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# Thermodynamic potential in quantum electrodynamics*

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The thermodynamic potential, $\Omega$, in quantum electrodynamics (QED) is derived using the path-integral formalism. Renormalization of $\Omega$ is shown by proving the following theorem: $\Omega_B(e_B,m_B,T,\mu) - \Omega_B(e_B,m_B,T=0,\mu=0) = \Omega_R(e_R,m_R,T,\mu,S)$, where $B$ and $R$ refer to bare and renormalized quantities, respectively, and $S$ is the Euclidean subtraction momentum squared. This theorem is proved explicitly to $e_R{}^4$ order and could be analogously extended to any higher order. Renormalization-group equations are derived for $\Omega_R$, and it is shown that perturbation theory in a medium is governed by effective coupling constants which are functions of the density. The behavior of the theory at high densities is governed by the Euclidean ultraviolet behavior of the theory in the vacuum.

## I. INTRODUCTION

We consider a medium of electrons, positrons, ions, and photons in thermodynamic equilibrium and wish to calculate perturbatively in a coupling-constant series the thermodynamic potential, $\Omega$, in relativistic many-body theory. $T$ denotes temperature and $\mu$ is the chemical potential corresponding to the conserved electric charge carried by the particles.[1] The formalism for computing $\Omega$ in nonrelativistic quantum field theory is well known (see Ref. 2, for example) and has many applications in solid-state and nuclear physics. However, relativistic mediums were throught only to appear in cosmology and, as such, little interest has been shown in relativistic many-body theory. However, this situation changed dramatically with the discovery of pulsars, interpreted as highly dense neutron stars (see Ref. 3 for a general review) since these astrophysical objects are *de facto* relativistic systems. We use here quantum electrodynamics as a laboratory for establishing various theorems about $\Omega$ including the derivation of $\Omega$ from a quantum field Lagrangian and the renormalization of $\Omega$.

These results will be true for any renormalizable theory including the non-Abelian gauge theories of the strong interaction.[4] In the latter, the concept of asymptotic freedom[5] (the property of the coupling constant becoming small) was shown to be true[6] for high densities and/or high temperatures) allowing for the first time reliable calculations of the equation of state and other properties of the high dense medium.[7] These results all center about obtaining a renormalized, infrared-finite $\Omega$ as a function of $T$ and $\mu$. In the following we present a historical background.

Matsubara[8] in 1955 proposed a thermodynamic perturbation theory which is the analog of the perturbation expansion of field theory. This was extended by Abrikosov, Gor'kov, and Dzyaloshinskii,[9] Martin and Schwinger,[10] and Fradkin[11] to momentum-space parametrization and use of Green's function methods. These works laid the foundation for finite-temperature field theory (FTFT) but important gaps were left. The first one concerns the renormalization of Green's functions in FTFT. As early as 1959, Fradkin[12] stated without proof that the FTFT Green's functions are renormalizable by vacuum-type counterterms appearing in the Lagrangian. However, there arises the interesting possibility that these counterterms, the bare charge, bare mass, and wave-function scale factors become $T$, $\mu$ dependent. This possibility was ruled out in the work of Ref. 6, where, for the first time, the proof of the renormalizability of Green's functions in FTFT by the vacuum values of the counterterms was presented.

The importance of this theorem in Ref. 6 lies in the following: Because the vacuum counterterms are independent of $T,\mu$, one can prove by renormalization-group arguments[6] that the behavior of Green's functions at large $T/\mu$ is identical to the Green's functions' behavior in the deep Euclidean region in the vacuum. In Sec. V we will show that $\Omega$ also has this property.

The second gap was in the nonrecognition that the partition function was a gauge-dependent quantity for gauge theories, e.g., quantum electrodynamics (QED). This complication was cleared up by Bernard,[13] who showed that a physically meaningful partition function which is gauge invariant can be defined in "physical gauges," ones with the correct number of degrees of freedom, and then functional methods may be used to determine Feynman rules for this partition function in other gauges. The most important omission of the earlier work is in the thermodynamic potential. Akhiezer and Peletminskii[14] were the first to consider the thermodynamic potential in relativistic field theory, where they computed the $e^2$ ($e$ is the physical charge of the electron) correction to the noninteraction thermodynamic potential in QED. However, they did not realize the theorem which covers the renormalization of $\Omega$ (and whose statement and proof constitute new results in this paper) and introduced a superfluous charge renormalization. Had they used a gauge-invariant regularization procedure, their infinity would have been regulated to zero. Norton and Cornwall[15] have considered the thermodynamic potential for spinless mesons. They showed that the thermodynamic potential can be constructed as an effective action functional of the various $N$-point amplitudes ($N \leq 4$). Their analysis is quite involved, and a consistent method for renormalization of $\Omega$ is obscure in their work. As will be seen below even after all charge and mass renormalizations are performed, $\Omega$ is still naively quadratically divergent, and only after precise cancellations of parts of Feynman graphs with parts of others is $\Omega$ seen to be finite after subtracting its vacuum value. This is why the renormalization of $\Omega$ is technically one of the most difficult procedures in quantum field theory. The last topic of importance is the question of infrared finiteness of $\Omega$ to all orders of perturbation theory. We have, at this time, no general proof to all orders.

The plan of this paper is as follows. In Sec. II a path-integral derivation for $\Omega$ is given with subsequent derivation of (known) Feynman rules for Green's functions and the new Feynman rules for $\Omega$. The violation of charge conjugation is discussed in Sec. III. In Sec. IV the proof of the theorem concerning the renormalization of $\Omega$ is given explicitly to $e^4$ order. It can be analogously extended (but burdensomely) to any higher order, but since fourth order contains all the complexities which one can encounter, this is unnecessary to demonstrate the theorem. Using the theorem on the renormalization of $\Omega$, we construct renormalization-group equations for $\Omega$ and show that perturbation theory is governed by effective constants which are functions of the density. Finally, a conclusion follows.

## II. PATH-INTEGRAL DERIVATION FOR $\Omega$

### A. Feynman rules for Green's functions

Since the number of particles is determined by equilibrium, we are interested in the grand canonical ensemble. The grand partition function is

$$Z = e^{-\beta\Omega} = \mathrm{Tr}\, e^{-\beta(H-\mu N)} , \tag{1}$$

where $H$ is the QED Hamiltonian operator in some gauge and $N$ is the number operator, $\bar{\psi}\gamma^0\psi$. We will evaluate Eq. (1) using functional integral techniques. Before we continue we discuss the gauge dependence of $Z$. The masslessness of the photon means that there are only two dynamical degrees of freedom associated with the Maxwell field (the transverse photon fields). Longitudinal and timelike photons must decouple from the grand partition function. However, if we use the Feynman gauge or any other gauge which has more than two independent degrees of freedom, we find that $H$ includes these extra degrees of freedom and we include their contributions to $\Omega$. Thus $e^{-\beta\Omega}$ is a gauge-dependent quantity for this reason. As Bernard has remarked,[13] the true gauge-invariant grand partition function, $Z_G$, must be defined to be $Z_G = e^{-\beta\Omega}|_{\text{physical}}$, where the gauge chosen is a physical gauge describing massless photons. If we wish to use other gauges, including the "nonphysical" gauges, we then must define their Feynman rules so that $Z_G$ is derived.

