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Poincaré- and gauge-invariant two-dimensional quantum chromodynamics

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We present a canonical formulation of two-dimensional quantum chromodynamics in the axial or Coulomb gauge $A_1^a = 0$. For consistency with the Lagrange equations of motion, the Hamiltonian must include a nontrivial dynamical background electric field term. This breaks translation invariance in the gauge-noninvariant sector. We argue, however, that for the purpose of calculating gauge-invariant physical quantities one can consider the naive theory defined by the Feynman rules without the background electric field. We show that the naive theory has an anomalous Poincaré algebra due to its non-Abelian character; the theory is Lorentz invariant only in the color-singlet sector. Because of this fact the quark propagator has a noncovariant pole, and the $i\epsilon$ prescription is different from the naive one. In the $N \to \infty$ limit (where N is the number of colors) we can set up a two-component Bethe-Salpeter equation in the color-singlet sector to determine the spectrum of the theory. The resulting equation has an obvious interpretation in terms of forward- and backward-moving strings, and leads to the same spectrum of bound states as that obtained by 't Hooft in the light-cone gauge.

I. INTRODUCTION

The purpose of this paper is to discuss the consistent formulation of two-dimensional quantum chromodynamics (QCD) [with an SU(N) gauge symmetry] in the limit $N \to \infty$ in an arbitrary axial gauge. The theory[1-3] was formulated by 't Hooft in the lightlike axial gauge, and there has been some speculation[4-6] about its consistency in other gauges. Our result differs from Refs. 4–6 but agrees with Ref. 1. We shall in fact discuss only the $A_1^a = 0$ gauge in this paper for reasons of clarity. The generalization to an arbitrary axial gauge does not exhibit any new problems.

The theory is defined by the Lagrangian

$$\mathcal{L} = -\tfrac{1}{4} F^a_{\mu\nu} F^{a\mu\nu} + \bar{\psi}(\tfrac{1}{2} i \overleftrightarrow{D} - m)\psi , \tag{1}$$

where

$$D_\mu = \partial_\mu - ig A^a_\mu \frac{\lambda^a}{2} \tag{2}$$

and

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu . \tag{3}$$

Our definition of g differs from that of Ref. 1 by a factor of $1/\sqrt{2}$. The (suppressed) indices on the quark spinor fields indicate N color components and n flavors, and the gluons have $N^2 - 1$ components.

The theory is set up in terms of canonical variables in Sec. II. We discover that in order to obtain a consistent set of equations of motion, it is necessary to add an extra term to the naive Hamiltonian. This term corresponds to the QCD analog of the background electric field term that arises in two-dimensional Abelian theories.[7-9] The presence of the extra term leads to a violation of translation invariance in the gauge-variant sector of the theory and, hence, the Poincaré algebra only closes on the color-singlet sector.

We can argue that (unlike in the Abelian case[7,10]) the extra term does not affect the physical gauge-invariant sector of the theory. We restrict our subsequent discussions to this sector and effectively use the naive theory defined by the Feynman rules in the absence of the background electric field. We show that the naive theory has an anomalous Poincaré algebra such that Lorentz invariance is satisfied only in the color-singlet sector. This happens only in a non-Abelian theory.

In Sec. III, we find the one-fermion eigenstates of the Hamiltonian and determine the renormalized energy of the quark in the $1/N$ approximation. This is done by normal ordering the interaction and diagonalizing the quadratic terms that remain. This method differs from that of Ref. 6. The result is an integral equation that determines the renormalized wave function and energy of the quark. We present an exact solution in the zero-bare-mass limit and determine the general behavior for arbitrary mass. As we explain in Sec. II, we expect to obtain noncovariant results for the renormalized energy. This is displayed by an explicit exact solution $E(p) = |p| - \gamma/|p|$ in the zero-bare-mass limit. We show that this

solution can be computed exactly in perturbation theory because higher-order diagrams are identically zero. A property of the renormalized energy in this theory is that it changes sign as a function of momentum. It is at this point that we differ significantly from the authors of Ref. 4 who invoke an $i\epsilon$ prescription that excludes such solutions. We show that the correct $i\epsilon$ prescription in this theory is obtained by replacing ϵ by $\epsilon E(p)$ in the propagator, and that, with this proviso, the same results can be obtained by the ordinary Feynman-rule techniques. There is no difficulty with the principal-value cutoff prescription in our approach.

In Sec. IV, we derive the Bethe-Salpeter equation for the quark-antiquark bound state. It takes the form of two coupled integral equations in terms of two unknown functions. We interpret these equations in terms of forward- and backward-moving strings. The equations are valid in an arbitrary frame and the total momentum appears as a parameter. Using the proven Poincaré invariance of the color-singlet sector we can evaluate our equation in the infinite-momentum frame. It then becomes identical to 't Hooft's equation[1] that was derived in the light-cone gauge. This proves the gauge invariance of the spectrum in this theory.

