M^{LR}(\pm) &= P\, VF(\pm)\, VP, \end{aligned} \tag{2}$$

where P is a diagonal matrix with channel eigenstate projectors

$$P_\alpha = \sum_{E,\, \eta_\alpha} |\, \phi_\alpha(\eta_\alpha, E)\rangle\langle \phi_\alpha(\eta_\alpha, E)|$$

as elements.

The three-body K matrices on the on the other hand are obtained from the operator $\tilde{C}$ satisfying the equation

$$\tilde{C} = \bar{\delta} G + \bar{\delta} G k \tilde{C} = \bar{\delta} G + \tilde{C} k \bar{\delta} G, \tag{3}$$

where G is the real part of $G_0(\pm)$, $G_0(\pm) = G \mp iD_0$, and k is a two-body k operator defined by [Eqs. (4.4) and (4.13) of Ref. 1]

$$t(\pm) = k \mp ikD_0 t(\pm) \mp iVDV. \tag{4}$$

In Eq. (4), D is the diagonal matrix with elements

$$D_\alpha = \sum_{E'_\alpha,\, \eta_\alpha} |\, \phi_\alpha(\eta_\alpha, E'_\alpha)\rangle\, \delta(E - E'_\alpha) \langle \phi_\alpha(\eta_\alpha, E'_\alpha)|\,,$$

so that the last term in (4) accounts for the imaginary part of $t(\pm)$ due to two-body bound states. In analogy with (2) we now define the three-body K matrix to be the on-shell matrix elements of the operators

$$
\begin{aligned}
K &= k\tilde{C}k,\\
K^R &= k\tilde{C}VP,\\
K^L &= PV\tilde{C}k,\\
K^{LR} &= PV\tilde{C}VP.
\end{aligned}
\tag{5}
$$

These on-shell matrix elements coincide with the on-shell matrix elements of the operator C of Ref. 1 [cf. Eq. (4.22)].

In the Cahill version of the unitary formalism the transition amplitudes are obtained from the K matrices by solving the two sets of Heitler-type equations (4.7), (4.18)–(4.20), and (4.24)–(4.27) of Ref. 1, while in the Kowalski version one has to solve the sets of Heitler-type equations (5.5)–(5.8) and (5.11)–(5.14).

In order to demonstrate that the unitary formalism naturally involves only one set of Heitler-type equations, we proceed with a simple rederivation of the K-matrix equations. We first write Eq. (1) as

$$F(\pm) = [1 - \bar{\delta}G_0(\pm)t(\pm)]^{-1}\bar{\delta}G_0(\pm). \tag{6}$$

Multiplying the numerator and denominator of this expression by $(1 \pm iD_0k)$ and making use of relation (4) we obtain

$$
\begin{aligned}
F(\pm) = (1 \pm iD_0k)\{1 - \bar{\delta}Gk + [(1+\bar{\delta})iD_0k + \bar{\delta}G\, iVDV]\}^{-1}&\\
\times \bar{\delta}G_0(\pm).&
\end{aligned}
\tag{7}
$$

Expanding the inverse operator, we can also write Eq. (7) as

$$
\begin{aligned}
F(\pm) = &(1 \pm iD_0k)(1 - \bar{\delta}Gk)^{-1}\bar{\delta}G_0(\pm)\\
&\mp (1 \pm iD_0k)(1 - \bar{\delta}Gk)^{-1}\\
&\quad\times[(1+\bar{\delta})iD_0t(\pm)F(\pm) + \bar{\delta}G\, iVDVF(\pm)].
\end{aligned}
\tag{8}
$$

From this expression it is only a matter of identifying terms in order to obtain the final expression for the M's of Eq. (2) in terms of the K's of Eq. (5). Dropping terms that will vanish upon taking *on-shell* matrix elements we get

$$
\begin{aligned}
M(\pm) + t(\pm) &\hat{=} K + k \mp K^R iDM^L(\pm)\\
&\quad \mp (K+k)(1+\bar{\delta})iD_0[M(\pm) + t(\pm)],\\
M^L(\pm) &\hat{=} K^L \mp K^{LR} iDM^L(\pm)\\
&\quad \mp K^L(1+\bar{\delta})iD_0[M(\pm) + t(\pm)],\\
M^R(\pm) &\hat{=} K^R \mp K^R iDM^{LR}(\pm)\\
&\quad \mp (K+k)(1+\bar{\delta})iD_0M^R(\pm),\\
M^{LR}(\pm) &\hat{=} K^{LR} \mp K^{LR} iDM^{LR}(\pm)\\
&\quad \mp K^L(1+\bar{\delta})iD_0M^R(\pm),
\end{aligned}
\tag{9}
$$

where the $\hat{=}$ sign has been used to indicate that the equalities are true only after on-shell matrix elements have been taken.

The set of (pairwise coupled) equations (9) is the main result of this paper. These equations certainly have the form to be expected from three-body Heitler-type equations and together with Eqs. (3) and (5) they define the three-body unitary formalism.

The previously obtained Heitler-type equations follow in a simple manner from Eq. (9). The simplification in form obtained here is therefore not entailed by any reduction in the computational work required to actually solve these equations.

The general motivation for and possible usefulness of unitary three-body formalisms has been discussed previously[1-3,4] and will not be repeated here.

To conclude, we have obtained a three-body unitary formalism that only involves one set of equations for the three-body K matrix, and one set of Heitler-type equations for the three-body T matrix in terms of this K matrix. This latter set of equations replaces the hierarchy of Heitler-type equations proposed by Cahill and by Kowalski.

We finally remark that the unitary formalism constructed by Sasakawa[4] is identical to the formalism developed here.[5]

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[1]K. L. Kowalski, Phys. Rev. D 10, 1271 (1974).
[2]R. T. Cahill, Nucl. Phys. A194, 599 (1972).
[3]K. L. Kowalski, Phys. Rev. D 5, 395 (1972).
[4]T. Sasakawa, Nucl. Phys. A203, 496 (1973).

[5]This follows from the fact that Eqs. (79) and (86) of Ref. 4 are equivalent to our Eq. (3). See also the reference to Sasakawa's paper in Ref. 1.