The expression for $Z_G$ from functional integration is the following:

$$\begin{aligned} Z_G = N_F(N(\beta))^2 \int_{\substack{\text{periodic } A\\ \text{antiperiodic } \psi}} & [dA][d\psi][d\bar{\psi}][d\psi_I][d\bar{\psi}_I] \\ & \times \exp\left\{ \int_0^\beta d\tau \int d^3x \left[ -\tfrac{1}{4}F_{\mu\nu}(\vec{x},\tau)F^{\mu\nu}(\vec{x},\tau) \right.\right. \\ & \left.\left. \qquad + \bar{\psi}(\vec{x},\tau)(i\vec{\partial}_{\vec{x}} - \gamma^0\partial_\tau - e_B \not{A} + \mu\gamma^0 - m_B)\psi(\vec{x},\tau) + \mathcal{L}_{\text{ion}}(\vec{x},\tau) \right] \right\} \\ & \times \det\left[\frac{\partial F(\vec{x},\tau)}{\partial\omega}\right] \delta(F(\vec{x},\tau)) . \end{aligned} \tag{2}$$

In Eq. (2), $A$, $\psi$, and $\psi_I$ are respectively the photon fields, lepton Fermi fields, and ion fields. $\mathcal{L}_{\text{ion}}$ is the Lagrangian for the ions. Hereafter we will ignore explicit reference to the ions except to take into account their neutralizing charge density. All that will be said about the lepton fields will apply analogously to them. $F(\vec{x},\tau)$ is the gauge-fixing function. For convenience we set

$$F(\vec{x},\tau) = \partial_\mu A^\mu(\vec{x},\tau) - f(\vec{x},\tau) , \tag{3}$$

where $f(\vec{x})\,\tau)$ is an arbitrary function and $0 \leq \tau \leq \beta$. Equation (2) is a generalization of $Z_G$ in Bernard[13]; it is manifestly gauge invariant since the combination

$$d[A]\det\left(\frac{\partial F}{\partial\omega}\right)\delta(F)$$

is the gauge-invariant measure. What has been done was to take the functional formula for the transition amplitude for going from one state function $|\phi_0\rangle$ at $t=0$ to $|\phi_1\rangle$ at $t=t_1$,

$$\langle\phi_1|e^{-it_1(H-\mu N)}|\phi_0\rangle \ ,$$

setting $it_1=\beta$ (in general $it=\tau$), and summing over all periodic (antiperiodic) paths where

$$|\phi_1\rangle|_{\tau=\beta} = +(-)|\phi_0\rangle|_{\tau=0}$$

to obtain the trace. This gives

$$Z_G = \sum_\phi \langle\phi|e^{-\beta(H-\mu N)}|\phi\rangle \ .$$

The change of variables $it=\tau$ is a "Wick" rotation of the theory into Euclidean space. $N_f$ is a $(T,\mu)$-independent infinity[16] and $[N(\beta)]^2$ is a $T$-dependent infinity arising from integration of the two independent photon conjugate momenta. The

$$\delta(F(\vec{x},\tau))\det\left(\frac{\partial F(\vec{x},\tau)}{\partial\omega}\right)$$

appears to avoid the double-counting problem in gauge theories: Fields $\{A_\mu(\vec{x},\tau)\}$ which are connected to each other by gauge transformations represent the same value of the action and should be counted only as one history. $\omega$ parametrizes the gauge transformation. For $F(\vec{x},\tau)$ of Eq. (3), $\delta A^\mu = -\partial^\mu\omega$ and

$$\det\left(\frac{\delta}{\delta\omega}\,(\partial_\mu A^\mu)\right) = \det(-\Box^2) \ .$$

We can multiply $Z_G$ by arbitrary constants (since this does not change physical quantities). Multiplying by

$$\exp\left[-\frac{1}{2\alpha}\int_0^\beta d\tau\int d^3x f^2(\vec{x},\tau)\right]$$

and integrating over $[df]$, we use the $\delta$ function $\delta(F)$ to obtain the desired form of $Z_G$ (remembering that we are dropping the reference to the ions)

$$Z_G = N_F'[N(\beta)]^2[\det(-\Box^2)]\int_{\substack{\text{periodic } A\\ \text{antiperiodic } \psi}}[dA][d\psi][d\bar\psi] \times\exp\left\{\int_0^\beta d\tau\int d^3x\left[-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2\alpha}(\partial_\mu A^\mu)^2 + \bar\psi(i\,\vec{\partial\!\!\!/}_{\vec{x}} - \gamma^0\partial_\tau - e_B A\!\!\!/ + \mu\gamma^0 - m_B)\psi\right]\right\} \ . \tag{4}$$

The periodicity (antiperiodicity) of $A$ ($\psi$) fields in the interval $0 \leq \tau \leq \beta$ is determined by their commutation relations. From this requirement we may Fourier-expand the $A$ and $\psi$ fields and plug into Eq. (4). By expanding the exponential of the interaction term in a power series to get a diagrammatic expansion for $Z_G$, we obtain the Feynman rules for Green's functions in FTFT:

Use the $T=0$, $\mu=0$ rules with the following replacements:

$$\begin{aligned}
&\int\frac{d^4K}{(2\pi)^4} \to \frac{i}{\beta}\sum_N\int\frac{d^3K}{(2\pi)^3}, \quad \beta = 1/T\\
&K^0=\omega_N, \quad \omega_N = 2\pi Ni/\beta \ \text{(bosons)} \ ,\\
&K^0=\omega_N+\mu, \quad \omega_N = (2N+1)i\pi/\beta \ \text{(fermions)} \ ,\\
&N = 0, \pm 1, \pm 2, \ldots \ ,\\
&(2\pi)^4\delta(K_1+K_2+\cdots) \to (1/i)(2\pi)^3\beta\delta_{\omega_{N_1}+\omega_{N_2}+\cdots}\delta^3(\vec{K}_1+\vec{K}_2+\cdots) \ .
\end{aligned} \tag{5}$$

The various frequency sums $(i/\beta)\sum_N$ are converted to contour integrals as in Ref. 6 for actual computations. These contour integrals will be frequently used in the renormalization of $\Omega$ below, so we remind the reader that for bosons[6]

$$\frac{i}{\beta}\sum_{N=-\infty}^{+\infty} f(\nu_N = 2Ni\pi/\beta) = \frac{1}{2\pi}\int_{-i\infty+\epsilon}^{+i\infty+\epsilon}\frac{dK^0 f(K^0)}{\exp(\beta K^0)-1} + \frac{1}{2\pi}\int_{-i\infty-\epsilon}^{+i\infty-\epsilon}\frac{dK^0 f(K^0)}{\exp(-\beta K^0)-1} + \frac{1}{2\pi}\int_{-i\infty}^{+i\infty} dK^0 f(K^0) \ , \tag{6}$$

and for fermions[6]

$$\frac{i}{\beta}\sum_{N=-\infty}^{+\infty} f(\nu_N+\mu,\ \nu_N=(2N+1)i\pi/\beta) = -\frac{1}{2\pi}\int_{-i\infty+\mu+\epsilon}^{i\infty+\mu+\epsilon}\frac{dK^0 f(K^0)}{\exp[\beta(K^0-\mu)]+1} - \frac{1}{2\pi}\int_{-i\infty+\mu-\epsilon}^{i\infty+\mu-\epsilon}\frac{dK^0 f(K^0)}{\exp[\beta(\mu-K^0)]+1} + \oint_{C'} dK^0 f(K^0) + \frac{1}{2\pi}\int_{-i\infty}^{+i\infty} dK^0 f(K^0)\ . \tag{7}$$

$C'$ is the contour running counterclockwise in the $K^0$ plane from $(\mu,-i\infty)$ to $(\mu,+i\infty)$ to $(0,+i\infty)$ to $(0,-i\infty)$ back to $(\mu,-i\infty)$. For convenience we will call the first two terms of Eq. (6) and the first three terms of Eq. (7) "finite" contours as opposed to the remaining "infinite" contours running up the imaginary axis.