II. POINCARE ALGEBRA

The generators of the Poincaré algebra can be derived by standard methods. The momentum P, the Hamiltonian H, and the boost $M^{01}=tP-K$ are given by

$$P=\int dx\,\theta^{01},\quad H=\int dx\,\theta^{00},\quad K=\int dx\,x\theta^{00},\tag{2.1}$$

where $\theta_{\mu\nu}$ is the symmetric energy-momentum tensor

$$\theta_{\mu\nu}=-F^a_{\mu\lambda}F^{a\lambda}_{\nu}+\tfrac{1}{4}i\bar\psi(\vec D_\mu\gamma_\nu+\vec D_\nu\gamma_\mu)\psi-g_{\mu\nu}\mathcal{L}.\tag{2.2}$$

In the present theory, if we set $A^a_1=0$, we obtain

$$\theta^{01}=-\frac{i}{2}\psi^\dagger\frac{\vec\partial}{\partial x}\psi,\tag{2.3a}$$

$$\theta^{00}=\psi^\dagger\left(-\frac{i}{2}\gamma_5\frac{\vec\partial}{\partial x}+\gamma^0 m\right)\psi+\tfrac{1}{2}(F^a_{01})^2,\tag{2.3b}$$

where $F^a_{01}=-\partial_1 A^a_0$ is given below. Our γ-matrix convention is $\gamma^0=\sigma_3$, $\gamma^1=i\sigma_2$, $\gamma_5=\gamma^0\gamma^1=\sigma_1$.

The equation of motion $D^\mu F^a_{\nu\mu}=J^a_\nu$ leads to a constraint for $\nu=0$. Denoting $J^a_0\equiv\rho^a=\psi^\dagger g\tfrac{1}{2}\lambda^a\psi$, one finds the constraint

$$\frac{\partial^2 A^a_0}{\partial x^2}=\rho^a,\tag{2.4}$$

whose general solution is given by

$$A^a_0(x,t)=\tfrac{1}{2}\int_{-\infty}^{\infty}dy\,|x-y|\,\rho^a(y,t)-xF^a(t)+B^a(t).\tag{2.5}$$

Note that B^a can be changed by an x-independent but time-dependent gauge transformation. From A^a_0 one obtains the electric field $F^a_{01}=-\partial_1 A^a_0$,

$$F^a_{01}=-\tfrac{1}{2}\int_{-\infty}^{\infty}dy\,\epsilon(x-y)\rho^a(y,t)+F^a(t).\tag{2.6}$$

As $x\to\pm\infty$ the electric field takes the values

$$\begin{aligned}&F^a_{01}(+\infty,t)=F^a(t)-\tfrac{1}{2}Q^a(t)\equiv\Pi^a(t),\\&F^a_{01}(-\infty,t)=F^a(t)+\tfrac{1}{2}Q^a(t)=\Pi^a(t)+Q^a(t),\end{aligned}\tag{2.7}$$

where

$$Q^a(t)=\int_{-\infty}^{\infty}dx\,\rho^a(x,t)\tag{2.8}$$

is the charge. Correspondingly, A^a_0 has the asymptotic behavior

$$A^a_0(x,t)\underset{x\to\pm\infty}{\sim}-xF^a_{01}(\pm\infty,t)+(B^a\mp\tfrac{1}{2}Q^a_1),\tag{2.9}$$

where $Q^a_1(t)$ is the first moment of the charge density

$$Q^a_1(t)=\int_{-\infty}^{\infty}dx\,x\rho^a(x,t).\tag{2.10}$$

Replacing these results in the field equation $D^0F^a_{10}=J^a_1$ as $x\to\pm\infty$ (assuming $J^a_1\to0$) we derive the time development of the electric field at $x\to\pm\infty$[9]:

$$\begin{aligned}&\frac{\partial\Pi^a}{\partial t}=-gf^{abc}(B^b-\tfrac{1}{2}Q^b_1)\,\Pi^c,\\&\frac{\partial}{\partial t}(\Pi^a+Q^a)=-gf^{abc}(B^b+\tfrac{1}{2}Q^b_1)(\Pi^c+Q^c).\end{aligned}\tag{2.11}$$

Since we can change $B^b(t)$ by a gauge transformation, we choose it conveniently so that $B^b+\tfrac{1}{2}Q^b_1=0$, leading to

$$\frac{\partial\Pi^a}{\partial t}=-\frac{\partial Q^a}{\partial t}=gf^{abc}Q^b_1\Pi^c.\tag{2.12}$$

As seen from the above equations, the electric field cannot be made time independent at both ends $x\to\pm\infty$ if $\Pi^a\neq0$. Furthermore, $\Pi^a(t)$ is a dynamical variable and its equation of motion must be reproduced by the Hamiltonian. We also note that Q^a is not conserved if $\Pi^a\neq0$, but Π^a+Q^a is conserved.

The Poincaré generators now take the form

$$P=-\frac{i}{2}\int dx\,\psi^\dagger\frac{\vec\partial}{\partial x}\psi,\tag{2.13a}$$

$$\begin{aligned}H=&\int dx\,\psi^\dagger\left(-\frac{i}{2}\gamma_5\frac{\vec\partial}{\partial x}+\gamma_0 m\right)\psi\\&-\tfrac{1}{4}\int dx\,dy\,\rho^a(x)\,|x-y|\,\rho^a(y)\\&+Q^a_1(\Pi^a+\tfrac{1}{2}Q^a),\end{aligned}\tag{2.13b}$$

$$K=\int dx\, x\psi^{\dagger}\left(-\frac{i}{2}\gamma_5\frac{\overleftrightarrow{\partial}}{\partial x}+\gamma^0 m\right)\psi$$
$$-\tfrac{1}{8}\int dx\,dy\,\rho^a(x)(x+y)|x-y|\rho^a(y)$$
$$+\tfrac{1}{2}Q_2^a(\Pi^a+\tfrac{1}{2}Q^a), \qquad (2.13c)$$

where $Q_n^a(t)=\int dx\, x^n\rho^a(x,t)$.

