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Fermion field theory on a lattice: Variational analysis of the Thirring model

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We extend variational techniques previously described to study the two-dimensional massless free fermion field and the Thirring model on a spatial lattice. The iterative block spin procedure of constructing an effective lattice Hamiltonian, starting by dissecting the lattice into three-site blocks, is shown to be successful in producing results known for the continuum Thirring model.

I. INTRODUCTION

In this paper we continue the development and application of techniques for finding the ground state and spectrum of low-lying physical states of field theories without recourse to either weak- or strong-coupling expansions. Following our earlier papers[1] we study lattice Hamiltonians by means of a constructive variational procedure. The methods, which in our preceding paper (Paper III) were described and applied to free-field theory of bosons in one space dimension and to the one-dimensional Ising model with a transverse applied magnetic field, are now applied to two fermion theories in one space and one time $(1x-1t)$ dimensions: massless free fermions and the Thirring model.[2-5] Our aim in this paper is to demonstrate that these methods are easily applied to fermion theories and that our simple constructive approach reproduces results known to hold in soluble continuum models. Our fundamental approach to the study of these models is the same as in Paper III. Namely, we dissect the lattice into blocks containing a few sites which are coupled together via the gradient terms in the Hamiltonian. The Hamiltonian for the resulting few-degree-of-freedom problem is diagonalized and the degrees of freedom "thinned" by keeping only an appropriate set of low-lying states. We then construct an effective Hamiltonian by computing the matrix elements of the original Hamiltonian in the space of states spanned by the lowest-energy eigenstates in each block. The process is then repeated for our new effective Hamiltonian. At each step, the coupling parameters of the effective Hamiltonian change and the procedure is repeated until we enter either a very-weak- or very-strong-coupling regime.

The specific features of these models which make them interesting are that (i) for the first time one *must* study the behavior of the first-order gradient term which we introduced for fermion theories[6] in order to avoid doubling of states and preserve chiral symmetry, and (ii) the wave-function renormalization, Z_2, in the Thirring model vanishes when the coupling strength exceeds a finite value $g_{\rm cr}\simeq 1.1$, even in the presence of a cutoff. We also analyze the lattice theory's Schwinger term and establish correspondence with the continuum theory.

As we show, our general procedure leads to the conclusion that there is a finite value of the coupling strength $g_{\rm cr}$ such that for $g<g_{\rm cr}$ one only needs to study the general properties of the massless fermion theory. On the other hand, for $g>g_{\rm cr}$, we are driven to study the theory of the Heisenberg antiferromagnet, which is also a theory with massless low-lying excitations. In particular, this analysis confirms a conjecture made in Paper II to the effect that the spectrum of the strong-coupling limit of our lattice Thirring model is that of the Heisenberg antiferromagnet with nearest-neighbor interactions even though the gradient appropriate to a fermion theory on the lattice includes long-range couplings. In addition to these results we show in the Appendix that the operator expression for the Schwinger term becomes, in the infinite-volume limit, the ground-state expectation value of the kinetic energy density divided by the square of the wave-function renormalization constant $Z_2(g)$.

II. GENERAL METHOD APPLIED TO FREE FIELD THEORY

To fix notation and to illustrate our general method as well as its accuracy, we construct the ground state and lowest-lying excited states of a lattice free fermion theory. The free massless lattice Hamiltonian is [see (3.1)–(3.4) of Paper II]

$$H=\Lambda\sum_{j_1,j_2} i\delta'(j_1-j_2)(b_{j_1}^\dagger b_{j_2}-d_{j_1}^\dagger d_{j_2})\ , \tag{2.1}$$

where

$$\delta'(j)=\begin{cases}\dfrac{-(-1)^j}{j}\ , & j\neq 0\\[4pt] 0\ , & j=0\ .\end{cases}$$

The two-component dimensionless lattice field χ_j

is specified in the Dirac basis

$$\chi_j = \begin{pmatrix} b_j \\ d_j^\dagger \end{pmatrix}$$

with $\alpha = \gamma_5 = \sigma_3$,[7] and satisfies the anticommutation relations

$$\{\chi_j^\dagger, \chi_{j'}\} = \delta_{jj'} \underline{1},$$
$$\{\chi_j, \chi_{j'}\} = 0$$

and so,

$$\{b_j, b_{j'}^\dagger\} = \{d_j, d_{j'}^\dagger\} = \delta_{jj'} ,$$
$$\{b_j, d_{j'}\} = 0, \text{ etc.} \tag{2.2}$$

