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Relativistic invariance as gauge invariance and high-intensity Compton scattering

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The problem of an electron interacting with the classical external field of a plane electromagnetic wave is reexamined. An explicit connection between symmetry properties of the free field and the behavior of an electron in the external field is exhibited. A new group of transformations leaving the electromagnetic field tensor unaltered is constructed and is used to derive the classical and quantum (semiclassical) evolution operators of an electron in the wave. The classical operator is shown to describe both the evolution of the electron and the evolution of its spin in the external field. The quantum operator is used to define a representation which seems well adapted to the calculation of scattering processes that occur in a laser beam. However, difficulties of uniqueness of this representation appear in the case of a monochromatic wave. The high-intensity Compton scattering amplitude is derived as a simple application of the new formalism.

INTRODUCTION

During the last decade, the Volkov states,[1] which are exact solutions of the Dirac equation for an electron in the external field of a plane electromagnetic wave, have received considerable attention. This attention is justified by the fact that they make it possible to calculate the cross section of various scattering processes that occur in the interaction between free or weakly bound electrons and the intense fields produced by powerful lasers.

The Volkov states have been used to calculate modifications of the photoelectric cross section induced by an intense maser field,[2] to calculate the relativistic cross section of the multiphotonic inverse bremsstrahlung process,[3] and to obtain a generalization of the Klein-Nishina cross section of the Compton effect to include an explicit dependence on the external field intensity.[4-7] However, their use is quite controversial.[5] Actually, difficulties of decoupling the electron from the beam are encountered when Volkov states are assumed to represent the incident and outgoing electron states in the intense field. As has been shown by Kibble,[6] the condition *sine qua non* for the nonlinear Compton effect to exist lies in the electron representation by free states before and after its interaction with the light beam.

Furthermore, it is known that Volkov states exhibit a classical character. Nonlinear effects obtained by using Volkov states can be explained in purely classical arguments. The origin of the intensity-dependent frequency shift in Compton scattering can merely be seen as a Doppler shift arising from the nonzero average velocity of the electron in the beam, while Compton scattering is, as is well known, a purely quantum effect. For these reasons, we felt the necessity of reexamining the electron behavior in the classical external field of a plane wave.

The purpose of this paper is to present some new features of the electron behavior in such a field and to propose a new method of treating laser-electron scattering problems. This method consists in using a representation in which the electron behaves as a free particle and which is obtained by transforming the original Dirac equation by means of the evolution operator of the particle in the field. This operator will be derived by using the quite remarkable properties of symmetry which exist in the problem and which do not seem to have been shown before. In fact, the symmetries of the free field will play a leading part. This is easy to understand remembering the basic assumption of the theory of the external field.

As is well known, in this theory it is assumed that the field is acting on the particle without reaction of the particle on the field. Let us now suppose that, for a given field, we exactly know the operator which transforms the state of a free electron into the state of the particle coupled to the external field, that is, the evolution operator of the particle. Since this operator shows the particle behavior without reacting on the field, it is reasonable to expect this operator to leave the electromagnetic field unaltered. Hence, when one aims at studying the behavior of a charged particle in a given external field, the investigation of symmetries of the free field is often worthwhile. Plane waves are a type of field for which the symmetries will appear as most simple space-time symmetries.

In the first part of the paper, we shall recall the form of the most general operator of the proper Lorentz group which leaves both the electric and magnetic fields of a plane wave unaltered. Then, we shall see that this operator can be generalized as a Lorentz-type operator having the same space-time dependence as the fields. We shall show that the set of such space-time-dependent operators is an invariance group of Maxwell equations for plane waves.

The second part will be devoted to the derivation of the classical evolution operator as an element of the new invariance group. We shall point out that the classical evolution operator shows not only the motion of the particle but also the motion of spin of a Dirac particle in the field. This question was the subject of a previous paper[8] concerning only a linearly polarized plane wave.

In the third part of the paper we shall derive the quantum (semiclassical) evolution operator by using a method essentially based on an analogy with the classical case. This method is founded upon the obvious remark that in the proper frame of the electron interacting with the external field, the electron behaves as a free particle, the evolution of which is therefore described by the field-free Dirac equation. This method will single out the part played by the above-mentioned symmetries of the free field.

By means of the evolution operator we shall then define a representation[9] which seems well adapted to laser-electron scattering problems but which is not uniquely determined in the case of a monochromatic wave. We shall examine this difficulty by considering high-intensity Compton scattering as an example of the application of the new formalism.

I. SYMMETRY PROPERTIES OF THE FIELD OF AN ARBITRARY PLANE ELECTROMAGNETIC WAVE

Let us consider a plane electromagnetic wave, traveling along the x axis of coordinates (whose unit vector is $\vec{n}$), the four-potential of which is $A(\tau)$. The relativistic invariant $\tau = n\cdot r = t - x$ is the scalar product of the four-vector $r = \binom{t}{\vec{r}}$ by the null four-vector

$$n = \begin{bmatrix} 1 \\ \vec{n} \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} .$$

(Throughout the paper we shall use a unit system where $\hbar = c = 1$.)

As is known,[8] an operator of the proper Lorentz group which leaves both the electric field $\vec{E}(\tau)$ and the magnetic field $\vec{H}(\tau)$ of a plane wave unaltered exists. This operator is the product of an operator of Lorentz transformation (without rotation) $\mathfrak{L}$, whose associated velocity $\vec{\beta}$ verifies the relation

$$1 - \vec{\beta}\cdot\vec{n} = (1-\beta^2)^{1/2}, \tag{1}$$

by an operator $\mathfrak{R}$ of rotation in space around the unit vector

$$\vec{u} = [(\vec{n}\times\vec{\beta})^2]^{-1/2}\vec{n}\times\vec{\beta} \tag{2}$$

and of angle

$$\alpha = \pi - 2\cos^{-1}\{\beta^{-1}[1-(1-\beta^2)^{1/2}]\}. \tag{3}$$

Let

$$\nu^2 = 2[(1-\beta^2)^{-1/2} - 1].$$

In the above-mentioned reference,[8] we chose the velocity $\vec{\beta}$ in the (x, y) plane. More generally, if we call θ the angle between the projection $\vec{\beta}_\perp$ of $\vec{\beta}$ on the (y, z) plane and the y axis, the operator of Lorentz transformation $\mathfrak{L}(\nu,\theta)$ and the operator of rotation $\mathfrak{R}(\nu,\theta)$ can be written in the following matrix forms:

$$\mathfrak{L}(\nu,\theta) = \begin{bmatrix} 1+\frac{1}{2}\nu^2 & -\frac{1}{2}\nu^2 & -\nu\cos\theta & -\nu\sin\theta \\ -\frac{1}{2}\nu^2 & 1+\frac{1}{8}(1+\frac{1}{4}\nu^2)^{-1}\nu^4 & \frac{1}{4}(1+\frac{1}{4}\nu^2)^{-1}\nu^3\cos\theta & \frac{1}{4}(1+\frac{1}{4}\nu^2)^{-1}\nu^3\sin\theta \\ -\nu\cos\theta & \frac{1}{4}(1+\frac{1}{4}\nu^2)^{-1}\nu^3\cos\theta & 1+\frac{1}{2}(1+\frac{1}{4}\nu^2)^{-1}\nu^2\cos^2\theta & \frac{1}{4}(1+\frac{1}{4}\nu^2)^{-1}\nu^2\sin(2\theta) \\ -\nu\sin\theta & \frac{1}{4}(1+\frac{1}{4}\nu^2)^{-1}\nu^3\sin\theta & \frac{1}{4}(1+\frac{1}{4}\nu^2)^{-1}\nu^2\sin(2\theta) & 1+\frac{1}{2}(1+\frac{1}{4}\nu^2)^{-1}\nu^2\sin^2\theta \end{bmatrix},$$