In the above expression for the Hamiltonian, we have dropped an infinite term proportional to $\frac{1}{2}\infty[\Pi^2+(\Pi+Q)^2]$. This term has the effect of giving infinite energy to any charged state. Similarly, in the boost, we have dropped an infinite term proportional to $\frac{1}{4}\infty^2[(\Pi+Q)^2-\Pi^2]$. These steps will be justified below when we show that the combinations $(\Pi+Q)^2\pm\Pi^2$ commute with all operators in the theory, up to an x-independent gauge transformation. In particular they commute with H, and therefore they can be diagonalized independently. We thus take the above finite operators as the *definitions* of the generators of the Poincaré algebra in this theory.

Let us verify that the Hamiltonian reproduces the Lagrange equations of motion. We have to specify the commutation rules of Π^a with ψ and with itself. Since Π^a is part of the gauge field, it is reasonable to take

$$[\Pi^a,\psi]=0. \qquad (2.14)$$

To find the commutation rules of Π^a with itself, we note that the combination Π^a+Q^a was conserved [Eq. (2.12)]. This conservation property will be guaranteed by the Hamiltonian, provided Π^a+Q^a acts like the *total* generator of global color transformations on all variables of the theory. This implies

$$[\Pi^a+Q^a,\Pi^b+Q^b]=igf^{abc}(\Pi^c+Q^c). \qquad (2.15a)$$

Furthermore, from the anticommutation rules of the ψ's we can immediately deduce that

$$[Q^a,Q^b]=igf^{abc}Q^c. \qquad (2.15b)$$

Equations 2.14 and 2.15 lead uniquely to

$$[\Pi^a,\Pi^b]=igf^{abc}\Pi^c. \qquad (2.16)$$

Using the canonical anticommutation rules for ψ and Eqs. (2.14) and (2.16), we can now verify that

$$\dot{\psi}=i[H,\psi]=-\left(\gamma_5\frac{\partial}{\partial x}+im\gamma_0\right)\psi$$
$$+\tfrac{1}{2}ig\tfrac{1}{2}\lambda^a\{A_0^a,\psi\}, \qquad (2.17)$$

where A_0^a is identical to Eq. (2.5) with $B^a=-\frac{1}{2}Q_1^a$ as before. Similarly the equation for Π^a is

$$\dot{\Pi}^a=i[H,\Pi^a]=gf^{abc}Q_1^b\Pi^c,$$

as in (2.12). We thus have the correct Hamiltonian and canonical rules.

We are now ready to check the closure of the Poincaré algebra. By direct commutation, using (2.14) and (2.16), we verify that the theory is Lorentz invariant for arbitrary Π^a:

$$[P,K]=-iH,$$
$$[H,K]=-iP. \qquad (2.18)$$

However, the theory is not translationally invariant in the charged sector owing to the presence of the Q_1^a term in the Hamiltonian. Thus, $[P,H]=-iQ^a(\Pi^a+\frac{1}{2}Q^a)$ does not vanish on charged states. The source of this violation is the background field $F^a=\Pi^a+\frac{1}{2}Q^a$. Since the operator $\int dx\psi^{\dagger}(\overleftrightarrow{\partial}/\partial x)\psi$ is not conserved an ordinary Fourier expansion of the field does not diagonalize the quadratic terms in the normal-ordered Hamiltonian. This clearly creates problems for the treatment of the quantum theory in the charged sector, such as the calculation of the full quark propagator, etc.

To circumvent this problem, one is tempted to put the background field $F^a=\Pi^a+\frac{1}{2}Q^a=0$ in order to get a translationally invariant theory. This, however, creates some problems if it is taken as an operator statement. First of all, one of the Lagrange equations is not satisfied by the equations of motion generated by the naive Hamiltonian H_1 with $F^a=0$. Namely, one obtains, by using the naive Hamiltonian in $\dot{\psi}=i[H_1,\psi]$,

$$D^0F_{10}^a-J_1^a=\tfrac{1}{8}gf^{abc}(Q_1^bQ^c+Q^cQ_1^b) \qquad (2.19)$$

instead of 0 on the right-hand side. As a direct consequence of this fact, $\theta_{\mu 0}$ is not conserved and the Lorentz algebra does not close, i.e.,

$$[H_1,K_1]=-iP-\frac{i}{4}f^{abc}Q^a(Q_1^bQ_2^c+Q_2^cQ_1^b), \qquad (2.20)$$

where K_1 is obtained from K by setting $F^a=0$. Although we have watched the order of the operators, this anomaly of the Poincaré algebra is not just a quantum ordering effect, but it occurs even if one neglects the order of the operators.

We note that the charge operator Q^a in Eq. (2.20) can be moved to the right or the left of $\{Q_1^b,Q_2^c\}$ and therefore, when applied to color-singlet states, the complete naive Poincaré algebra closes.

We shall argue now that no physics is lost in the quark-antiquark singlet sector if we consider a modified theory in which the background field is set equal to 0: $F^a=\Pi^a+\frac{1}{2}Q^a=0$.[10] Suppose $F^a\neq 0$, then the time development of a singlet operator constructed of quarks will be given by

$$O=i[H,O]=i[H_1,O]+i[H_2,O],$$

where H_1 is translationally invariant and H_2 is defined as the part of the Hamiltonian due to the background field $H_2 = Q_1^a(\Pi^a + \frac{1}{2}Q^a)$. Since O is constructed out of quarks it commutes with Π^a, and since it is a singlet, it commutes with Q^a, but it does not necessarily commute with Q_1^a. When it is sandwiched between $q\bar{q}$-singlet states one obtains for the time development of O

$$\langle \text{singlet}|O|\text{singlet}\rangle = \langle \text{singlet}|i[H_1, O]|\text{singlet}\rangle + i\langle \text{singlet}|(\Pi^a + \tfrac{1}{2}Q^a)[Q_1^a, O]|\text{singlet}\rangle .$$

Using $Q^a|\text{singlet}\rangle = 0$, and $\Pi^a|\text{singlet}\rangle = (c\text{-number}) \times |\text{singlet}\rangle$, we see that

$$\Pi^a\langle \text{singlet}|[Q_1^a, O]|\text{singlet}\rangle = 0 .$$

since $[Q_1^a, O]$ is not a singlet. Therefore, the time development of O in the physical sector is given by H_1 only.