### B. Feynman rules for $\Omega_R$

From Eq. (4) we will derive the expression for $\Omega$. Upon taking the operation $e_B d/de_B$ on $Z_G$ we obtain

$$e_B\frac{d\Omega}{de_B} = \frac{1}{\beta}\int_0^\beta d\tau \int d^3x \langle\bar\psi(\vec{x},\tau)A\!\!\!/\psi(\vec{x},\tau)\rangle e_B\ , \tag{8}$$

where $\langle O\rangle$ denotes the statistical average of an operator $O$:

$$\langle O\rangle = \mathrm{Tr}(e^{-\beta(H-\mu N)}O)/\mathrm{Tr}e^{-\beta(H-\mu N)}\ .$$

We now use the equation of motion from the Lagrangian in Eq. (4),

$$(i\vec{\partial\!\!\!/}_{\vec{x}} - \gamma^0\partial_\tau + \mu\gamma^0 - m_B)\psi(\vec{x},\tau) = e_B A\!\!\!/\psi(\vec{x},\tau)\ , \tag{9}$$

to obtain

$$e_B\frac{d\Omega}{de_B} = \frac{1}{\beta}\int_0^\beta d\tau\int d^3x \lim_{\substack{\vec{y}\to\vec{x}\\ \tau'\to\tau^+}} \mathrm{Tr}[(i\vec{\partial\!\!\!/}_{\vec{x}} - \gamma^0\partial_\tau - m_B + \mu\gamma^0)\langle\bar\psi(\vec{y},\tau')\psi(\vec{x},\tau)\rangle]\ .$$

Now with $\tau'\to\tau^+$, $\langle\bar\psi(\vec{y},\tau')\psi(\vec{x},\tau)\rangle$ is the temperature fermion position-space Green's function, $G_B(\vec{x},\tau;\vec{y},\tau')$, in the presence of interactions. We Fourier-transform it to momentum space with

$$G_\beta(\vec{x},\tau;\vec{y},\tau') = \frac{1}{\beta}\sum_{N=-\infty}^{+\infty}\int\frac{d^3p}{(2\pi)^3}e^{i\vec{p}\cdot(\vec{x}-\vec{y})-\omega_N(\tau-\tau')}G_\beta(p_0,\vec{p})\ ,$$

$$\omega_N = (2N+1)i\pi/\beta,\quad p_0=\omega_N+\mu\ .$$

$G_\beta(p_0,\vec{p})$ is the momentum-space interacting fermion Green's function. Noting that $(1/\beta)\int_0^\beta d\tau = 1$, $\int d^3x = V$, we easily obtain

$$e_B\frac{d\Omega}{de_B} = -iV\left(\frac{i}{\beta}\sum_N\right)\int\frac{d^3p}{(2\pi)^3}\lim_{\epsilon\to 0^+} e^{(2N+1)i\pi\epsilon/\beta}\,\mathrm{Tr}G_F^{-1}G_\beta\ , \tag{10}$$

where the free propagator is $G_F=(p\!\!\!/-m_B)^{-1}$ and $\epsilon=0^+$. Equation (10) can be set in a more convenient form. Dyson's equation is $G_\beta = G_F + G_F\Sigma G_\beta$ with solution $G_\beta=(G_F^{-1}-\Sigma)^{-1}$. $\Sigma$ is the sum of all proper self-energy diagrams. Then $\mathrm{Tr}G_F^{-1}G_\beta = \mathrm{Tr}(1-G_F\Sigma)^{-1}$. Since $\sum_N e^{(2N+1)i\pi\epsilon/\beta}=0$ we can set

$$\mathrm{Tr}(1-G_F\Sigma)^{-1}\to\mathrm{Tr}\left(\frac{1}{1-G_F\Sigma}-1\right) = \mathrm{Tr}(\Sigma G_\beta)\ .$$

So we obtain the final answer for $\Omega_B$,

$$\Omega_B(e_B,m_B,T,\mu) - \Omega_B(0,m_B,T,\mu) = -iV\int_0^{e_B}\frac{de_B}{e_B}\left[\frac{i}{\beta}\sum_N\int\frac{d^3p}{(2\pi)^3}\right]\mathrm{Tr}\left[\Sigma(p_0,\vec{p})G_\beta(p_0,\vec{p})\right]\ , \tag{11}$$

where $\Omega_B$ is the unrenormalized ("bare") $\Omega$. Equation (11) is an expansion of $\Omega_B$ in terms of propagators. We also see that only closed loops contribute to $\Omega$. To $e_B^2$ only Fig. 1 appears while in Fig. 2 and Fig. 3 are respectively the $e_B^4$ and $e_B^6$ graphs. We learn, considering only even numbers of photons attached to fermion-antifermion loops, that there are $3^{N-1}$ graphs for $e_B^{2N}$ order.

In the vacuum odd numbers of photon lines attached to fermion-antifermion loops are zero by charge conjugation, but in FTFT ($\mu\neq 0$), charge conjugation is broken and these graphs have finite nonzero values. As will be discussed in Sec. III these graphs are ultraviolet (UV) finite so we need only consider an even number of photon lines attached to fermion loops for consideration of UV infinities.

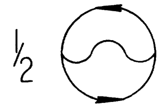


FIG. 1. The order-$e^2$ correction to the thermodynamic potential in QED. The wavy line is the photon and the smooth line is the fermion.

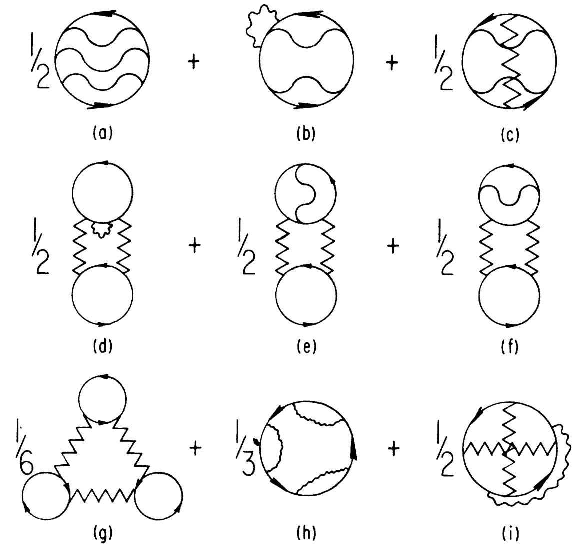


FIG. 3. The order-$e^6$ correction to the thermodynamic potential.

In Fig. 1–Fig. 3 each closed loop comes in with a combinatoric factor which is computed as follows. From (11) the $e_B^{2N}$ contribution to the integrand will bring about a factor $1/2N$ from the $e_B'$ integration. There exists also another factor. When we consider all the propagator diagrams to a desired order and then close the fermion line onto itself, the same closed loop may appear many times. For example, in Fig. 4 closing all the fermion propagators gives rise to the same loop [graph (c) in Fig. 3]. Thus the combinatoric factor for each loop is $(1/2N \times D$, where $D$ is the number of distinct propagators giving rise to the same loop. It so happens that the factor $D/2N$ can be determined for each loop by inspection of the graph. We define the automorphism factor, $A$, of a closed loop as follows: It is the number of ways in which the same diagram could be drawn if the internal fermion lines were regarded as distinct. In considering this transposition of fermion lines, one must be careful so that the directed arrows of transposed lines are consistent with the directed arrows of lines not interchanged. Then $D/2N = 1/A$. This follows from the Dyson-Wick expansion for the closed loops. For example, the graphs (d)–(f) in Fig. 3 have $A = 2$ since the fermion lines in the lower bubble could be interchanged; however, the lines in the upper bubble cannot be interchanged since the arrows cannot match up. It is clear then that by opening up all the fermion lines in the closed-loop expansion at the $e_B^{2N}$ order, with the proper combinatoric factors, one just reobtains the $e_B^{2N}$-order Feynman graphs for the propagator and each propagator appears with factor 1. The question of renormalization of $\Omega$ is dealt with in Sec. IV. Here we will give the rules for obtaining the renormalized $\Omega$ in quantum electrodynamics and leave their derivation to Sec. IV. The renormalized thermodynamic potential is

$$\Omega_B(e_B, m_B, T, \mu) - \Omega_B(e_B, m_B, T=0, \mu=0) = \Omega_R(e_R, m_R, T, \mu, S) . \quad (12)$$

$e$ and $m$ are the coupling constant and mass parameters appearing in the Lagrangian in the relativistic field theory. The subscripts $B$ and $R$ refer to the bare (unrenormalized) and renormalized quantities respectively and $S$ is the subtraction momentum squared. To obtain $\Omega_R(e_R, m_R, T, \mu, S)$ as a power series in $e_R$ do the following:

(1) Draw all topologically distinct closed loop ("vacuum") diagrams.

(2) Multiply each diagram by $1/A$, where $A$ is its automorphism factor.

(3) Evaluate each diagram as if it were an ordinary FTFT Green's function using the rules for Green's functions above. The mass and coupling constant used are $m_B, e_B$.