$$\mathfrak{R}(\nu,\theta) = \begin{bmatrix} 1+\frac{1}{4}\nu^2 & 0 & 0 & 0 \\ 0 & 1-\frac{1}{4}\nu^2 & -\nu\cos\theta & -\nu\sin\theta \\ 0 & \nu\cos\theta & 1-\frac{1}{4}\nu^2\cos(2\theta) & -\frac{1}{4}\nu^2\sin(2\theta) \\ 0 & \nu\sin\theta & -\frac{1}{4}\nu^2\sin(2\theta) & 1+\frac{1}{4}\nu^2\cos(2\theta) \end{bmatrix}(1+\tfrac{1}{4}\nu^2)^{-1}, \tag{4}$$

and are easily obtained using the operators $\mathfrak{L}(\nu,\theta=0)$ and $\mathfrak{R}(\nu,\theta=0)$ of the above-mentioned reference by means of the relations

$$\mathcal{L}(\nu,\theta)=\mathcal{P}(\theta)\mathcal{L}(\nu,0)\mathcal{P}^{-1}(\theta),\quad \mathcal{R}(\nu,\theta)=\mathcal{P}(\theta)\mathcal{R}(\nu,0)\mathcal{P}^{-1}(\theta),$$

where $\mathcal{P}(\theta)$ is a 4×4 rotation matrix of angle θ around the x axis.

The most general operator of Lorentz transformation $\mathfrak{M}(\nu,\theta)$, which leaves both the electric and magnetic fields of a plane wave unaltered, can thus be written in the following form:

$$\mathfrak{M}(\nu,\theta)=\mathcal{R}(\nu,\theta)\mathcal{L}(\nu,\theta)=\begin{bmatrix}1+\frac{1}{2}\nu^2 & -\frac{1}{2}\nu^2 & -\nu\cos\theta & -\nu\sin\theta\\ \frac{1}{2}\nu^2 & 1-\frac{1}{2}\nu^2 & -\nu\cos\theta & -\nu\sin\theta\\ -\nu\cos\theta & \nu\cos\theta & 1 & 0\\ -\nu\sin\theta & \nu\sin\theta & 0 & 1\end{bmatrix}. \tag{5}$$

The operators $\mathcal{R}(\nu,\theta)$, $\mathcal{L}(\nu,\theta)$, and $\mathfrak{M}(\nu,\theta)$ can be easily written in tensor form. In order to do so, we define the following four-vectors:

$$f^\mu=(1+\tfrac{1}{4}\nu^2)^{-1/2}\begin{bmatrix}0\\ \frac{1}{2}\nu\\ \cos\theta\\ \sin\theta\end{bmatrix},$$

$$h^\mu=(1+\tfrac{1}{4}\nu^2)^{-1/2}\begin{bmatrix}0\\ -1\\ \frac{1}{2}\nu\cos\theta\\ \frac{1}{2}\nu\sin\theta\end{bmatrix},$$

$$k^\mu=\begin{bmatrix}1\\0\\0\\0\end{bmatrix};\; l^\mu=\begin{bmatrix}0\\0\\-\sin\theta\\ \cos\theta\end{bmatrix}$$

which are unit four-vectors ($k^2=-f^2=-h^2=-l^2=1$) orthogonal to each other ($f\circ h=k\circ f=f\circ l=h\circ k=h\cdot l=k\cdot l=0$). We also define the two following spacelike four-vectors:

$$a^\mu=\begin{bmatrix}0\\ \vec{a}\end{bmatrix}=\begin{bmatrix}0\\0\\ \cos\theta\\ \sin\theta\end{bmatrix};\quad j^\mu=n^\mu+a^\mu=\begin{bmatrix}1\\1\\ \cos\theta\\ \sin\theta\end{bmatrix}$$

which are such that $j^2=a^2=-1$ and $j\cdot n=a\circ n=0$.

By means of these definitions, the operators $\mathcal{R}$, $\mathcal{L}$, and $\mathfrak{M}$ can be rewritten in the following forms:

$$\mathcal{L}^\sigma{}_\rho(\nu,\theta)=(1+\tfrac{1}{2}\nu^2)\delta^\sigma_\rho+\tfrac{1}{2}\nu^2h^\sigma h_\rho+\tfrac{1}{2}\nu^2l^\sigma l_\rho + (1+\tfrac{1}{4}\nu^2)^{1/2}\nu g^{\sigma\omega}\epsilon_{\omega\rho\alpha\beta}h^\alpha l^\beta, \tag{6}$$

$$\mathcal{R}^\mu{}_\sigma(\nu,\theta)=(1+\tfrac{1}{4}\nu^2)^{-1}[(1-\tfrac{1}{4}\nu^2)\delta^\mu_\sigma+\tfrac{1}{2}\nu^2k^\mu k_\sigma-\tfrac{1}{2}\nu^2l^\mu l_\sigma - \nu g^{\mu\omega}\epsilon_{\omega\sigma\alpha\beta}k^\alpha l^\beta], \tag{7}$$

$$\mathfrak{M}^\mu{}_\rho(\nu,j)=\mathcal{R}^\mu{}_\sigma\mathcal{L}^\sigma{}_\rho =\delta^\mu_\rho+\nu(n^\mu j_\rho-j^\mu n_\rho)+\tfrac{1}{2}\nu^2n^\mu n_\rho, \tag{8}$$

where $\epsilon_{\omega\rho\alpha\beta}$ is the completely antisymmetric unit four-tensor (we have $\epsilon_{0123}=-\epsilon^{0123}=-1$) and $g^{\mu\rho}$ is the metric tensor ($g^{00}=-g^{11}=-g^{22}=-g^{33}=1$). Incidentally, we can verify that $\mathcal{R}^{-1}(\nu,\theta)=\mathcal{R}(-\nu,\theta)$ and that $\mathfrak{M}^{-1}(\nu,j)=\mathfrak{M}(-\nu,j)$.

As the operator of Lorentz transformation $\mathfrak{M}$ is given in the form (8), it is immediate to see that it leaves the four-vector n unaltered:

$$n'^\mu\equiv\mathfrak{M}^\mu{}_\rho n^\rho=n^\mu. \tag{9}$$

Acting on the four-potential $A(\tau)$ of the plane wave (which is submitted to the condition $n\cdot A=0$) it makes a gauge transformation:

$$A'^\mu\equiv\mathfrak{M}^\mu{}_\rho A^\rho=A^\mu+\nu(j\cdot A)n^\mu=A^\mu+\partial^\mu\chi \tag{10}$$

with

$$\chi(\tau)=\int_{-\infty}^{\tau}\nu[j\cdot A(\tau')]d\tau'.$$

It follows from relations (9) and (10) and from the relativistic invariance of the retarded time τ that the operator $\mathfrak{M}(\nu,j)$ leaves the tensor of the plane-wave field $F^{\alpha\beta}=\partial^\alpha A^\beta-\partial^\beta A^\alpha=(d/d\tau)(n^\alpha A^\beta-n^\beta A^\alpha)$ unchanged. $\mathfrak{M}(\nu,j)$ is the most general operator of the proper Lorentz group which leaves both the fields $\vec{E}(\tau)$ and $\vec{H}(\tau)$ of a plane wave unaltered.