The argument given above applies in particular to the Bethe-Salpeter wave function which is of main interest in this paper,

$$\phi_{\alpha\beta}(x_1, x_2) = \langle 0|T(\psi_\alpha^i(x_1)\psi_\beta^i(x_2))|E, P\rangle ,$$

where the color indices are contracted to make a singlet. Since H_2 has no effect on the time development of $\phi_{\alpha\beta}$, it cannot affect the spectrum of hadrons in the $q\bar{q}$ channel, and therefore can be completely dropped. This is in contrast to the Abelian case treated by Coleman[7] where the background field had physical consequences in the neutral sector.

Another way of arriving at the same result is to consider H_2 as a perturbation in a translationally invariant theory defined by H_1 to zeroth order. It is not difficult to convince oneself that in the singlet sector of H_1 the perturbation H_2 will vanish order by order. This is due to the non-Abelian property of the operators.

Therefore, from now on we will set the background field $F^a = 0$ and treat a modified, translationally invariant theory given by H_1. Note that this is precisely the theory defined by the naive Feynman rules which do not take into account a background field. *However, because of Eq. (2.20), now we should expect noncovariant results in the charged sector of this theory. In particular, the pole of the full quark propagator in the* $1/N$ *expansion will not be covariant as we shall display below. We emphasize that, as seen in Eq. (2.20), this phenomenon is due to the non-Abelian character of the theory, it would not occur in an Abelian theory.* Furthermore, as already argued, the physical singlet sector is Poincaré covariant in the naive theory.

III. THE SELF-ENERGY EQUATION

In this section, we will discuss the derivation of the "self-energy" equation that arises by normal ordering of the Hamiltonian (excluding the additional background electric field terms) and diagonalizing the quadratic terms. We will also derive the self-energy equation from the Feynman perturbation series. Furthermore, for the case $m = 0$ we exhibit an explicit solution, while for the $m \neq 0$ case we give the general behavior.

A. The Hamiltonian method

The Hamiltonian is not in normal-ordered form. We wish first to choose a vacuum, $|0\rangle$, such that $H|0\rangle = 0$ in the $1/N$ approximation. To do so, we expand $\psi(x)$ in terms of eigenstates of the momentum operator e^{ikx} and obtain

$$\psi_\alpha^i(x, t) = \frac{1}{(2\pi)^{1/2}}\int_{-\infty}^{\infty} dk[b^i(k, t)u_\alpha(k) + d^{\dagger i}(-k, t)v_\alpha(-k)]e^{ikx} . \quad (3.1)$$

The vacuum state is defined by

$$b^i|0\rangle = 0 = d^i|0\rangle . \quad (3.2)$$

The wave functions $u(k)$ and $v(-k)$ will be defined below. They are not ordinary Dirac wave functions for a free particle. We normalize them so that

$$u^\dagger(k)u(k) = 1 = v^\dagger(-k)v(-k) , \quad (3.3)$$

$$u^\dagger(k)v(-k) = 0 , \quad (3.4)$$

$$u_\alpha(k)u_\beta^\dagger(k) + v_\alpha(-k)v_\beta^\dagger(-k) = \delta_{\alpha\beta} . \quad (3.5)$$

It is clear that a general form of the wave function is

$$u(k) = T(k)\begin{pmatrix}1\\0\end{pmatrix}, \quad v(-k) = T(k)\begin{pmatrix}0\\1\end{pmatrix}, \quad (3.6)$$

where $T(k)$ is a unitary 2×2 matrix in Dirac space. With these definitions, the canonical anticommutation relations become

$$\{b^i(k, t), b^{\dagger j}(k', t)\} = \delta(k - k')\delta^{ij} , \quad (3.7)$$

$$\{d^i(k, t), d^{\dagger j}(k', t)\} = \delta(k - k')\delta^{ij} , \quad (3.8)$$

and all other anticommutators are zero.

One can now write the interaction term in the Hamiltonian in normal ordered form plus the normal ordering pieces (leaving out the background electric field term as described previously) as follows:

$$\int dx\,dy\rho^a(x)|x-y|\rho^a(y)=\int dx\,dy:\rho^a(x)|x-y|\rho^a(y):$$
$$+\frac{g^2}{4\pi}\left(N-\frac{1}{N}\right)\int dk\,dx\,dy\,e^{ik(x-y)}|x-y|\psi^\dagger(x)[u(k)u^\dagger(k)-v(-k)v^\dagger(-k)]\psi(y), \tag{3.9}$$

where N is the number of colors in the SU(N) theory. The newly generated quadratic terms may be combined with the original kinetic terms in the free part of the Hamiltonian to give the quadratic form:

$$\int dx\,dy\,\psi^\dagger(x)M(x,y)\psi(y), \tag{3.10}$$

where $M(x,y)$ is a 2×2 traceless, Hermitian operator. As $N\to\infty$, in order for the Hamiltonian to annihilate the vacuum defined by b^i and d^i we must demand that this term be diagonal, i.e., it must take the form