(4) Include one extra overall factor of $-i$.

(5) There will be an overall three-momentum-conserving $\delta$ function with zero argument times $(2\pi)^3$; eliminate this factor (this factor is as-

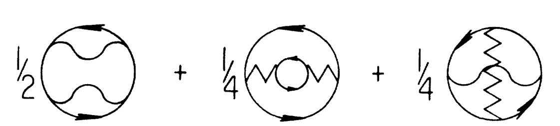


FIG. 2. The order-$e^4$ correction to the thermodynamic potential.

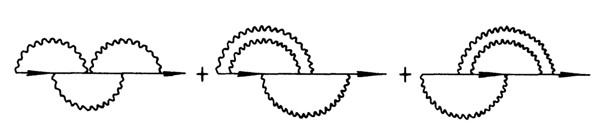


FIG. 4. The propagator expansion of graph (c) in Fig. 3.

sociated with infinite volume and dividing by it gives $\Omega/V$).

The above rules give

$$\frac{\Omega_B}{V}(e_B, m_B, T, \mu) - \frac{\Omega_B}{V}(0, m_B, T, \mu) .$$

(6) Subtract its vacuum value (i.e., $T,\mu = 0$) and express $e_B, m_B$ in terms of $e_R$, $m_R$, and $S$.

All infinities associated with vertex renormalization disappear. The only remaining infinities are associated with fermion self-mass insertions on closed fermion lines.

(7) Add

$$\frac{\Omega_B}{V}(0, m_B, T, \mu) - \frac{\Omega_B}{V}(0, m_B, 0, 0) ,$$

which, when expressed in terms of $e_R$, $m_R$, and $S$ gives $\Omega_R(0, m_R, T, \mu, S)$ plus mass counterterms which precisely cancel the fermion self-mass insertion infinities mentioned above. Equation (12) results. These rules are proved in Sec. IV.

## III. VIOLATION OF CHARGE CONJUGATION

The ordinary vacuum QED Lagrangian is invariant under the operation of charge conjugation. The new term, $\mu\bar{\psi}\gamma^0\psi$, which appears as a result of the presence of the medium, changes sign under charge conjugation. Thus this symmetry is broken. It follows then that closed fermion loops with an odd number of photon vertices no longer vanish. Let us see how this comes about for the tadpole diagram of Fig. 5. This has the value

$$D_{\text{tadpole}} = -e \operatorname{Tr} \int \gamma_\mu \frac{(\not{p} + m)}{p^2 - m^2} \frac{d^4 p}{(2\pi)^4} . \tag{13}$$

Equation (13) vanishes (as do all odd photon vertices) in the vacuum because the only surviving $\gamma$ matrices are multiplied by odd polynomial functions of $p$. For FTFT with $\mu \neq 0$, Eq. (13) has the value (at $T = 0$, for example)

$$D_{\text{tadpole FTFT } T=0} = -ie\delta_{\mu 0} n , \tag{14}$$

where the number density, $n$, is $p_F{}^3/3\pi^2$ and the Fermi momentum is defined as $p_F{}^2 + m^2 = \mu^2$. Equation (14) is found just by using Eq. (7). All graphs which violate charge conjugation are ultraviolet finite. An outline of a proof is as follows. Infinities can only come from the "infinite" contours of Eqs. (6) and (7). We deform these contours to the Feynman contour which runs along the real axis, below it in the left half plane, and above it in the right. As shown in Sec. IV, this results in the loops' $\sum \int d^3 p_i$ integration going to the usual vacuum $\int d^4 p_i$ + finite pieces. But all vacuum $\int d^4 p_i$ are zero so infinities vanish. These graphs which violate charge conjugation, though ultraviolet finite, may have infrared infinities. We will show that the lowest-order infrared infinities of these graphs cancel. To the $e^2$ order we have the following graphs shown in Fig. 6. The double lines on the fermion loops represent the ions. To avoid the infrared problem associated with the masslessness of the photon, we introduce an infrared cutoff, $\lambda$, defined as a fictitious photon mass. If we call $c = e^2 n^2/\lambda^2$ then it is easy to see that for Fig. 6 we have for $\Omega/V$ graph (a) $= -c/2$, graph (b) $= -c/2$, graph (c) $= +c$ so the sum of the graphs is zero.

At this point an observation can be made concerning the embedding of QED within unified gauge theories of the weak and electromagnetic forces based on spontaneous symmetry breaking. It should be recalled that the result of the broken or "hidden" symmetry is that the gauge vectors of the weak force acquire large masses. In Ref. 17 it was stated that the broken symmetry can be restored at a high temperature, $T_c$, in analogy with superconductivity, and so the weak gauge bosons would become massless above some $T > T_c$. The basic requirement for this is that one has thermodynamic equilibrium. The question arises whether or not massless weak vector bosons occur in the early universe at high temperature if there is a *net weak charge*. Let us assume this. Now one of the weak vector bosons is the $Z$ boson which mediates the weak neutral currents; it couples to particles just as the photon does so graph (a) in Fig. 6 exists where the wavy line is now the $Z$ boson. This graph diverges as $\lambda \to 0$, so if the $Z$ becomes massless for $T > T_c$ then $\Omega \to \infty$ because graphs (b) and (c) do not cancel (a) by the hypothesis of nonneutrality. However, the limit $\Omega \to \infty$ shows nonequilibrium, contrary to the assumption. Therefore the weak vector bosons cannot become massless at any temperature if there is a net weak

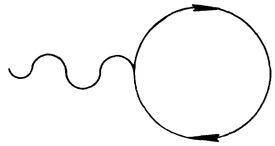

FIG. 5. Tadpole diagram in QED which is nonzero in FTFT.

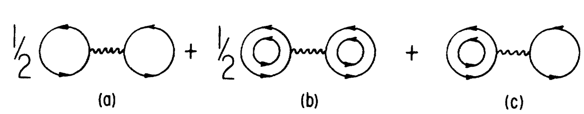


FIG. 6. Lowest-order graphs which violate charge conjugation. The double lines are ions.

charge. The only physical case then is the one of weak charge neutrality, but as was demonstrated in Ref. 18, the plasma effect cancels the long-range forces for all massless vector bosons in a neutral gas so the weak interaction remains short-ranged at all temperatures.

## IV. RENORMALIZATION OF $\Omega$

We now come to the primary interest, that of the renormalization of $\Omega$. This will proceed in the three steps given in the Feynman rules: (1) Find $\Omega_B(e_B, m_B, T, \mu) - \Omega_B(0, m_B, T, \mu)$ [this is given by Eq. (11)]. (2) Subtract its vacuum value. (3) Add $\Omega_B(e_B, m_B, T, \mu) - \Omega_B(0, m_B, 0, 0)$. This results in obtaining $\Omega_B(e_B, m_B, T, \mu) - \Omega_B(e_B, m_B, 0, 0)$, which will be shown to be finite. The proof is tedious and will be shown only to $e_R^4$ order. It can be extended to higher order but becomes increasingly burdensome. Since the fourth order has all the renormalization problems of higher orders, the proof to fourth order demonstrates the theorem. To begin, we first want to compute $\Omega_B(0, m_B, T, \mu) - \Omega_B(0, m_B, 0, 0)$. In Eq. (4) we set $e_B = 0$ and $m_B = m_R - \Delta m$. Operating by $\Delta m\, d/d\Delta m$ on the resultant $Z_G$ we obtain in the usual way

$$-\beta\Delta m \frac{d\Omega}{d\Delta m}\bigg|_{e_B=0} = \int_0^\beta d\tau \int d^3x \lim_{\substack{\vec{y}\to\vec{x}\\ \tau'\to\tau^+}} \mathrm{Tr}\langle \Delta m \psi(\vec{y}, \tau')\psi(\vec{x}, \tau)\rangle\bigg|_{e_B=0}$$

$$= -iV\left(\frac{i}{\beta}\sum_N \int \frac{d^3p}{(2\pi)^3}\right)\mathrm{Tr}\langle \Delta m G_F\rangle \ ,$$

where we Fourier-transformed the position-space–free Green's function momentum space. So

$$\Omega_B(0, m_B, T, \mu) - \Omega_B(0, m_R, T, \mu) = iV\int \frac{d\Delta m}{\Delta m}\left(\frac{i}{\beta}\sum_N \int \frac{d^3p}{(2\pi)^3}\right)\mathrm{Tr}\langle \Delta m G_F\rangle \ . \tag{15}$$