The operator $\mathfrak{M}(\nu,j)$ depends on two arbitrary and independent parameters ν and j (that is, a). The parameter $\vec{a}$ is related to the direction of the Lorentz transformation ($\vec{a}=\beta_\perp^{-1}\vec{\beta}_\perp$), while the parameter ν characterizes its amplitude $[1+\frac{1}{2}\nu^2=(1-\beta^2)^{-1/2}]$. In the following, we shall see that it is sufficient to choose conveniently these parameters to get the classical and quantum evolution operators of an electron interacting with the classical external field of a plane wave. The parameters ν and j will then become operators and functions of τ. Thus the operator $\mathfrak{M}(\nu,j)$ will no longer be a mere Lorentz transformation. Nevertheless, it will still satisfy the following conditions (which are those of the proper Lorentz group):

$$\mathfrak{M}^{\mu}{}_{\alpha}\mathfrak{M}_{\rho}{}^{\alpha}=\mathfrak{M}^{\beta}{}_{\rho}\mathfrak{M}_{\beta}{}^{\mu}=\delta^{\mu}_{\rho};$$

$$\det\mathfrak{M}=1;\quad \mathfrak{M}^{0}{}_{0}\geq 1.$$

The parameters ν and j are now generalized and assumed to depend on $\tau=n\cdot r$. The Lorentz-type operators $\mathfrak{M}(\nu(\tau),j(\tau))$ still show other noticeable properties. At once, it is evident that the set of operators $\mathfrak{M}(\nu(\tau),j(\tau))$ is a commutative group. As a matter of fact, if we remember that $\mathfrak{M}^{-1}(\nu,j)=\mathfrak{M}(-\nu,j)$, we can easily check that

$$\mathfrak{M}^{-1\mu}{}_{\rho}(\nu'(\tau),j'(\tau))\mathfrak{M}^{\rho}{}_{\sigma}(\nu(\tau),j(\tau))=\mathfrak{M}^{\mu}{}_{\sigma}(\nu''(\tau),j''(\tau)),$$

where $\nu''(\tau)=[-(\nu j-\nu' j')^2]^{1/2}$ and $j''^{\sigma}(\tau)=\nu''^{-1}(\nu j^{\sigma}-\nu' j'^{\sigma})$. When $\nu(\tau)$ and $j(\tau)$ are constant, this group obviously reduces to the set of Lorentz transformations which leaves the fields of a plane wave unaltered. However, in the general case this group is no longer a subgroup of the proper Lorentz group.

Furthermore, the operators $\mathfrak{M}(\tau)$ obeys the following relations:

$$\partial^{\rho}\mathfrak{M}^{\mu}{}_{\rho}(\nu(\tau),j(\tau))=0,$$

$$\partial_{\mu}\mathfrak{M}^{\mu}{}_{\rho}(\nu(\tau),j(\tau))=0$$

since $\partial^{\rho}\nu(\tau)=n^{\rho}d\nu(\tau)/d\tau$ and $\partial^{\rho}j^{\mu}(\tau)=n^{\rho}dj^{\mu}(\tau)/d\tau$.

These relations allow us to define in an unequivocal way the operator $\partial'^{\rho}=\mathfrak{M}(\tau)^{\rho}{}_{\sigma}\partial^{\sigma}$. Let $A'^{\sigma}=\mathfrak{M}(\tau)^{\sigma}{}_{\rho}A^{\rho}(\tau)$. It can be easily checked that we have

$$F'^{\mu\rho}=\partial'^{\mu}A'^{\rho}-\partial'^{\rho}A'^{\mu}=\partial^{\mu}A^{\rho}-\partial^{\rho}A^{\mu}=F^{\mu\rho}.$$

Hence, the electromagnetic field tensor is still left unaltered by the group of transformations $\mathfrak{M}(\tau)$.

Furthermore, we have the following equations:

$$\epsilon_{\mu\rho\sigma\omega}\partial'^{\rho}F'^{\sigma\omega}=\epsilon_{\mu\rho\sigma\omega}\partial^{\rho}F^{\sigma\omega}=0,$$

$$\partial'_{\mu}F'^{\mu\rho}=\partial_{\mu}F^{\mu\rho}=0.$$

That is to say, the Maxwell equations for plane waves conserve their form under the transformations $\mathfrak{M}(\tau)$.

The interesting points of the mathematical transformations $\mathfrak{M}(\tau)$ will appear in the following sections. We shall see that the behavior of an electron in the external field of a plane wave is connected to the above-mentioned symmetries of the electromagnetic field.

II. THE CLASSICAL EVOLUTION OPERATOR

In classical electrodynamics, the equation which describes the evolution of a charged particle in a given external field is, of course, the Lorentz force equation. In the case of plane waves, it is well known that it can be solved exactly. Our problem is to find the classical evolution operator of an electron interacting with the external field of a plane electromagnetic wave. We shall show that this operator is an element of the above-mentioned group.

Let p be the four-momentum of a particle of charge-e ($e>0$) and of mass m, interacting with the external field, and let p_i be its four momentum before the field was switched on. The Lorentz force equation written in covariant form is

$$\frac{dp^{\mu}}{ds}=-\frac{e}{m}F^{\mu\nu}(\tau)p_{\nu}=-\frac{e}{m}F(\tau)^{\mu}{}_{\alpha}p^{\alpha}, \tag{11}$$

where s is the proper time of the particle. Let $\mathfrak{M}$ be the sought-after classical evolution operator. By definition, we let

$$p^{\alpha}=\mathfrak{M}^{\alpha}{}_{\beta}p_i^{\beta}.$$

If we use the fact that the four quantities p_i^{β} are arbitrary, Eq. (11) yields

$$\left(\frac{d}{ds}\mathfrak{M}\right)^{\mu}{}_{\alpha}\mathfrak{M}^{-1\alpha}{}_{\rho}=-\frac{e}{m}F(\tau)^{\mu}{}_{\rho} \tag{12}$$

Now, from Eq. (11) we have immediately $(d/ds)(n\cdot p)=0$; that is, $n\cdot p=n\cdot p_i$. Since, by definition, $p^{\mu}=mdr^{\mu}/ds$ it follows that $ds=(n\cdot p)^{-1}m\,d\tau$, whence, using Eq. (12) and the obvious fact that $F(\tau)$ commutes with $\int F(\tau)d\tau$, the operator $\mathfrak{M}$ can be written in the following form:

$$\mathfrak{M}(\tau)=\exp\left[-e(n\cdot p)^{-1}\int_{-\infty}^{\tau}F(\tau')d\tau'\right].$$

Since F is an antisymmetric tensor the operator $\mathfrak{M}$ is a Lorentz-type operator.[10] Furthermore, it obviously commutes with F. Therefore, the Lorentz-type operator $\mathfrak{M}$ leaves the fields of the plane wave unaltered. It follows that it can be written in the form (8), whence, if we use Eqs. (8) and (12), we obtain

$$(n\cdot p)\frac{d}{d\tau}\{\nu(\tau)[n^{\mu}j_{\rho}(\tau)-n_{\rho}j^{\mu}(\tau)]\}=-e\frac{d}{d\tau}[n^{\mu}A_{\rho}(\tau)-n_{\rho}A^{\mu}(\tau)]$$

whence it follows that

$$\nu_p(\tau)=-e(n\cdot p)^{-1}[-A^2(\tau)]^{1/2}, \tag{13}$$

$$j^{\rho}(\tau)=[-A^2(\tau)]^{1/2}A^{\rho}(\tau), \tag{14}$$

and consequently

$$\theta(\tau)=\tan^{-1}[A_2(\tau)A_y{}^{-1}(\tau)] \ . \tag{15}$$

The relations (8), (13), and (14) determines completely the evolution operator $\mathfrak{M}$. The four-momentum of the particle in the field[11] follows immediately:

$$p^\mu(\tau)=\mathfrak{M}^\mu{}_\rho(\nu_p(\tau),j(\tau))p_i^\rho$$
$$=p_i^\mu+eA^\mu(\tau)-(n\cdot p)^{-1}[eA(\tau)\cdot p_i+\tfrac{1}{2}e^2A^2(\tau)]n^\mu \ . \tag{16}$$

The construction of operator $\mathfrak{M}$ as the product of the two operators $\mathfrak{R}$ and $\mathfrak{L}$ leads to an interesting result.