$$\int dk\,E(k)[b^\dagger(k)b(k)+d^\dagger(k)d(k)] \tag{3.11}$$

up to a constant. This requires eigenvalue conditions on the wave functions $u_\alpha(p)$ and $v_\alpha(-p)$ with real eigenvalues $E(p)$ and $-E(p)$, respectively,

$$\left\{p\gamma_5+m\gamma_0+\frac{\gamma}{2}\int\frac{dk}{(p-k)^2}[u(k)u^\dagger(k)-v(-k)v^\dagger(-k)]\right\}u(p)=E(p)u(p) \tag{3.12}$$

and similarly, for $v(-p)$ with $E(p)$ replaced by $-E(p)$. The constant γ is given by (γ is fixed as $N\to\infty$)

$$\gamma=\frac{g^2}{4\pi}\left(N-\frac{1}{N}\right), \tag{3.13}$$

and the integral is defined by a principal-value prescription.

We note that Eq. (3.11) corresponds to a completely diagonalized Hamiltonian in the one-fermion sector, in the $1/N$ approximation: The states $b_i^\dagger|0\rangle$, $d_i^\dagger|0\rangle$ are *exact eigenstates of the Hamiltonian* in this limit, since the remaining normal-ordered interaction is of order $1/N$ when applied on these states. Therefore, the operators $b_i^\dagger$, $d_i^\dagger$ create renormalized quarks with renormalized energy. The two eigenvalue conditions can be written in matrix form

$$E(p)T(p)\gamma_0T^\dagger(p)=p\gamma_5+m\gamma_0+\frac{\gamma}{2}\int\frac{dk}{(p-k)^2}T(k)\gamma_0T^\dagger(k). \tag{3.14}$$

We take for the unitary transformation

$$T(k)=\exp[-\tfrac{1}{2}\theta(p)\gamma^1], \tag{3.15}$$

which is sufficiently general for our considerations. We find two equations:

$$E(p)\cos\theta(p)=m+\frac{\gamma}{2}\int\frac{dk}{(p-k)^2}\cos\theta(k), \tag{3.16}$$

$$E(p)\sin\theta(p)=p+\frac{\gamma}{2}\int\frac{dk}{(p-k)^2}\sin\theta(k), \tag{3.17}$$

which imply that $\theta(p)$ is odd under $p\to -p$ if $E(p)$ is even. By taking appropriate linear combinations we obtain the decoupled forms

$$p\cos\theta(p)-m\sin\theta(p)=\frac{\gamma}{2}\int\frac{dk}{(p-k)^2}\sin[\theta(p)-\theta(k)], \tag{3.18}$$

$$E(p)=m\cos\theta(p)+p\sin\theta(p)+\frac{\gamma}{2}\int\frac{dk}{(p-k)^2}\cos[\theta(p)-\theta(k)]. \tag{3.19}$$

Therefore, the eigenvalue $E(p)$ is completely determined by the solution of the $\theta(p)$ equation.

The fermion propagator can be obtained by direct calculation,

$$\langle 0|T[\psi_\alpha^i(x)\psi_\beta^j(y)]|0\rangle=i\delta^{ij}\int\frac{dp\,dp^0}{(2\pi)^2}e^{ip_\mu(x-y)^\mu}S(p,p_0)_{\alpha\beta}, \tag{3.20}$$

with

$$S_{\alpha\beta}(p,p_0)=\frac{u(p)\overline{u}(p)}{p^0-E(p)+i\epsilon}+\frac{v(-p)\overline{v}(-p)}{p^0+E(p)-i\epsilon}, \tag{3.21}$$

$$S_{\alpha\beta}(p,p_0)=\frac{p^0\gamma^0-\gamma^1E(p)\sin\theta(p)+E(p)\cos\theta(p)}{[p^0-E(p)+i\epsilon][p^0+E(p)-i\epsilon]}, \tag{3.22}$$

where we have made use of Eqs. (3.6) and (3.15).

We note that the denominator in Eq. (3.22) takes the form $p_0^2 - E^2 + 2iE\epsilon$ and not $p_0^2 - E^2 + i\epsilon$. In a covariant theory $E = (p^2 + c^2)^{1/2}$ is positive, so that the two forms are equivalent. However, we have argued that in the quark sector we expect non-covariant energies to appear, and in particular $E(p)$ may not always be positive. Thus, we must retain the form $E\epsilon$. This is where we depart from all previous treatments of this problem. As we shall see the inconsistencies pointed out in Ref. 4 are removed by this simple observation.

B. The Feynman-diagram method

For completeness, we derive the results of the previous section with Feynman-diagram techniques. The self-energy equation in the $1/N$ approximation is derived by standard methods (see Fig. 1):

$$\Sigma(p) = -i\frac{\gamma}{\pi}\int \frac{dk_0 dk}{(p-k)^2}\gamma^0 S(k,k_0)\gamma^0 , \tag{3.23}$$

where the integral is defined with a principal-value prescription. The propagator $S(k,k_0)$ can always be parametrized in the form (3.22). If we write the self-energy in the notation of Ref. 4 as

$$\Sigma(p) = A + \gamma^1 B , \tag{3.24}$$

we can identify

$$A = E\cos\theta - m , \tag{3.25}$$

$$B = E\sin\theta - p . \tag{3.26}$$

We take care to keep the form $\epsilon E(p)$ that appears in the fermion propagator. In covariant perturbation theory $E(p)$ is positive and one normally replaces $\epsilon E(p) \to \epsilon$. However, in our case it is crucial to keep this form. Clearly, dropping the $E(p)$ at this point will not allow for the possibility of solutions with negative $E(p)$. We can now derive equations for $E(p)$ and $\theta(p)$ by means of Eqs. (3.23)–(3.26). As usual,[1,4] this involves evaluating the k^0 integral in Eq. (3.23) explicitly, taking great care with the $i\epsilon E(p)$ prescription. The resulting equations are those previously derived [Eqs. (3.16) and (3.17)] where no conditions appear on the sign of $E(p)$.