Now $\Omega_B(0, m_R, T, \mu)$ is just $V\Omega_R(0, m_R, T, \mu)$ [where $\Omega_R(0, m_R, T, \mu)$ is the ideal-gas thermodynamic potential per unit volume] plus its zero-point pressure. Therefore when we subtract the vacuum value of Eq. (15) we obtain for our answer

$$\Omega_B(0, m_B, T, \mu) - \Omega_B(0, m_B, 0, 0) = V\Omega_R(0, m_R, T, \mu) + iV\int \frac{d\Delta m}{\Delta m}\left(\frac{i}{\beta}\sum_N{}' \int \frac{d^3p}{(2\pi)^3}\right)\mathrm{Tr}(\Delta m G_F) \ . \tag{16}$$

In Eq. (16) we must also include a minus sign from the trace. In all other equations which follow, this minus sign will have already been included in the factors multiplying the integrals. $\sum_N'$ means that in evaluating the energy sums via Eq. (7) only the "finite contours" are to be used. Inspection of Eq. (16) shows that, as stated in rule (7) of Sec. II B, this quantity is just the ideal-gas term plus mass counterterms. Using Eq. (16) we show some of the lowest-order mass counterterms in Fig. 7. Working these out we have that up to $e_R^4$ Eq. (16) is ($T = 0$)

$$\begin{aligned}\Omega_B(0, m_B, T, \mu)|_{T=0} - \Omega_B(0, m_B, 0, 0) = {}& V\Omega_R(0, m_R, T, \mu)|_{T=0} \\ & + 2Vm_R(\Delta m^{(2)} + \Delta m^{(4)})\int \frac{d^3p}{(2\pi)^3}\frac{\Theta(\mu - (\vec{p}^2 + m_R^2)^{1/2})}{(\vec{p}^2 + m_R^2)^{1/2}} \\ & + V(\Delta m^{(2)})^2\int \frac{d^3p}{(2\pi)^3}\Theta(\mu - (\vec{p}^2 + m_R^2)^{1/2})\left[\frac{m_R^2}{(\vec{p}^2 + m_R^2)^{3/2}} - \frac{1}{(\vec{p}^2 + m_R^2)^{1/2}}\right],\end{aligned} \tag{17}$$

where $\Delta m^{(2)}$ and $\Delta m^{(4)}$ are respectively the $e_R^2$- and the $e_R^4$-order mass counterterms. We are now in a position to prove Eq. (12) and the Feynman rules for $\Omega$.

### A. Renormalization proof to second order

For convenience we renormalize on the mass shell which means $e_R$ and $m_R$ are the physical parameters denoted by $e$ and $m$. We first find $\Omega_B(e_B, m_B, T, \mu) - \Omega_B(0, m_B, T, \mu)$; this is given by Eq. (11) so we must calculate Fig. 1. Call the value of this graph, $I$; we have

$$I = V\frac{e_B^2}{2}\left(\frac{i}{\beta}\sum_N \int \frac{d^3p}{(2\pi)^3}\right)\left(\frac{i}{\beta}\sum_m \int \frac{d^3K}{(2\pi)^3}\right)\mathrm{Tr}\left\{\frac{\gamma_\mu(\not p - \not K + m_B)\gamma^\mu(\not p + m_B)}{[(p-K)^2 - m_B^2]K^2(p^2 - m_B^2)}\right\} \ . \tag{18}$$

In considering the $p$ integration let us do the finite contour of Eq. (7) first. Then we close over the two fermion poles $(p-K)^2 - m_B^2$ and $p^2 - m_B^2$. By symmetry, or if one explicitly works it out, these poles give identical results so we will close over the $p^2 - m_B^2$ pole and multiply it by 2. We will call the operation of closing over a fermion line with the finite contours "opening up the fermion line" and decompose the closed loops into their propagator expansion. In Fig. 8 we have the propagator expansion for Fig. 1. For convenience we work at zero temperature. The simple modification for nonzero temperature will be pointed out. Thus for the finite $p_0$ contour we have (setting $e_B = e$, $m_B = m$ to this order)

$$I = V\int \frac{d^3p}{(2\pi)^3}\,\frac{O(\mu-(\vec{p}^2+m^2)^{1/2})}{2(\vec{p}^2+m^2)^{1/2}}\,(ie^2)\left(\frac{i}{\beta}\sum_m\int\frac{d^3K}{(2\pi)^3}\right)\mathrm{Tr}\left\{\frac{\gamma_\mu(\not{p}-\not{K}+m)\gamma^\mu}{[(p-K)^2-m^2]K^2}\,(\not{p}+m)\right\}\bigg|_{p_0=(\vec{p}^2+m^2)^{1/2}} . \tag{19}$$

In the following, the notation $|_{p_0=(\vec{p}^2+m^2)^{1/2}}$ will be abbreviated to just $|_{p_0=\surd}$. Now we do the $K$ contours. From Eq. (6) at $T=0$, we see that only the infinite contour up the imaginary axis contributes for the photon loop. To help separate the finite piece from the infinite piece we deform this contour to the Feynman contour used in the vacuum propagator definition. This contour runs along the real axis from $-\infty$ to $+\infty$ and is below the real axis in the left-hand plane and above the real axis in the right-hand plane. We can accomplish this deformation by setting $K_0 = K_0' + i\epsilon K_0'$, $\epsilon > 0$. Then Eq. (19) becomes (dropping the prime on $K_0'$)

$$I = V\int \frac{d^3p}{(2\pi)^3}\,\frac{O(\mu-(\vec{p}^2+m^2)^{1/2})}{2(\vec{p}^2+m^2)^{1/2}}\,(ie^2)\int\frac{d^4K}{(2\pi)^4}\,\mathrm{Tr}\left\{\frac{\gamma_\mu(\not{p}-\not{K}+m)\gamma^\mu(\not{p}+m)}{(K^2+i\epsilon)[(p-K)^2-m^2-i\epsilon K_0(p_0-K_0)]}\right\}\bigg|_{p_0=\surd} . \tag{20}$$

The $d^4K$ integration is now almost the second-order fermion vacuum proper self-energy except that the $(p-K)^2-m^2$ propagator has the $-i\epsilon K_0(p_0-K_0)$ imaginary piece rather than the $+i\epsilon$. It is just this difference which gives rise to a finite $\mu$-dependent self-energy: Subtract and add to Eq. (20) the expression

$$I' = V\int \frac{d^3p}{(2\pi)^3}\,\frac{O(\mu-(\vec{p}^2+m^2)^{1/2})}{2(\vec{p}^2+m^2)^{1/2}}\,(ie^2)\int\frac{d^4K}{(2\pi)^4}\,\mathrm{Tr}\left\{\frac{\gamma_\mu(\not{p}-\not{K}+m)\gamma^\mu(\not{p}+m)}{(K^2+i\epsilon)[(p-K)^2-m^2+i\epsilon]}\right\}\bigg|_{p_0=\surd} \tag{21}$$

to obtain

$$I = V\int \frac{d^3p}{(2\pi)^3}\,\frac{O(\mu-(\vec{p}^2+m^2)^{1/2})}{2(\vec{p}^2+m^2)^{1/2}}\,(ie^2)\int_0^{p_0}\frac{dK_0}{2\pi}\int\frac{d^3K}{(2\pi)^3}\,\mathrm{Tr}\,[\gamma_\mu(\not{p}-\not{K}+m)\gamma^\mu(\not{p}+m)]\,(2\pi i)\delta((p-K)^2-m^2)\bigg|_{p_0=\surd} + I' . \tag{22}$$