From the relation $\mathfrak{M}(\nu,j)=\mathfrak{R}(\nu,\theta)\mathfrak{L}(\nu,\theta)$ it follows that

$$\mathfrak{M}(\nu,j)=\mathfrak{M}^{-1}(-\nu,j)=\mathfrak{L}^{-1}(-\nu,\theta)\mathfrak{R}^{-1}(-\nu,\theta)$$
$$=\mathfrak{L}^{-1}(-\nu,\theta)\mathfrak{R}(\nu,\theta) \ .$$

Now, let us assume that the particle was at rest before the field was switched on. The initial four-momentum of the particle is then $p_i=\binom{m}{0}$ and the parameter $\nu(\tau)$ becomes

$$\nu_0(\tau)=-\frac{e}{m}[-A^2(\tau)]^{1/2} \ . \tag{17}$$

Relation (16) then becomes

$$p(\tau)=\mathfrak{M}(\nu_0(\tau),j(\tau))\binom{m}{0}$$
$$=\mathfrak{L}^{-1}(-\nu_0(\tau),\theta(\tau))\mathfrak{R}(\nu_0(\tau),\theta(\tau))\binom{m}{0}$$
$$=\mathfrak{L}^{-1}(-\nu_0(\tau),\theta(\tau))\binom{m}{0} \ .$$

Therefore, the operator $\mathfrak{R}(\nu_0(\tau),\theta(\tau))$ appears to be unnecessary to transform the state of a free particle (whose state would uniquely be determined by its four-momentum p_i) into the state of the particle coupled to the external field of a plane wave. However, the operator $\mathfrak{R}$ is an operator which depends on the characteristics of the particle (e and m) and on the external field through $A(\tau)$. Thus, it is clear that $\mathfrak{R}(\nu_0(\tau),\theta(\tau))$ shows an interaction between the particle and the field. Now, since $\mathfrak{R}$ acts in the three-dimensional space, it shows the evolution of a dynamic quantity represented by a spacelike four-vector. That is precisely the case of the intrinsic angular moment of the particle, that is, its spin $\binom{0}{\vec{\zeta}}$ which is defined in the "instantaneous rest frame" of the particle. We are thus led to expect the operator $\mathfrak{R}(\nu_0(\tau),\theta(\tau))$ to be the evolution operator of the spin of the particle which is at rest in the frame that is transformed from the initial frame by the operator $\mathfrak{L}(\nu_0(\tau),\theta(\tau))$.

If $\overline{\mathfrak{R}}$ is the restriction of operator $\mathfrak{R}$ given by (4) or (7) [taking into account (15) and (17)] in the ordinary three-dimensional space, this operator can be written in the following form:

$$\overline{\mathfrak{R}}_{\alpha\beta}=[1+\tfrac{1}{4}\nu_0{}^2(\tau)]^{-1}$$
$$\times\{[1-\tfrac{1}{4}\nu_0{}^2(\tau)]\delta_{\alpha\beta}+\tfrac{1}{2}b_\alpha b_\beta-\epsilon_{\alpha\beta\gamma}b_\gamma\} \ ,$$

where $\epsilon_{\alpha\beta\gamma}$ is the Levi-Civita symbol and where β_α is the α component ($\alpha=1,2,3$) of the three-vector

$$\vec{b}=\nu_0(\tau)\vec{1}(\tau)=-\frac{e}{m}\vec{n}\times\vec{A}(\tau)$$

or in matrix form, by the matrix (4) where the first line and the first column are obviously omitted.

The equation which describes the evolution of the spin vector $\vec{\zeta}$ of a charged particle (whose gyromagnetic ratio is g) classically moving in the fields $\vec{E}(\tau)$ and $\vec{H}(\tau)$ is the Thomas equation[12,13]:

$$\frac{d\vec{\zeta}}{dt}=-\frac{e}{2m}[g-2+2m(p^0)^{-1}]\vec{\zeta}\times\vec{H}$$
$$-\frac{e}{2m}(g-2)p^0(p^0+m)^{-1}(\vec{v}\cdot\vec{H})\vec{v}\times\vec{\zeta}$$
$$-\frac{e}{2m}[g-2p^0(p^0+m)^{-1}]\vec{\zeta}\times(\vec{E}\times\vec{v}) \ . \tag{18}$$

In this equation p^0 and $\vec{v}$ are, respectively, the energy and the velocity of the particle (which has been assumed to be at rest before the field was switched on) given by the solution of the Lorentz force equation. In the case of plane electromagnetic waves, we have from (16), here with $p_i=\binom{m}{0}$,

$$p^0(\tau)=m[1+\tfrac{1}{2}\nu_0{}^2(\tau)] \ ,$$

$$\vec{v}(\tau)=\frac{\vec{p}(\tau)}{p^0(\tau)}=[1+\tfrac{1}{2}\nu_0{}^2(\tau)]^{-1}[\nu_0(\tau)\vec{a}+\tfrac{1}{2}\nu_0{}^2(\tau)\vec{n}] \ .$$

A quite long but elementary calculation shows that

$$\vec{\zeta}(\tau)=\overline{\mathfrak{R}}(\nu_0(\tau),\theta(\tau))\vec{\zeta}_0$$

(where $\vec{\zeta}_0$ is the spin vector of the free particle) is the exact solution of Eq. (18) *if the particle gyromagnetic ratio* $g=2$; that is, in the case of a Dirac particle. It is a result that has already been shown in the above-mentioned previous paper in the special case of a linearly polarized plane wave.[8]

In conclusion, the evolution operator $\mathfrak{M}(\nu_0,j)=\mathfrak{R}(\nu_0,\theta)\mathfrak{L}(\nu_0,\theta)$ which is an element of the above-mentioned group, shows the motion of an electron and the motion of its spin in the plane-wave field.

The result seems restricted to the case of a weak and slowly varying plane wave so that the Thomas equation can be valid. Actually, we shall see that this is not so. The evolution operator $\mathfrak{M}$ will be the same in the quantum (semiclassical) case for which the evolution of the electron (and its spin) is described by the Dirac equation.