C. Solution of the self-energy equation

Since the self-energy equations are not easy to solve in general, it may be instructive to consider the explicitly solvable $m = 0$ case. The solution is simply given by[11]

$$\theta(p) = \tfrac{1}{2}\pi\epsilon(p) , \tag{3.28}$$

$$E(p) = |p| - \mathrm{P}(\gamma/|p|) , \tag{3.29}$$

where

$$\epsilon(p) = \begin{cases} +1 & p > 0 \\ -1 & p < 0 , \end{cases}$$

and P denotes the principal value.

The form of the solution for $m \neq 0$ can now be guessed as plotted in Fig. 2. This clearly interpolates between the two limits $m^2/\gamma \to 0$ given by Eq. (3.28) and $m^2/\gamma \to \infty$ given by $\tan\theta = p/m$. Notice that $E(p)$ is not positive for the whole range of p, in particular for small m^2/γ, $E \to -\gamma/m$ as $p \to 0$, and this explains why there is no such solution for the equations of Ref. 4 as derived with the principal-value cutoff for the gluon propagator. Since the quark is confined, there is no physical objection to negative energies. In any event we will not use any explicit solutions in order to arrive at the spectrum. We will need only the general behavior $\theta(p) \to \pm\pi/2$ as $p \to \pm\infty$, which can be arrived at directly by analyzing Eqs. (3.16) and (3.17).

The origin of the negative sign in the self-energy $E(p)$ is presumably due to the confining potential $|x-y|$ which is equivalent to the principal-value prescription. If, instead, a small momentum cutoff λ is introduced to regulate the gluon propagator, one gets a nonconfining potential at large distances:

$$\begin{aligned} V(x-y) &= \int_{-\infty}^{\infty} dp\, e^{ip(x-y)}\frac{1}{p^2}\theta(p^2-\lambda^2) \\ &= \frac{2}{\lambda}\cos\lambda(x-y) + \pi|x-y| - 2(x-y)\mathrm{Si}[\lambda(x-y)] , \\ &\xrightarrow[|x-y|\to\infty]{} \frac{2}{\lambda}\cos\lambda(x-y) , \end{aligned} \tag{3.30}$$

where

$$\mathrm{Si}(x) = \int_0^x dy\,\frac{\sin y}{y} .$$

FIG. 1. Self-energy equation.

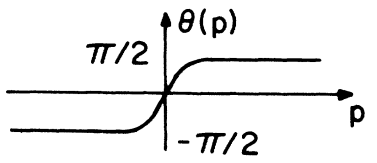

FIG. 2. General behavior of the solution to Eq. (3.18).

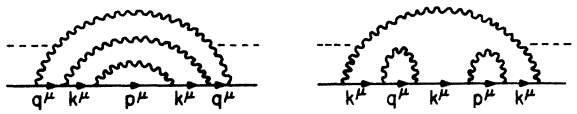

FIG. 3. Calculation of the self-energy in perturbation theory.

This prescription gives a positive-definite $E(p)$ (Ref. 4).

The $m=0$ solution for $E(p)$ [Eq. (3.29)] involves only the first power of γ. Indeed, it is easy to see that it is exactly given by the one-loop contribution to the perturbation series for the self-energy in Fig. 1. We will now show that the rest of the series is zero. The zero-mass fermion propagator in perturbation theory may be written as

$$S=S^{+}+S^{-}=\frac{[1+\gamma_5\epsilon(p)]\gamma^0}{p^0-|p|+i\epsilon}+\frac{[1-\gamma_5\epsilon(p)]\gamma^0}{p^0+|p|-i\epsilon}$$
$$=\frac{1}{\not{p}+i\epsilon}. \tag{3.31}$$

In any Feynman diagram of order higher than g^2 contributing to the quark self-energy, there is at least one loop with two or more fermion propagators in it carrying the same momentum (Fig. 3). For example, the order g^4 correction contains the structure

$$S(k_\mu)\gamma^1 B_2(k)S(k_\mu), \tag{3.32}$$

where $B_2(k)$ is the lowest-order contribution to the self-energy which is $B_2=-P(\gamma/k)$ [giving the order $-\gamma$ contribution to $E(p)$ in Eq. (3.29)]. The momentum k_μ is a loop momentum which is to be integrated. In performing the k_0 integration, the $i\epsilon$ prescription causes the terms in the integral containing only the S^+ or only the S^- parts of the propagators to vanish. The remaining terms then vanish by simple γ-matrix algebra. Thus, a factor of $[1+\gamma_5\epsilon(k)]$ can always be pulled through an even number of γ matrices until it is adjacent to a $[1-\gamma_5\epsilon(k)]$, which it causes to vanish. The argument is the same to all orders, and only the g^2 contribution survives.

We note that there are apparently other solutions to the equations with $m=0$. These are solutions in which $|\theta(p)|=\frac{1}{2}\pi$ but $\theta(p)$ is odd in p and changes sign an arbitrary number of times. Since they do not reduce to free field theory in the $\gamma\to 0$ limit, we do not understand the relevance of these solutions and we shall not consider them further.