Λ is the inverse lattice spacing and the lattice has $2N+1 = \Lambda L$ sites. This theory possesses two conserved "charges" which in this representation are

$$Q \equiv \sum_j :\chi_j^\dagger \chi_j: = \sum_j Q_j = \sum_j [n_b(j) - n_d(j)] ,$$
$$\tag{2.3}$$
$$Q_5 \equiv \sum_j \chi_j^\dagger \gamma_5 \chi_j = \sum_j Q_{5_j} = \sum_j [n_b(j) + n_d(j) - 1] ,$$

where

$$n_b(j) = b_j^\dagger b_j ,$$
$$n_d(j) = d_j^\dagger d_j .$$

The Hamiltonian (2.1) is also invariant under three discrete symmetry operations. Two are unitary, and analogous to P and C, and one is antiunitary corresponding to time reversal Θ. We define them as follows:

(1) Charge conjugation:

$$C b_j C^{-1} = b_j^\dagger, \quad C Q C^{-1} = -Q ,$$
$$C d_j C^{-1} = d_j^\dagger, \quad C Q_5 C^{-1} = -Q_5 , \tag{2.4}$$
$$C^2 = 1;$$

(2) parity:

$$P b_j P^{-1} = d_{-j}^\dagger, \quad P Q P^{-1} = Q ,$$
$$P d_j P^{-1} = b_{-j}^\dagger, \quad P Q_5 P^{-1} = -Q_5 , \tag{2.5}$$
$$P^2 = 1;$$

(3) time reversal:

$$\Theta = TK,$$

where K is the antilinear operator of complex conjugation and the unitary T operator is defined by

$$T b_j T^{-1} = d_j, \quad \Theta Q \Theta^{-1} = -Q ,$$
$$T d_j T^{-1} = b_j, \quad \Theta Q_5 \Theta^{-1} = Q_5 , \tag{2.6}$$
$$\Theta^2 = 1 .$$

TABLE I. Single-site basis states.

State	Q_j	Q_{5j}
$\lvert 0_j\rangle$: $b_j \lvert 0_j\rangle = d_j \lvert 0_j\rangle = 0$	0	-1
$\lvert +_j\rangle \equiv b_j^\dagger \lvert 0_j\rangle$	$+1$	0
$\lvert -_j\rangle \equiv d_j^\dagger \lvert 0_j\rangle$	-1	0
$\lvert \pm_j\rangle \equiv b_j^\dagger d_j^\dagger \lvert 0_j\rangle$	0	$+1$

Note that this problem shares one feature in common with the Ising calculation: there are only a finite number of states associated with each lattice site. In this case for each lattice site there are four basis states, summarized in Table I.

The transformation properties of these states under P, C, and Θ are given once we adopt the conventions

$$C\lvert 0_j\rangle = i\lvert \pm_j\rangle ,$$
$$P\lvert 0_j\rangle = \lvert \pm_{-j}\rangle , \tag{2.7}$$

and

$$\Theta\lvert 0_j\rangle = \lvert 0_j\rangle .$$

Note that these conventions imply

$$P\lvert +_j\rangle = -\lvert +_{-j}\rangle ,$$
$$P\lvert -_j\rangle = \lvert -_{-j}\rangle ,$$

Also, it follows from Eq. (2.7) that

$$C\lvert +_j\rangle = i\lvert -_j\rangle .$$

In particular, fermions and antifermions have opposite intrinsic parity as usual.

Our purpose in this section is to study the free-fermion model in order to establish the notation and methods to be used in the discussion of the Thirring model. For this reason, the calculation will be done in a way which is more cumbersome than it needs to be for this simple case. Aside from matters of notation, we will illustrate the fact that although $\delta'(j_1 - j_2)$ is nonvanishing for all $j_1 \neq j_2$, it is not more difficult to work with than a gradient term expressed simply as the difference of fields at neighboring sites.

To begin our iterative procedure, we dissect the lattice into three-site blocks. We write

$$H = \sum_p H_p + \sum_{p' \neq p} H_{pp'}$$
$$= \Lambda \sum_p \left[\sum_{\alpha\alpha'} i\,\delta'(\alpha - \alpha')(b_{p\alpha}^\dagger b_{p\alpha'} - d_{p\alpha}^\dagger d_{p\alpha'}) \right]$$
$$+ \Lambda \sum_{p \neq p'} \left[\sum_{\alpha\alpha'} i\,\delta'(3(p - p') + \alpha - \alpha') \times (b_{p\alpha}^\dagger b_{p'\alpha'} - d_{p\alpha}^\dagger d_{p'\alpha'}) \right] , \tag{2.8}$$

where now p specifies the block and $\alpha=-1,\ 0,\ 1$ to specify a site within a block. Each single-block H_p operates only on that factor of the product basis which pertains to block p, and is diagonal otherwise.