In Eq. (22) we used ($\epsilon > 0$)

$$\frac{1}{(p-K)^2-m^2-i\epsilon K_0(p_0-K_0)} - \frac{1}{(p-K)^2-m^2+i\epsilon} = \begin{cases} 0; & K_0<0,\quad K_0>p_0 \\ 2\pi i\delta((p-K)^2-m^2); & 0<K_0<p_0 . \end{cases} \tag{23}$$

The first term in Eq. (22) is now finite[19] and the last term, $I'$, has its $K$ integration just giving us the normal vacuum self-energy. Thus for $I$ we have the result

$$I = -V\int \frac{d^3p}{(2\pi)^3}\,\frac{O(\mu-(\vec{p}^2+m^2)^{1/2})}{2(\vec{p}^2+m^2)^{1/2}}\,\mathrm{Tr}[\Sigma^{(2)}(p_0,\vec{p})(\not{p}+m)]\bigg|_{p_0=\surd} , \tag{24}$$

where $\Sigma^{(2)} = \Sigma^{(2)}_{\rm vac} + \Sigma^{(2)}_{f\mu}$. $\Sigma^{(2)}_{\rm vac}$ is the vacuum value of the second-order proper self-energy and $\Sigma_{f\mu}$ is the finite $\mu$-dependent piece. Now for $p^2 = m^2$

$$\Sigma^{(2)}_{\rm vac}|_{p^2=m^2} = \Delta m^{(2)} + (1-Z_2^{-1})(\not{p}-m) + \Sigma^{(2)}_{fV} ,$$

where $\Sigma^{(2)}_{fV}$ is the vacuum finite piece of $\Sigma^{(2)}_{\rm vac}$; this has the property that for on-mass-shell renormalization

$$\Sigma^{(2)}_{fV}(\not{p}+m)|_{p^2=m^2} = 0 . \tag{25}$$

$Z_2$ is the fermion wave-function renormalization. Thus the final value for $I$ is[20]

$$I = -2(\Delta m^{(2)})mV\int\frac{d^3p}{(2\pi)^3}\,\frac{O(\mu-(\vec{p}^2+m^2)^{1/2})}{(\vec{p}^2+m^2)^{1/2}} - V\int\frac{d^3p}{(2\pi)^3}\,\frac{O(\mu-(\vec{p}^2+m^2)^{1/2})}{2(\vec{p}^2+m^2)^{1/2}}\,\mathrm{Tr}[\Sigma^{(2)}_{f\mu}(\not{p}+m)]\bigg|_{p_0=\surd} . \tag{26}$$

Now we consider the infinite $p_0$ contour which remains. For $T=0$, this contour gives a $\mu$-independent infinity. For $T\neq 0$, we get the same previous $\mu$-independent infinity and no $T$-dependent infinities. This can be seen from Eq. (6) where the two temperature contours are both identically zero for no pole in $f(K^0)$. This absence of $T$-dependent infinities coming from the outer infinite $p_0$ contour is true in general at all orders. The reason is that the infinity coming from the infinite $p_0$ contour is of the form $K^2\times$(infinite constant) in every order, so that there are no $K^2$ poles remaining in doing the finite contours in (6).

We now subtract $\Omega_B(e_B, m_B, 0, 0) - \Omega_B(0, m_B, 0, 0)$. This just subtracts out the infinity coming from the infinite $p_0$ contour. Now we add $\Omega_B(0, m_B, T, \mu) - \Omega_B(0, m_B, 0, 0)$. This is given by Eq. (17) and we see that the infinities cancel to this $e^2$ order. What remains is just the ideal-gas and exchange term,

$$\Omega_R(e,m,T,\mu)|_{T=0} = V\Omega_R(0,m,T,\mu)|_{T=0} - V\int \frac{d^3p}{(2\pi)^3}\,\frac{\Theta(\mu-(\vec{p}^{\,2}+m^2)^{1/2})}{2(\vec{p}^{\,2}+m^2)^{1/2}}\,\mathrm{Tr}[\Sigma^{(2)}_{f\mu}(\not{p}+m)]\Big|_{p_0=\sqrt{\phantom{x}}}\ . \tag{27}$$

This completes the proof to $e^2$ order.

### B. Renormalization proof to fourth order

We again follow the sequence of obtaining

$$\Omega_B(e_B,m_B,T,\mu)-\Omega_B(0,m_B,T,\mu)\ ,$$

subtracting out its vacuum value, and adding

$$\Omega_B(0,m_B,T,\mu)-\Omega_B(0,m_B,0,0)\ .$$

The integrals are arranged so that the outermost integration, call it $d^4p$, is a fermion loop, and we again consider its finite and infinite contours separately. For the finite $p_0$ integration we get the propagator expansion of Fig. 9. There are two remarks to be made. First, the middle graph of Fig. 2 has two fermion loops, and when we open each one separately and add we get graph (b) of Fig. 9. So Fig. 9 represents all the finite integrations. Second, the dashed lines in graphs (d) and (f) mean close over the double poles. The $\times$ in the vertex of graph (g) is $e^3f(\Lambda)$, where $e_B$ in Fig. 1 has been expanded as $e_B=e+e^3f(\Lambda)$. $e^3f(\Lambda)$ is the vertex counterterm to lowest order and $\Lambda$ is some gauge-invariant ultraviolet regulator. The $\times$ in graphs (e) and (f) are just $-i\Delta m^{(2)}$ mass insertions coming from expanding to lowest order $i\times$ (propagator in Fig. 1). To separate out the finite and infinite pieces of these graphs, we again fold down the "infinite" photon $K$ integration(s) to the Feynman contour by the usual change of variables $K_{j0}\rightarrow K'_{j0}+i\epsilon K'_{j0}$ ($\epsilon>0$ and we drop the primes). Now each graph has a finite outside $d^3p$ integration and one or two $K$ inner integration(s) with an integrand of strange $i\epsilon$ propagators. To each graph we add and subtract within each integrand the vacuum expression for the propagators with their $+i\epsilon$ imaginary parts, just as we did in second order. Now each graph in Fig. 9 has two pieces, one with the integrand having all propagators with their vacuum $+i\epsilon$ and one with the integrand which is the difference between the rotated contour propagators and the vacuum propagators. We now are in a position to analyze Fig. 9.

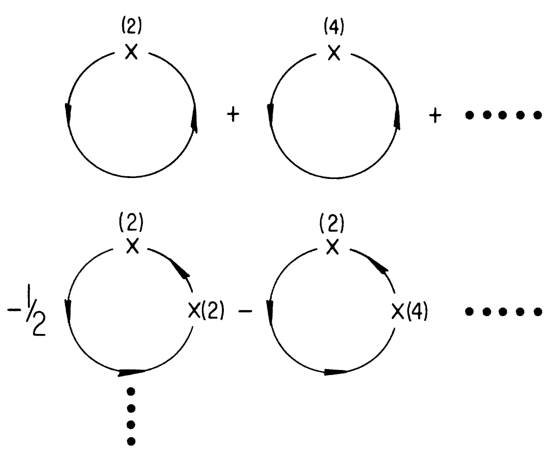


FIG. 7. Mass counterterms given by Eq. (16). (2), (4), ... refer to $\Delta m^{(2)}$, $\Delta m^{(4)}$, ..., which are the second-, fourth-, ... order mass counterterms.

Let us take from graphs (a), (b), (c), (e), (g) that piece which involves the integrand with only $+i\epsilon$ propagators. Calling the sum of these pieces $S$, we have

$$S=-V\int \frac{d^3p}{(2\pi)^3}\,\frac{\Theta(\mu-(\vec{p}^{\,2}+m^2)^{1/2})}{2(\vec{p}^{\,2}+m^2)^{1/2}} \times \mathrm{Tr}[\Sigma^{(4)}_{\mathrm{vac}}(\not{p}+m)]\Big|_{p_0=\sqrt{\phantom{x}}} \tag{28}$$

because graphs (a), (b), (c), (e), (g) with $+ie$ propagators are just the fourth-order vacuum self-energy. As before, only $\Delta m^{(4)}$ survives in Eq. (25) and the final value for $S$ is

$$S=-2\Delta m^{(4)}m\int \frac{d^3p}{(2\pi)^3}\,\frac{\Theta(\mu-(\vec{p}^{\,2}+m^2)^{1/2})}{(\vec{p}^{\,2}+m^2)^{1/2}}\ . \tag{29}$$

What remains of each of the graphs (a), (b), (c), (e), (g) is the piece which has in the integrand the difference of propagators. We will look at these graphs individually and start with (g) which is the simplest. In (g) there is only one propagator which has the strange $i\epsilon$ (the internal fermion line) so the integrand of the one remaining piece of (g) has Eq. (23) in it. This just gives rise to $\Sigma^{(2)}_{f\mu}$ so the last remaining piece of (g) [call it (g′)] is

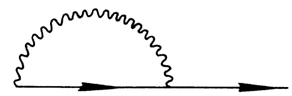

FIG. 8. Propagator expansion for Fig. 1.