If the $m\neq 0$ self-energy equations [(3.16) and (3.17)] are solved perturbatively[5] [i.e., for $\gamma \ll (p^2+m^2)$] we do not recover the $m=0$ solution by setting $m=0$ in this expansion. This is because the expansion is singular at $m=0$ (at any p) and is invalid for small enough values of m.[12]

IV. THE BOUND-STATE EQUATION

Having obtained the form of the quark propagator, it is straightforward to derive the equation for the color-singlet bound states of quark-antiquark pairs. The equation for the wave function, $\Gamma(r^\mu,p^\mu)$, is given by the usual construction[1]

$$\Gamma(r^\mu,p^\mu)=\frac{i\gamma}{2\pi}\int\frac{dk^0dk}{(p-k)^2}S(p^\mu)\gamma^0\Gamma(r^\mu,k^\mu)\times\gamma^0S(p^\mu-r^\mu), \tag{4.1}$$

where r^μ is the total momentum of the bound states and p^μ is the momentum in one of the quark legs (Fig. 4). Since the interaction is instantaneous, we may perform the p^0 integration as usual by defining

$$\phi(r,p)=\int dp^0\Gamma(r^\mu,p^\mu). \tag{4.2}$$

Substituting for $S(p^\mu)$ from Eq. (3.21) and doing the p^0 integration, we obtain

$$\phi(r,p)=\gamma\int\frac{dk}{(p-k)^2}\times\left[\frac{u(p)\bar{u}(p)\gamma^0\phi(r,k)\gamma^0v(r-p)\bar{v}(r-p)}{E(p)+E(r-p)-r^0}+\frac{v(-p)\bar{v}(-p)\gamma^0\phi(r,p)\gamma^0u(p-r)\bar{u}(p-r)}{E(p)+E(r-p)+r^0}\right]. \tag{4.3}$$

Using the definition of $T(p)$ [Eq. (3.6)] and $\theta(-p)=-\theta(p)$ to write

$$u(p)\bar{u}(p)=T(p)\frac{1+\gamma^0}{2}T^\dagger(p)\gamma^0,$$
$$v(r-p)\bar{v}(r-p)=\gamma^0T(r-p)\frac{1-\gamma^0}{2}T^\dagger(r-p), \tag{4.5}$$

we have

$$\tilde{\phi}(r,p)=\gamma\int\frac{dk}{(p-k)^2}\left[\frac{1+\gamma^0}{2}\frac{T^\dagger(p)T(k)\tilde{\phi}(r,k)T^\dagger(r-k)T(r-p)}{E(p)+E(r-p)-r^0}\frac{1-\gamma^0}{2}+\frac{1-\gamma^0}{2}\frac{T^\dagger(p)T(k)\tilde{\phi}(r,k)T^\dagger(r-k)T(r-p)}{E(p)+E(r-p)+r^0}\frac{1+\gamma^0}{2}\right], \tag{4.6}$$

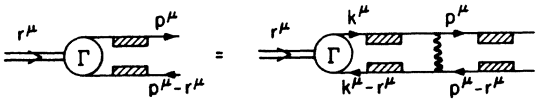

FIG. 4. Diagrammatic representation of Eq. (4.1).

where $\bar{\phi}$ is obtained from ϕ by unitary transformations

$$\bar{\phi}(r,p) = T^{\dagger}(p)\phi(r,p)T(r-p). \tag{4.7}$$

The general structure of $\bar{\phi}$, as given by Eq. (4.6), is $[(1+\gamma_0)A(1-\gamma_0)+(1-\gamma_0)B(1+\gamma_0)]$. The 2×2 matrices A and B could contain only γ_5 or γ^1. Since $\gamma^1=\gamma_0\gamma_5$ can be replaced by γ_5 due to the $(1\pm\gamma_0)$ factors, we can finally express $\bar{\phi}$ in terms of two unknown functions ϕ_+ and ϕ_-, such that

$$\bar{\phi} = \phi_+ M^+ + \phi_- M^-, \tag{4.8}$$

where the matrices $M^{\pm}$ are given by

$$M^{\pm} = \tfrac{1}{2}(1\pm\gamma_0)\gamma_5. \tag{4.9}$$

We can arrive at a pair of coupled integral equations for $\phi_{\pm}$ by substituting Eq. (4.8) in Eq. (4.6). Thus defining

$$C(p,k,r) = \cos\tfrac{1}{2}[\theta(p)-\theta(k)]\cos\tfrac{1}{2}[\theta(r-p)-\theta(r-k)], \tag{4.10}$$

$$S(p,k,r) = \sin\tfrac{1}{2}[\theta(p)-\theta(k)]\sin\tfrac{1}{2}[\theta(r-p)-\theta(r-k)], \tag{4.11}$$

we arrive at

$$[E(p)+E(r-p)-r^0]\phi_+(r,p) = \gamma\int\frac{dk}{(p-k)^2}[C(p,k,r)\phi_+(r,k) + S(p,k,r)\phi_-(r,k)], \tag{4.12}$$

$$[E(p)+E(r-p)+r^0]\phi_-(r,p) = \gamma\int\frac{dk}{(p-k)^2}[C(p,k,r)\phi_-(r,k) + S(p,k,r)\phi_+(r,k)].$$

These equations, together with (3.16) and (3.17),

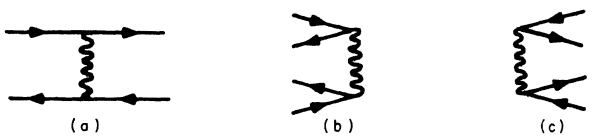

FIG. 5. Time-ordered interactions that dominate the $N\to\infty$ limit.