III. THE QUANTUM (SEMICLASSICAL) EVOLUTION OPERATOR

We consider now the case of a quantum particle with spin$\frac{1}{2}$ interacting with the classical external field of a plane wave. The evolution of the state of such a particle is, of course, described by the Dirac equation

$$\{\gamma_\mu[i\partial^\mu + eA^\mu(\tau)] - m\}\psi = 0\,. \tag{19}$$

A solution of this equation, due to Volkov,[1] is

$$\psi_p = T_p(\tau)\exp[-ie\lambda_p(\tau)]\varphi_p\,,$$

where

$$T_p(\tau) = [1 + \tfrac{1}{2}e(n\cdot p)^{-1}\gamma\cdot A\gamma\cdot n]\,,$$

$$\lambda_p(\tau) = -(n\cdot p)^{-1}\int_{-\infty}^{\tau}[p\cdot A(\tau') + \tfrac{1}{2}eA^2(\tau')]\,d\tau'\,,$$

and

$$\varphi_p = (2p^0)^{-1/2}e^{-ip\cdot r}u_p$$

(u_p being a normalized bispinor: $\bar{u}_p u_p \equiv u_p^\dagger\gamma^0 u_p = 2m$) is the "plane-wave" solution of the Dirac equation for a free electron.

Our aim is now to derive the quantum (semiclassical) evolution operator $\hat{U}$; that is, the operator which will transform an arbitrary solution φ of the field-free Dirac equation

$$(i\gamma\cdot\partial - m)\varphi = 0 \tag{20}$$

into the corresponding solution $\psi = \hat{U}\varphi$ of Eq. (19).

Arguing from analogy with the classical case, we shall now look for a Lorentz-type operator $\hat{\mathfrak{M}}$ which will allow us to write the Dirac equation in the proper frame of the electron interacting with the plane-wave field. Relativistic invariance of the problem is therefore needed but in a somewhat generalized manner. Besides, the sought-after operator $\hat{\mathfrak{M}}$ will be assumed to depend on the momentum operator $i\partial$ since the quantum particle is not generally in a pure state of given four-momentum p.

To use the relativistic invariance of the Dirac equation, we shall use the relativistic invariance of the products $\gamma_\mu\partial^\mu$ and $\gamma_\mu A^\mu(\tau)$. If $\mathfrak{N}$ is a matrix of Lorentz transformation, we have, of course, $\gamma_\mu(i\partial^\mu + eA^\mu) = \gamma'_\rho(i\partial'^\rho + eA'^\rho)$, where $\gamma'_\mu = \mathfrak{N}_\mu{}^\rho\gamma_\rho$, $\partial'^\alpha = \mathfrak{N}^\alpha{}_\sigma\partial^\sigma$, and $A'^\rho = \mathfrak{N}^\rho{}_\omega A^\omega$. However, the present situation is more complicated in that the sought-after Lorentz-type operator $\hat{\mathfrak{M}}$ will, *a priori*, depend on t and on operators $\vec{r}$ and ∂^μ. It follows that we can generally have $\hat{\mathfrak{M}}^\mu{}_\rho\partial^\rho \neq \partial^\rho\hat{\mathfrak{M}}^\mu{}_\rho$ and $\hat{\mathfrak{M}}^\sigma{}_\alpha A^\alpha \neq A^\alpha\hat{\mathfrak{M}}^\sigma{}_\alpha$. To use explicitly the relativistic invariance of the Dirac equation, we are therefore led to require, for a Lorentz-type operator which depends on r and on $i\partial$, the following additional conditions:

$$[\partial^\rho, \hat{\mathfrak{M}}^\mu{}_\rho] = 0\,, \tag{21}$$

$$[A^\rho, \hat{\mathfrak{M}}^\mu{}_\rho] = 0\,. \tag{22}$$

We shall also use gauge invariance. The Dirac equation is, of course, invariant under the gauge transformations $A^\mu \to A^\mu + \partial^\mu\lambda(r)$, $\psi \to \exp[ie\lambda(r)]\psi$. More generally, if $\hat{\lambda}$ is now an operator which depends on t and on operators $\vec{r}$ and ∂^μ, the Dirac equation (19) is still invariant under the "generalized" gauge transformations

$$A^\mu \to A^\mu + [\partial^\mu, \hat{\lambda}], \quad \psi \to \exp(ie\hat{\lambda})\psi$$

if $\hat{\lambda}$ is such that

$$[\hat{\lambda}, A^\mu] = 0\,, \qquad [\hat{\lambda}, [\partial^\mu, \hat{\lambda}]] = 0\,. \tag{23}$$

Hence, if we call $u = t + x$, the operators $\hat{\mathfrak{M}}, \hat{\lambda}, A$ depend generally on u, τ, y, z. In addition, the operators $\hat{\mathfrak{M}}$ and $\hat{\lambda}$ depend generally on $\partial/\partial u = \frac{1}{2}(\partial/\partial t + \partial/\partial x) = \frac{1}{2}n\cdot\partial$, $\partial/\partial\tau = \frac{1}{2}(\partial/\partial t - \partial/\partial x)$, $\partial/\partial y$, $\partial/\partial z$. Now, we deal with plane waves $A(\tau)$. Operators $\hat{\mathfrak{M}}$ and $\hat{\lambda}$ depend only on τ. We will show now that they do not depend on $\partial/\partial\tau$.

Since A^μ is an arbitrary function of τ, it follows from Eqs. (23) that $[\hat{\lambda}, \tau] = 0$, and therefore $\hat{\lambda}(\tau, \partial/\partial u, \partial/\partial y, \partial/\partial z)$. Furthermore, it follows from Eq. (21) that

$$[\partial^\rho, \hat{\mathfrak{M}}^\mu{}_\rho] = n^\rho\left[\frac{\partial}{\partial\tau}, \hat{\mathfrak{M}}^\mu{}_\rho\right] = 0$$

and consequently

$$\left[\hat{\mathfrak{M}}^\mu{}_0 + \hat{\mathfrak{M}}^\mu{}_1, \frac{\partial}{\partial\tau}\right] = 0\,. \tag{24}$$

Furthermore, since for plane waves, $A^0(\tau) = A_x(\tau)$, $A_y(\tau)$, and $A_z(\tau)$ are three independent functions of τ, it follows from Eq. (22) that

$$[\hat{\mathfrak{M}}^\mu{}_0 + \hat{\mathfrak{M}}^\mu{}_1, \tau] = [\hat{\mathfrak{M}}^\mu{}_2, \tau] = [\hat{\mathfrak{M}}^\mu{}_3, \tau] = 0\,. \tag{25}$$

Equations (24) and (25) obviously show that $\hat{\mathfrak{M}}^\mu{}_0 + \hat{\mathfrak{M}}^\mu{}_1$, $\hat{\mathfrak{M}}^\mu{}_2$, and $\hat{\mathfrak{M}}^\mu{}_3$ do not depend on the operator $\partial/\partial\tau$ and that

$$\hat{\mathfrak{M}}^\mu{}_0 + \hat{\mathfrak{M}}^\mu{}_1 = \hat{C}^\mu\left(\frac{\partial}{\partial u}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\,. \tag{26}$$

Finally, taking into account that $\hat{\mathfrak{M}}^\beta{}_0 g_{\beta\rho} = \hat{\mathfrak{M}}_\rho{}^0$ and $g_{\alpha\rho}\hat{\mathfrak{M}}^\alpha{}_{1,2,3} = -\hat{\mathfrak{M}}_\rho{}^{1,2,3}$ the equation $\hat{\mathfrak{M}}^\mu{}_\alpha\hat{\mathfrak{M}}_\rho{}^\alpha = \delta^\mu_\rho$ gives a relation between $\hat{\mathfrak{M}}^\mu{}_0, \hat{\mathfrak{M}}^\mu{}_1$ [which are connected by Eq. (26)] and $\hat{\mathfrak{M}}^\mu{}_2, \hat{\mathfrak{M}}^\mu{}_3$ which are operators which do not depend on $\partial/\partial\tau$. Consequently,

$$[\hat{\mathfrak{M}}^\mu{}_0, \tau] = [\hat{\mathfrak{M}}^\mu{}_1, \tau] = 0\,.$$

All the elements of the matrix $\hat{\mathfrak{M}}$ commute with τ and depend only on $\tau, \partial/\partial u, \partial/\partial y, \partial/\partial z$, which commute with τ. In short, the operators $A(\tau)$, $\hat{\mathfrak{M}}$, $\hat{\lambda}$, and $[\partial^\mu, \hat{\lambda}] = n^\mu[\partial/\partial\tau, \hat{\lambda}]$ commute.