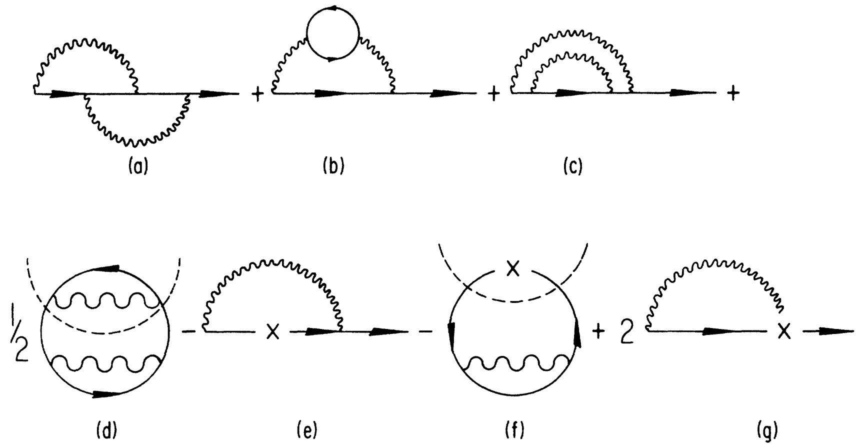


FIG. 9. Propagator expansion for Fig. 2.

$$(\mathrm{g}') = -2(Z_3^{-1/2} - 1)\, V \int \frac{d^3p}{(2\pi)^3} \frac{O(\mu - (\vec{p}^2 + m^2)^{1/2})}{2(\vec{p}^2 + m^2)^{1/2}} \mathrm{Tr}[\Sigma_{f\mu}^{(2)}(\not{p} + m)] \Big|_{p_0 = \sqrt{\ }} .$$

Now to this order $Z_3^{-1/2} - 1 = -\frac{1}{2}(Z_3 - 1)$ so the final value for (g′) is

$$(\mathrm{g}') = V(Z_3 - 1) \int \frac{d^3p}{(2\pi)^3} \frac{O(\mu - (\vec{p}^2 + m^2)^{1/2})}{2(\vec{p}^2 + m^2)^{1/2}} \mathrm{Tr}[\Sigma_{f\mu}^{(2)}(\not{p} + m)] \Big|_{p_0 = \sqrt{\ }} . \tag{30}$$

Let us continue on to the last remaining piece of (b) [call it (b′)]. As was said before, graph (b) includes opening up both fermion loops of the middle graph in Fig. 2. Looking at this graph we see that there are two fermion lines which have the strange $i\epsilon$ (one of the lines in the closed loop and the bottom fermion line). Thus the integrand of the last remaining piece in (b) has the form

$$AB - A'B' = (A - A')(B - B') + A'(B - B') + B'(A - A') , \tag{31}$$

where $A$, $B$ are the two strange $i\epsilon$ propagators and $A'$, $B'$ are the $+i\epsilon$ vacuum propagators. As before, any term such as $A - A'$, $B - B'$, where $A$, $B$ are propagators, will give a finite loop integration by Eq. (23). Thus the first term on the right-hand side is finite. The last two terms in Eq. (31) make the bottom fermion line a vacuum propagator with the upper loop a $\delta$ function (giving rise to a finite $K$ integration) and vice versa, where the upper loop has now pure vacuum propagators and the bottom fermion line is finite (on the mass shell from the $\delta$ function). The former term is finite and the latter contributes a $(Z_3 - 1)$ infinity. Working out the minus signs we have

$$(\mathrm{b}') = -V(Z_3 - 1) \int \frac{d^3p}{(2\pi)^3} \frac{O(\mu - (\vec{p}^2 + m^2)^{1/2})}{2(\vec{p}^2 + m^2)^{1/2}} \mathrm{Tr}[\Sigma_{f\mu}^{(2)}(\not{p} + m) \Big|_{p_0 = \sqrt{\ }} . \tag{32}$$

We continue to the last remaining term in graphs (c) and (e). In (c) the integrand is of the form of Eq. (31), but now one of the symbols is a propagator squared and the other one is a single propagator. Now by taking the derivative of Eq. (23) with respect to one of the loop momenta we see immediately that the difference of two squared propagators is the derivative of a $\delta$ function. Thus in graph (c) the first term in Eq. (31) is finite, while the last terms make the single propagator have the vacuum value times the derivative of the $\delta$ function and the double propagators have their vacuum value and the single propagators become a $\delta$ function [by Eq. (23)]. Only the former gives rise to an infinity. It makes the inner photon loop give rise to a $\Sigma_{\mathrm{vac}}^{(2)}$. We evaluate the derivative of the $\delta$ function by integration by parts; after that the $\delta$ function makes the two fermion lines on the mass shell. In the remaining term of graphs (e), (e′), we have a double propagator which becomes the derivative of the $\delta$ function. The $\Delta m^{(2)}$ piece of (e′) cancels the $\Delta m^{(2)}$ piece in $\Sigma_{\mathrm{vac}}^{(2)}$ of (c′) leaving the only infinity in the sum of the two diagrams a $(1 - Z_2^{-1})$ from (c′). Working out the signs we have

$$(\mathrm{c}') + (\mathrm{e}') = (Z_2^{-1} - 1)\, V \int \frac{d^3p}{(2\pi)^3} \frac{O(\mu - (\vec{p}^2 + m^2)^{1/2})}{2(\vec{p}^2 + m^2)^{1/2}} \mathrm{Tr}[\Sigma_{f\mu}^{(2)}(\not{p} + m)] \Big|_{p_0 = \sqrt{\ }} . \tag{33}$$

Next we turn our attention to the last remaining piece in (a). There are three different fermion propagators in (a) which have the strange $i\epsilon$. We need the counterpart to Eq. (31) in three variables. It is

$$ABC - A'B'C' = (A-A')(B-B')(C-C') + A'(B-B')(C-C') + B'(A-A')(C-C') + C'(A-A')(B-B') + A'B'(C-C') + B'C'(A-A') + A'C'(B-B') . \tag{34}$$

Each difference of an unprimed and primed variable produces a $\delta$ function by Eq. (23) giving us a finite loop integration. The only possible divergences represented in Eq. (34) are the last three terms, each of which gives rise to a single $\delta$ function. They put one of the three internal fermion lines in (a) on the mass shell while the other two are the vacuum propagators. When the two vacuum fermion lines are just separated by one vertex they give rise to a $(Z_1^{-1}-1)$-loop infinity. This happens twice. The one remaining infinity in (a′) is

$$(\mathrm{a}') = -2V(Z_1^{-1}-1)\int \frac{d^3p}{(2\pi)^3}\,\frac{O(\mu-(\vec{p}^2+m^2)^{1/2})}{2(\vec{p}^2+m^2)^{1/2}}\,\mathrm{Tr}[\Sigma_{f\mu}^{(2)}(\not p+m)]\Big|_{p_0=\sqrt{\ }} . \tag{35}$$