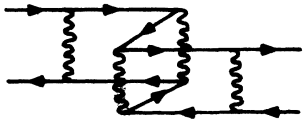

FIG. 6. Forward- and backward-moving string structure.

determine the bound-state spectrum in an arbitrary reference frame characterized by the total momentum r. The structure of the two equations has an immediate interpretation in terms of time-ordered diagrams. Thus, the right-hand side of Eq. (4.12) has two terms. The term containing $C(p,k,r)$ arises from diagrams which do not change particle number after interaction [Fig. 5(a)], while the term containing $S(p,k,r)$ comes from diagrams that create or destroy four particles [Figs. 5(b) and 5(c)]. This interpretation becomes particularly evident by writing

$$\begin{aligned} C(p,k,r) &= u^{\dagger}(p)u(k)v^{\dagger}(p-r)v(k-r), \\ S(p,k,r) &= u^{\dagger}(p)v(-k)v^{\dagger}(p-r)u(r-k). \end{aligned} \tag{4.13}$$

In a general frame both types of interactions occur and allow the $q\bar{q}$ pair to move forward or backward in the manner of a string forming a sheet structure, as shown in Fig. 6. Thus, $\phi_+(p,r)$ is the amplitude for moving forward and $\phi_-(p,r)$ is the amplitude for moving backward. This is evident from the fact that the total energy r^0 has opposite signs in the two equations (4.12).

In general, the equations are rather complicated and we have not tried to solve them. However, using our formal proof of Lorentz invariance in the color-singlet sector, we can evaluate the equations at any value of r and expect that $r^0 = (r^2+M^2)^{1/2}$, where M^2 is an invariant eigenvalue. In particular, we will take the $r\to\infty$ limit, because we expect that the backward motion should disappear and simplify the equations in this limit.

We define the rescaled variables $p=xr$ and $k=yr$. As we let $r\to+\infty$, the right-hand side of Eq. (4.12) is of order $1/r$. Therefore, any term on the left-hand side which grows with r must be set equal to zero and the $1/r$ terms on both sides must be matched. Thus we find that $\phi_-(x)=0$ for all values of x, and $\phi_+(x)\neq 0$ only when $0\leq x\leq 1$. Furthermore, when $0\leq x\leq 1$,

$$S(p,k,r)\to 0,$$

$$C(p,k,r)\to 1 \text{ only for } 0\leq y\leq 1,$$

$$E(p)+E(r-p)-(r^2+M^2)^{1/2} \to -\frac{1}{2r}\left(\frac{m^2-2\gamma}{x}+\frac{m^2-2\gamma}{1-x}-M^2\right), \tag{4.14}$$

where we have used the general behavior of $\theta(p) \to \pi/2$, for $p \to +\infty$ as shown in Fig. 2. Therefore, as expected, backward motion disappears and Eq. (4.12) reduces to

$$\left(\frac{m^2-2\gamma}{x}+\frac{m^2-2\gamma}{1-x}-M^2\right)\phi_+(x) = 2\gamma\int_0^1 \frac{dy}{(x-y)^2}\,\phi_+(y). \quad (4.15)$$

This is precisely the equation derived by 't Hooft in a different gauge[1] ($A_+=0$). We have thus proved the gauge invariance of the meson spectrum in two-dimensional QCD.

V. CONCLUSION

We have shown that two-dimensional QCD can be formulated in a consistent canonical formalism in the $A_1^a=0$ gauge. This requires a modification of the naive Hamiltonian by the addition of a background electric field in order to ensure Lorentz invariance and consistent equations of motion. These extra terms do not affect the derivation of the color-singlet bound-state equation. Therefore, they can be dropped in order to simplify the theory. In the absence of these terms the structure of the Poincaré algebra implies that the quark propagator need not have a covariant pole with energy $E=(p^2+c^4)^{1/2}$. In particular, quark energies can become negative and great care must be taken to enforce the correct $i\epsilon E$ prescription.

The spectrum of bound states is given by an eigenvalue equation which becomes identical with 't Hooft's equation in the infinite-momentum frame. In a general frame the equation describes forward- and backward-moving quark-antiquark pairs forming a sheet structure. Although we have not explicitly demonstrated the Lorentz covariance of our bound-state equations (4.12), we have demonstrated that the formalism is covariant in the color-singlet sector. This indicates that our spectrum must be frame independent, and thus equivalent to that of 't Hooft. To confirm frame independence directly from (4.12), one could differentiate the equations with respect to the parameter r and take the scalar product with the wave function $\phi_\pm(p,r)$. The $d\phi_\pm/dr$ terms drop out with this prescription. Using the eigenvalue condition, one must then show that the equations obtained through this procedure are satisfied only if $\partial r^0/\partial r = r/r^0$, implying $r^0=(r^2+M^2)^{1/2}$. This procedure is algebraically very involved for our equations and therefore we have deferred it to a later investigation. However, because of our formal proof, we expect to obtain the desired results.

Note added in proof. After this paper was submitted for publication, it was pointed out to us that other groups arrived independently at similar conclusions with respect to the Poincaré algebra [L.-F. Li and J. F. Willemsen, Phys. Rev. D 10, 4087 (1974): 13, 531 (E) (1976); N. Pak and P. Senjanović, Report No. SLAC-PUB-1975, 1977 (unpublished)]. However, our Hamiltonian Eq. (2.13b) is the only one that correctly reproduces the Lagrange equations of motion and is Lorentz covariant for *any* value of Π^a. This difference comes from the operator $\Pi^a(t)$ which was not introduced by the other groups.

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