Then, let us use the relativistic invariance of the Dirac equation. If we use the fact that the operator $\hat{\mathfrak{M}}$ sought is of the Lorentz type, we can rewrite Eq. (20) in the following form:

$$(\gamma_\rho \hat{\mathfrak{M}}_\mu{}^\rho \hat{\mathfrak{M}}^\mu{}_\alpha i\partial^\alpha - m)\varphi = 0\,. \tag{27}$$

Now, if we introduce the spinor operator $\hat{T}$ which corresponds to the Lorentz-type operator $\hat{\mathfrak{M}}$, that is, the operator $\hat{T}$ such that[14]:

$$\hat{\mathfrak{M}}_\mu{}^\rho \gamma_\rho = \hat{T}^{-1}\gamma_\mu \hat{T}\,, \tag{28}$$

$$\hat{T}^{-1} = \gamma^0 \hat{T}^\dagger \gamma^0 \tag{29}$$

and which does not depend on $\partial/\partial\tau$, Eq. (27) becomes

$$(\gamma_\mu \hat{\mathfrak{M}}^\mu{}_\alpha i\partial^\alpha - m)\hat{T}\varphi = i\gamma_\mu [\hat{\mathfrak{M}}^\mu{}_\rho \partial^\rho, T]\varphi. \tag{30}$$

If we now use the "generalized" gauge invariance of the Dirac equation, Eq. (19) can be written in the following form:

$$\left\{\gamma_\mu\left(i\partial^\mu + eA^\mu(\tau) + en^\mu\left[\frac{\partial}{\partial\tau}, \hat{\lambda}\right]\right) - m\right\}\exp[ie\hat{\lambda}]\psi = 0\,. \tag{31}$$

Equation (30) will be identical to Eq. (31) if we have, simultaneously,

$$\hat{\mathfrak{M}}^\mu{}_\alpha i\partial^\alpha = i\partial^\mu + eA^\mu(\tau) + en^\mu\left[\frac{\partial}{\partial\tau}, \hat{\lambda}\right], \tag{32}$$

$$\gamma_\mu\left[i\partial^\mu + eA^\mu(\tau) + en^\mu\left[\frac{\partial}{\partial\tau}, \hat{\lambda}\right], \hat{T}\right] = 0\,. \tag{33}$$

However, $\hat{T}$, which does not depend on $\partial/\partial\tau$, commutes with $A^\mu(\tau)$ and $[\partial/\partial\tau, \hat{\lambda}]$. Thus, Eq. (33) reduces to

$$\gamma_\mu[\partial^\mu, \hat{T}] = 0\,. \tag{34}$$

The Dirac conjugation of Eq. (34) leads to

$$[\partial^\mu, \hat{T}^{-1}]\gamma_\mu = 0\,. \tag{35}$$

Now, if we take into account Eq. (28) and the linear independence of the γ matrices, Eqs. (34) and (35) are equivalent to

$$[\partial_\mu, \hat{\mathfrak{M}}^\mu{}_\rho] = n_\mu\left[\frac{\partial}{\partial\tau}, \hat{\mathfrak{M}}^\mu{}_\rho\right] = 0\,. \tag{36}$$

Then, from Eqs. (32) and (36), we have

$$i\partial^\rho = \hat{\mathfrak{M}}_\mu{}^\rho\left(i\partial^\mu + eA^\mu + en^\mu\left[\frac{\partial}{\partial\tau}, \hat{\lambda}\right]\right) = \left(i\partial^\mu + eA^\mu + en^\mu\left[\frac{\partial}{\partial\tau}, \hat{\lambda}\right]\right)\hat{\mathfrak{M}}_\mu{}^\rho$$

and therefore,

$$(i\partial^\rho)(i\partial_\rho) = \left(i\partial^\mu + eA^\mu + en^\mu\left[\frac{\partial}{\partial\tau}, \hat{\lambda}\right]\right) \times\left(i\partial_\mu + eA_\mu + en_\mu\left[\frac{\partial}{\partial\tau}, \hat{\lambda}\right]\right),$$

whence

$$(n\cdot\partial)\left[\frac{\partial}{\partial\tau}, \hat{\lambda}\right] = -A(\tau)\cdot\partial + \tfrac{1}{2}ieA^2(\tau)\,. \tag{37}$$

If we let

$$Z^\mu(\tau) = \int_{-\infty}^{\tau} A^\mu(\tau')d\tau'\,, \qquad B(\tau) = \int_{-\infty}^{\tau} A^2(\tau')d\tau'\,, \tag{38}$$

the solution of Eq. (37) is

$$\hat{\lambda} = -(n\cdot\partial)^{-1}[Z(\tau)\cdot\partial - \tfrac{1}{2}ieB(\tau)]\,. \tag{39}$$

[The operator $(n\cdot\partial)^{-1}$ is well defined on the plane-wave basis φ_p and we have $(n\cdot\partial)^{-1}\varphi_p = i(n\cdot p)^{-1}\varphi_p$. Since the electron is a nonzero mass particle, $n\cdot p$ is always different from zero.]

Hence, Eq. (32) becomes

$$\hat{\mathfrak{M}}^\mu{}_\rho i\partial^\rho = i\partial^\mu + eA^\mu(\tau) - n^\mu[eA(\tau)\cdot\partial - \tfrac{1}{2}ie^2A^2(\tau)](n\cdot\partial)^{-1}\,. \tag{40}$$

We shall now see that the symmetries of the free field are extremely useful to find the evolution operator. As a matter of fact, Eq. (40) involves

$$n_\mu \hat{\mathfrak{M}}^\mu{}_\rho i\partial^\rho = n_\rho i\partial^\rho\,,$$

whence it follows immediately that $\hat{\mathfrak{M}}^\mu{}_\rho n_\mu = n_\rho$ and, consequently,

$$\hat{\mathfrak{M}}^\mu{}_\rho n^\rho = n^\mu\,. \tag{41}$$

Thus, the sought-after Lorentz-type operator $\hat{\mathfrak{M}}$ must leave the null four-vector n unchanged. Furthermore, Eq. (40) involves the following relation:

$$A_\mu \hat{\mathfrak{M}}^\mu{}_\rho i\partial^\rho = A_\mu i\partial^\mu + eA^2 = [A_\rho - i(n\cdot\partial)^{-1}eA^2 n_\rho]i\partial^\rho\,,$$

whence

$$A_\mu \hat{\mathfrak{M}}^\mu{}_\rho = A_\rho - i(n\cdot\partial)^{-1}eA^2 n_\rho\,,$$

whence, using (41) it follows that

$$\hat{\mathfrak{M}}^\sigma{}_\rho A^\rho(\tau) = A^\sigma(\tau) + i(n\cdot\partial)^{-1}eA^2(\tau)n^\sigma = A^\sigma(\tau) + [\partial^\sigma, \hat{\chi}]\,, \tag{42}$$