We go on to the last remaining diagrams, (d) and (f). We deform the photon contours and add and subtract from their integrand the vacuum propagators so each diagram has two pieces. Let us look at the piece of each which just involves the vacuum propagators. Now in (d) the two photon loops will give rise to a product of $\Sigma_{\mathrm{vac}}^{(2)}$ which is to be differentiated by $d/dp_0$ from the residue theorem for a double pole. The $(\Delta m^{(2)})^2$ of these $(\Sigma_{\mathrm{vac}}^{(2)})^2$ pieces would be exactly canceled by the vacuum propagator values of graph (f) if the factor $\frac{1}{2}$ multiplying (d) were changed to 1. Thus the vacuum value of (d) + (f) leaves a resultant mass infinity which is easily worked out to be [call it $(\mathrm{d}_1)+(\mathrm{f}_1)$]

$$(\mathrm{d}_1)+(\mathrm{f}_1) = -\tfrac{1}{2}V(\Delta m^{(2)})\int \frac{d^3p}{(2\pi)^3}\,O(\mu-(\vec{p}^2+m^2)^{1/2})\left[\frac{\mathrm{Tr}(\not p+m)(\not p+m)}{4(\vec{p}^2+m^2)^{3/2}} - \frac{\mathrm{Tr}(2\not p\gamma_0)}{4(\vec{p}^2+m^2)}\right]\Big|_{p_0=\sqrt{\ }} ,$$

and which has the final value

$$(\mathrm{d}_1)+(\mathrm{f}_1) = -(\Delta m^{(2)})^2 V\int \frac{d^3p}{(2\pi)^3}\left[\frac{m^2}{(\vec{p}^2+m^2)^{3/2}} - \frac{1}{(\vec{p}^2+m^2)^{1/2}}\right] . \tag{36}$$

What remains in each of the graphs (d) + (f) is the piece which has in the integrand the difference of the strange $i\epsilon$ and the vacuum $+i\epsilon$. For (d) there are two propagators so we use Eq. (31). The first term of Eq. (31) is of course finite while the remaining two terms of Eq. (31) make one of the photon loops give a $\Sigma_{\mathrm{vac}}^{(2)}$ and the other a $\Sigma_{f\mu}^{(2)}$, and vice versa; there are then two $\Delta m^{(2)}\Sigma_{f\mu}^{(2)}$ and $\Delta m^{(2)}(1-Z_2^{-1})$ infinities in (d) which, when multiplied by $\frac{1}{2}$, are canceled by the last remaining piece of (f) which has minus these infinities. The only infinity not canceled between the two remaining pieces is a $(1-Z_2^{-1})\Sigma_{f\mu}^{(2)}$ piece in (d) which does not exist in (f). Thus the sum of the two remaining pieces of (d) and (f) [call it $(\mathrm{d}_1)+(\mathrm{f}_1)$] is

$$(\mathrm{d}_2)+(\mathrm{f}_2) = V(Z_2^{-1}-1)\int \frac{d^3p}{(2\pi)^3}\,\frac{O(\mu-(\vec{p}^2+m^2)^{1/2})}{2(\vec{p}^2+m^2)^{1/2}}\,\mathrm{Tr}[\Sigma_{f\mu}^{(2)}(\not p+m)]\Big|_{p_0=\sqrt{\ }} . \tag{37}$$

This finishes the enumeration of the infinities in Fig. 9 and completes the outer finite contour of $p_0$. Looking at Eqs. (28)–(37) we see that, after the Ward identity $Z_1=Z_2$ is used, all infinities cancel except mass insertion infinities of Eq. (29) and (36). Now we take the infinite outer $p_0$ contour which just gives us pure vacuum infinities which disappear after subtracting out the vacuum value of

$$\Omega_B(e_B, m_B, T, \mu) - \Omega_B(0, m_B, T, \mu) .$$

Next we add

$$\Omega_B(0, m_B, T, \mu) - \Omega_B(0, m_B, 0, 0)$$

whose infinities are given by Eq. (17). This just cancels the remaining mass infinities that were left over in Eqs. (29) and (36). This completes the proof of the theorem, Eq. (12), to $e^4$ order.

This method of proof could be extended to a higher order but becomes increasingly tedious. One sees that there are precise cancellations of infinities which come about from parts of graphs canceling parts of others in an intricate, delicate manner.

## V. RENORMALIZATION-GROUP EQUATION FOR $\Omega_R$

From Eq. (12) we have the renormalization-group equations for $\Omega_R$,

$$S\frac{d}{dS}\,\Omega_R(e_R, m_R, T, \mu, S) = 0 . \tag{38}$$

Equation (38) can be used to obtain important information. Let us choose the following renormalization scheme: We make all wave-function and coupling-constant renormalizations at a Euclidean momentum $p^2=-M^2$, but we renormalize the fermion mass on the mass shell (for simplicity, one mass), so that $m_R$ is the position of the pole in the fermion propagator. For the interesting case $T=0$ (we will drop the $T$ dependence of $\Omega_R$ for this discussion) Eq. (38) becomes

$$\left[M\frac{\partial}{\partial M}+\beta\left(e_R,\frac{\mu}{M}\right)\frac{\partial}{\partial e_R}\right]\Omega_R\left(e_R,m_R,\frac{\mu}{M}\right)=0\ ,$$
$$\left[-\mu\frac{\partial}{\partial\mu}+\beta\left(e_R,\frac{\mu}{M}\right)\frac{\partial}{\partial e_R}\right]\Omega_R\left(e_R,m_R,\frac{\mu}{M}\right)=0\ . \tag{39}$$

The Callan-Symanzik function $\beta$ depends on $e_R$ and $\mu/M$, for large $p_F\gg m_R$. The general solution to Eq. (39) is

$$\Omega_R\left(e_R,m_R,\frac{\mu}{M}\right)=\Omega_R\left(\bar{e}_R\left(\frac{\mu}{M},e_R\right),m_R\right)\ , \tag{40}$$

where

$$\mu\frac{\partial\bar{e}_R}{\partial\mu}=\beta\left(\bar{e}_R,\frac{\mu}{M}\right)\ ,$$
$$\bar{e}_R(1,e_R)=e_R\ . \tag{41}$$

Equation (41) states that the effective coupling constants in a renormalizable field theory are functions of the density (through their dependence on $\mu$) for matter at $T=0$. Actually, this discussion makes clear the fact that $\mu$ plays the identical role here in FTFT that $Q$, Euclidean momentum, does for the vacuum. From Eq. (41) it follows that the behavior of matter, vis-à-vis its strength of interactions, is the same as the theory's ultraviolet behavior in the vacuum. Thus the strength of electromagnetic interactions increases with density[21] while strongly interacting matter, describable by a non-Abelian gauge theory, becomes weak at high densities.[21] It is easy to see that one must have densities of matter approaching the initial cosmological big-band singularity before perturbation theory in QED breaks down. This is due to the smallness of the electric charge. However, for the strong force to become weak enough so that perturbation theory may be done, one need only have densities of the order of $6\times10^{14}$ g/cc.[22] These large densities are in fact reached in the heavier neutron stars, and one has the astounding discovery that all quantities of interest in these heavier mass neutron stars, which involve the strong force, may be perturbatively calculated. The reason for this is that the quark fine-structure constant ~2 at nuclear densities (~$2\times10^{14}$ g/cc) and decreases rapidly above that density.[22]

## VI. CONCLUSION

The primary result in this paper is the renormalization of the thermodynamic potential, Eq. (12). The exact Feynman rules for constructing $\Omega_R$ as a power series in $e$ for QED are given in Sec. II B; analogous rules can be derived for any renormalizable theory following the procedure here by first obtaining $e_B d\Omega_B/de_B$ and following through with the same calculations.

An immediate corollary is that the use of perturbation theory for a renormalizable field theory in a medium results in an effective coupling constant that varies with density. Equation (41) gives $\bar{e}_R$ as a function of $\mu$. To obtain $\bar{e}_R$ as a function of density, $\rho$, one must have $\rho=\rho(\mu)$. While this is relatively unimportant for electromagnetism, it has important implications for the strong force.[22]

An outstanding problem in FTFT is the proof of the infrared finiteness of $\Omega_R$ to all orders of perturbation theory. Research on this is in progress.

Parallel work on finite-temperature field theory, including the explicit QED calculation of $\Omega_R$ to $e$ , has been done in a series of reports from the MIT group.[23]

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