where

$$\hat{\chi} = i(n\cdot\partial)^{-1}eB(\tau)\,.$$

Therefore, the Lorentz-type operator $\hat{\mathfrak{M}}$ is equivalent to a "generalized gauge transformation" whose "generalized gauge function" is $\hat{\chi}$. It follows from (41) and (42) that $\hat{\mathfrak{M}}$ leaves the tensor $F^{\mu\rho}$ of the field unaltered. Consequently, it can be written in the form (8) where $\hat{\nu}$ and $\hat{j}$ are now operators. In order to determine them, it is sufficient to use Eqs. (8) and (40). We easily find

$$\hat{\nu} = (n \circ \partial)^{-1} ie[-A^2(\tau)]^{1/2}, \quad (43)$$

$$j^\mu(\tau) = [-A^2(\tau)]^{-1/2} A^\mu(\tau). \quad (44)$$

It remains now to find the spinor operator $\hat{T}$ which corresponds to the Lorentz-type operator $\hat{\mathfrak{M}}$ and which is defined by relations (28) and (29). It is simpler to use the construction of operator $\hat{\mathfrak{M}}$ as the product of the two operators $\hat{\mathfrak{R}}$ and $\hat{\mathfrak{L}}$ [defined by relations (1), (2), and (3)], than to use the general methods of spinor calculus. As is known,[15] the spinor operator corresponding to an operator of Lorentz transformation $\mathfrak{L}$ (without rotation), whose associated velocity is $\vec{\beta}$, is

$$S(\mathfrak{L}) = \cosh(\tfrac{1}{2}\tanh^{-1}\beta) - \gamma^0(\vec{\gamma}\cdot\vec{\beta})\beta^{-1}\sinh(\tfrac{1}{2}\tanh^{-1}\beta)$$

and the spinor operator $S(\mathfrak{R})$ corresponding to a rotation operator $\mathfrak{R}$ of angle α around the unit vector $\vec{u}$ is

$$S(\mathfrak{R}) = \cos(\tfrac{1}{2}\alpha) + i\vec{\Sigma}\cdot\vec{u}\sin(\tfrac{1}{2}\alpha),$$

where $\vec{\Sigma} = \frac{1}{2}i\vec{\gamma}\times\vec{\gamma}$. The velocity $\vec{\beta}$ which is associated with the operator $\mathfrak{L}$ can be easily found by calculating the following four-vector:

$$\mathfrak{L}\begin{pmatrix}1\\ \vec{0}\end{pmatrix} = (1-\beta^2)^{-1/2}\begin{pmatrix}1\\ -\vec{\beta}\end{pmatrix}. \quad (45)$$

From (6) and (45) we easily obtain

$$\vec{\beta} = (1+\tfrac{1}{2}\nu^2)^{-1}(\nu\vec{a} + \tfrac{1}{2}\nu^2\vec{n}).$$

Therefore, the spinor operator which corresponds to operator $\hat{\mathfrak{L}}$ is

$$\hat{S}(\hat{\mathfrak{L}}) = (1+\tfrac{1}{4}\hat{\nu}^2)^{1/2}[1 - \tfrac{1}{2}\hat{\nu}(1+\tfrac{1}{4}\hat{\nu}^2)^{-1} \times (\gamma^0\vec{\gamma}\cdot\vec{a} + \tfrac{1}{2}\hat{\nu}\gamma^0\vec{\gamma}\cdot\vec{n})].$$

The space rotation $\hat{\mathfrak{R}}$ is defined by relations (2) and (3). We obtain immediately the corresponding spinor operator

$$\hat{S}(\hat{\mathfrak{R}}) = (1+\tfrac{1}{4}\hat{\nu}^2)^{-1/2}[1 + \tfrac{1}{2}\hat{\nu}(\vec{\gamma}\cdot\vec{n})(\vec{\gamma}\cdot\vec{a})].$$

Consequently, the spinor operator $\hat{T}$ corresponding to the operator $\hat{\mathfrak{M}} = \hat{\mathfrak{R}}\hat{\mathfrak{L}}$ is

$$\hat{T} = \hat{S}(\hat{\mathfrak{R}})\hat{S}(\hat{\mathfrak{L}}) = 1 + \tfrac{1}{2}\hat{\nu}\gamma\cdot n\gamma\cdot j. \quad (46)$$

If we now compare Eqs. (30) and (31), using the fact that the conditions (32) and (33) are satisfied, we obtain the general solution of the Dirac equation (19) in the following form:

$$\psi = \hat{T}\exp(-ie\hat{\lambda})\varphi \equiv \hat{U}\varphi,$$

where the operators $\hat{T}$ and $\hat{\lambda}$ are defined by relations (38), (39), (43), (44), and (46).

If φ is an eigenstate of operator $i\partial$, that is, if $\varphi = \varphi_p$. Eq. (27) shows that the operator $\hat{\nu}$ given by (43) is then diagonal and reduces to its classical eigenvalue ν_p given by (13). The operator $\hat{\mathfrak{M}}(\hat{\nu}, j)$ becomes identical to the classical evolution operator $\mathfrak{M}(\nu_p, j)$. We can say that the electron is moving according to the Lorentz force equation while its spin is moving according to the Thomas equation. Thus, the Thomas equation is rigorously valid in the case of plane waves. Furthermore, the operators $\hat{T}(\tau)$ and $\hat{\lambda}(\tau)$ reduce to their diagonal form and we find Volkov's solution again: $\psi = \psi_p$. These remarks exhibit the "classical character" of the Volkov states.

It is to be noted that the classical operator $\mathfrak{M}(\nu_p, j)$, which allows us to transform a "plane-wave" solution φ_p into a Volkov solution ψ_p, was given about 30 years ago by Taub[16] who took the form of the spinor operator T_p of the Volkov solution as the starting point of the derivation.

Hence, the operator $\hat{U} = \hat{T}\exp(-ie\hat{\lambda})$, which is a unitary operator (on the γ^0 Dirac metric: $\hat{U}^{-1} = \gamma^0\hat{U}^\dagger\gamma^0$), is the "variable spinor image" of the Lorentz-type operator $\hat{\mathfrak{M}}$. Since it is the quantum (semiclassical) evolution operator of an electron interacting with the external field of a plane wave, it allows us to obtain a representation in which the electron behaves as a free particle. In a certain sense, we may roughly say that the change of representation by means of operator $\hat{U}^{-1}$ physically corresponds to a "variable Lorentz transformation" by means of the "quantum" Lorentz-type operator $\hat{\mathfrak{M}}^{-1}$. In the "transformed frame" the electron is no longer interacting with the external field and behaves like a free Dirac particle. Furthermore, in this frame the fields are the same as in the initial frame.

IV. APPLICATION TO HIGH-INTENSITY COMPTON SCATTERING

We consider now an electron which is simultaneously interacting with the intense external field of a plane wave whose four-potential is $A(\tau)$ and with a second field (which will be supposed to be weak) whose four-potential is $A_1(r)$. As is known, the hypothesis of an intense plane wave allows us to consider the operator $A(\tau)$ as a classical potential.[17] The Dirac equation which describes the evolution of the electron state is

$$\{\gamma_\mu[i\partial^\mu + eA^\mu(\tau) + eA_1^\mu(r)] - m\}\psi = 0. \quad (47)$$

If we perform the unitary transformation

$$\psi' = \gamma^0\hat{U}^\dagger\gamma^0\psi = \hat{U}^{-1}\psi = (1 - \tfrac{1}{2}\hat{\nu}\gamma . n\gamma . j)\exp(ie\hat{\lambda})\psi,$$

Eq. (47) becomes, in the new representation,

$$\left\{\hat{T}^{-1}\gamma_\mu\hat{T}\left(i\partial^\mu + eA^\mu + en^\mu\left[\frac{\partial}{\partial\tau}, \hat{\lambda}\right]\right) - m + \hat{U}^{-1}\gamma_\mu eA_1^\mu(r)\hat{U}\right\}\psi' = 0. \quad (48)$$

Taking into account Eqs. (28), (29), and (32), Eq. (48) can be written in the form

$$[\gamma_\mu i\partial^\mu - m + \hat{U}^{-1}\gamma_\sigma e A_1^\sigma(r)\hat{U}]\psi' = 0\,. \qquad (49)$$

The interest of the unitary transformation $\hat{U}$ stands out in Eq. (49). In the new representation, the unperturbed states (by the operator A_1) are obviously free states and the propagator of the electron is merely a free electron propagator.

We are here interested in the simplest process: the emission of a photon (whose four-momentum is ω' and whose four-polarization is ϵ') by the electron interacting with the intense field of a plane wave. Thus, the operator A_1 corresponds to radiation reaction. In the new representation, this process can be represented by the diagram shown in Fig. 1.

The S matrix element S_{fi} corresponding to this diagram is

$$S_{fi} = (2\omega'^0)^{-1/2} ie \int \overline{\varphi}_{(p',\xi')}(r)\hat{U}^{-1}\gamma\cdot\epsilon' \times\exp(i\omega'\cdot r)\hat{U}\varphi_{(p,\xi)}(r)d^4r\,. \qquad (50)$$

Now, the states $\hat{U}\varphi_{(p,\xi)}$ and $\overline{\varphi}_{(p',\xi')}\hat{U}^{-1}$ are nothing other than the Volkov states $\psi_{(p,\xi)}$ and $\overline{\psi}_{(p',\xi')}$ so that the scattering amplitude (50) is strictly equivalent to

$$S_{fi} = (2\omega'^0)^{-1/2} ie \int \overline{\psi}_{(p',\xi')}(r)\gamma\cdot\epsilon' \times\exp(i\omega'\cdot r)\psi_{(p,\xi)}(r)d^4r\,. \qquad (51)$$

We find again the matrix element which is the starting point of most of the works devoted to the calculation of the cross section of nonlinear Compton scattering.[4] Thus, the scattering amplitude (51) in which the incident and outgoing electron states are two nonstationary Volkov states is equivalent to the scattering amplitude (50) in which the unperturbed electron states are nothing more than free states.

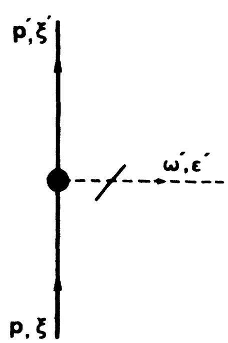

FIG. 1. Diagram of high-intensity Compton scattering. p, ξ and p', ξ' are, respectively, the four-momentum and four-polarization of the ingoing and outgoing free-electron states. The crossed dashed line means that the emerging photon is emitted in the electron "proper frame". The blob emphasizes vertex modification.

Hitherto, we have assumed that the plane-wave potential $A(\tau)$ was zero when $\tau=\pm\infty$, thus assuming that the light was somehow switched on and off. Obviously, when the laser beam is described by a monochromatic wave, this condition cannot be required. It follows that the evolution operator $\hat{U}$ cannot be uniquely determined in the monochromatic case.

Explicitly, if we let $Z^\mu(\tau) = \int A^\mu(\tau)d\tau + K^\mu$ and $B(\tau) = \int A^2(\tau)d\tau + B_0$, where B_0 is a constant and K a constant four-vector such that $n\cdot K = 0$ [as follows from Eqs. (23)], the general solution of Eq. (37) reads

$$\hat{\lambda} = \hat{\lambda}_1 + \hat{\lambda}_0\,,$$

where

$$\hat{\lambda}_1 = -(n\cdot\partial)^{-1}\{[Z(\tau)-K]\cdot\partial - \tfrac{1}{2}i[B(\tau)-B_0]\}$$

and

$$\hat{\lambda}_0 = (n\cdot\partial)^{-1}(K\cdot\partial - \tfrac{1}{2}iB_0)\,.$$

The operator $\hat{U}$ can then be written in factorized form: $\hat{U} = \hat{T}\exp(-ie\hat{\lambda}_1)\exp(-ie\hat{\lambda}_0)$. Now, since $\hat{\lambda}_0$ commutes with $(i\gamma\cdot\partial - m)$ it follows that the field-free electron state φ cannot be defined in an unequivocal way. In mathematical form this difficulty exhibits the impossibility of decoupling the particle from the field of a monochromatic wave which has, of course, an infinite extent in space-time.

Hence, the present method is an alternative to the use of (nonmonochromatic) Volkov states in scattering theory, provided an adiabatic switching of the field is assumed in order to satisfy the necessary requirement $A(\pm\infty)=0$ that ensures uniqueness for the evolution operator $\hat{U}$.

In the new representation, that is, in the new frame, the initial and final electron states are then free-electron states; the unperturbed propagator is, therefore, a free-electron propagator, while the perturbation operator is the following operator:

$$e\hat{U}^{-1}\gamma\cdot A_1(r)\hat{U} = e\gamma'\cdot A_1(\hat{U}^{-1}r\hat{U})$$

with

$$\gamma'^\mu \equiv \hat{U}^{-1}\gamma^\mu\hat{U} = \mathfrak{M}^\mu{}_\alpha\gamma^\alpha$$
$$= \gamma^\mu - (n\cdot\partial)^{-1} ie(\gamma\cdot nA^\mu - \gamma\cdot An^\mu)$$
$$+\tfrac{1}{2}(n\cdot\partial)^{-2}e^2A^2\gamma\cdot nn^\mu$$

and

$$\hat{U}^{-1} r^\mu \hat{U} = r^\mu - (n\circ\partial)^{-1} ie(Z^\mu + \hat{\lambda} n^\mu) + \tfrac{1}{2}(n\circ\partial)^{-2} ie\gamma\cdot n\gamma\cdot A n^\mu .$$

Obviously, since $\hat{U}$ is a unitary operator, scattering amplitudes which are calculated in the ordinary representation or in the new representation are rigorously identical.

CONCLUDING REMARKS

The interest of the formalism presented here is that it allows calculation of scattering processes that occur in a laser beam without using Volkov states. The ingoing and outgoing electron states are here free states and the unperturbed propagator is merely a free electron propagator. Moreover, the representation in question in this paper is quite analogous to the Furry representation,[18] valid when the external field is a static one, which is known to be very useful for bound-state problems. We must, however, take notice of the fact that the method cannot apply when the laser is roughly represented by a monochromatic wave. Therefore, we may be reasonably doubtful about the physical meaning of the results obtained by using monochromatic Volkov states in the traditional formalism.

The symmetries of the free field have played an essential part in this paper. In particular, the quantum evolution operator has been derived by using explicitly those symmetries and the relativistic invariance of the Dirac equation in a somewhat generalized manner. The natural question is whether our method is valid for arbitrary electromagnetic fields.

It is our opinion that the symmetries of the free field are always intimately connected with the behavior of an electron interacting with the external field. However, it is only in a few cases that we may expect the classical evolution operator to be a Lorentz-type operator and, therefore, the Dirac equation to be transformed into the field-free Dirac equation by an operator of the Lorentz type. Nevertheless, the present method can be considered as a helpful model upon which a more general theory can be carried out.

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