We choose to use three-site rather than two-site blocks when dealing with fermions so that the lowest eigenstates of the block Hamiltonian will have the same quantum numbers for Q and Q_5 that are displayed for single sites in Table I, as will be clear immediately. Our simple algorithm for "thinning" our space of states will be first to solve the $4^3=64$ degree-of-freedom problem, which can be done very simply by grouping the states in the different (Q, Q_5) sectors which do not mix. We anticipate that the block states of lowest energy will be in the $(Q=0;\ Q_5=-1,+1)$ sectors, degenerate by C symmetry, and we name them $\lvert 0_p\rangle$ and $\lvert \pm_p\rangle$, respectively, in analogy with the single-site states of like quantum numbers. (That these are indeed the lowest block eigenstates is verified in all cases by explicit computation.) However, when we apply $H_{pp'}$ between two blocks, it is apparent that it has no matrix element between $\lvert 0_p\rangle$ and $\lvert \pm_p\rangle$; the charge-selection rule governing b_j and d_j indicates that we have to keep $(Q=+1,\ -1;\ Q_5=0)$ states in order to have nonzero block-block coupling. We call these $\lvert +_p\rangle$ and $\lvert -_p\rangle$; again, they are degenerate. Since these four sectors are the sectors of greatest multiplicity, they are generally those containing the lowest eigenlevels for the three-site problem.

To be specific, let us consider the 20 states in the $Q=0$ sector of the three-site problem. These are listed in Table II along with the states into which they transform under iCP [note $(iCP)^2=1$]. The notation used is an obvious generalization of that in Table I: the creation operators appear in an order from left to right corresponding to the ordering of the sites in the lattice, and at each site the $b_j^\dagger$ is to the left of $d_j^\dagger$; thus for the block p

$$\lvert -\pm+\rangle_p \equiv d^\dagger_{3p-1} b^\dagger_{3p} d^\dagger_{3p} b^\dagger_{3p+1} \lvert 000\rangle_p . \tag{2.9}$$

Under C, P, and Θ these states will have transformation properties derived from (2.7); viz., starting from the $(Q=0,\ Q_5=-3)$ sector,

$$\begin{aligned} C\lvert 000\rangle_p &= -i\lvert \pm\pm\pm\rangle_p , \\ P\lvert 000\rangle_p &= \lvert \pm\pm\pm\rangle_p , \\ \Theta\lvert 000\rangle_p &= \lvert 000\rangle_p . \end{aligned} \tag{2.10}$$

As a general rule, the over-all phases to be introduced in the $(Q=0,\ Q_5=\pm1)$ and $(Q=\pm1,\ Q_5=0)$ sectors as we proceed will be such that the phase convention (2.7) will hold for the lowest-lying block states (with $\lvert 0_j\rangle$ replaced by $\lvert 0_p\rangle$, etc.).

TABLE II. Three-site basis states, $Q=0$ sector.

Q_5	State	iCP transform
-3	$\lvert 000>$	$-\lvert 000>$
-1	$\lvert \pm00>$	$\lvert 00\pm>$
	$\lvert 00\pm>$	$\lvert \pm00>$
	$\lvert 0\pm0>$	$\lvert 0\pm0>$
	$\lvert +-0>$	$\lvert 0+->$
	$\lvert 0+->$	$\lvert +-0>$
	$\lvert +0->$	$\lvert +0->$
	$\lvert -+0>$	$\lvert 0-+>$
	$\lvert 0-+>$	$\lvert -+0>$
	$\lvert -0+>$	$\lvert -0+>$
$+1$	$\lvert 0\pm\pm>$	$-\lvert \pm\pm0>$
	$\lvert \pm\pm0>$	$-\lvert 0\pm\pm>$
	$\lvert \pm0\pm>$	$-\lvert \pm0\pm>$
	$\lvert -+\pm>$	$-\lvert \pm-+>$
	$\lvert \pm-+>$	$-\lvert -+\pm>$
	$\lvert -\pm+>$	$-\lvert -\pm+>$
	$\lvert +-\pm>$	$-\lvert \pm+->$
	$\lvert \pm+->$	$-\lvert +-\pm>$
	$\lvert +\pm->$	$-\lvert +\pm->$
$+3$	$\lvert \pm\pm\pm>$	$\lvert \pm\pm\pm>$

We do this because we wish to identify the new block states in these sectors with the original single-site ones so that the only thing changing in the iteration is the Hamiltonian itself.

Of the nine states in the $Q_5=-1$ sector, six combinations that are even under iCP and three that are odd can be formed. The lowest eigenstate of the three-site Hamiltonian is even and can be written as follows:

$$\begin{aligned} \lvert 0_p\rangle &= \tfrac{1}{18}(3-4i)\lvert +0-\rangle_p + \tfrac{1}{18}(3+4i)\lvert -0+\rangle_p \\ &\quad - \tfrac{1}{9}(3+i)[\,\lvert +-0\rangle + \lvert 0+-\rangle\,]_p \\ &\quad - \tfrac{1}{9}(3-i)[\,\lvert 0-+\rangle + \lvert -+0\rangle]_p \\ &\quad + \tfrac{4}{9}i[\,\lvert 0\pm0\rangle + \tfrac{5}{8}(\lvert \pm00\rangle + \lvert 00\pm\rangle)]_p , \\ H_p\lvert 0_p\rangle &= -3\Lambda\lvert 0_p\rangle . \end{aligned} \tag{2.11}$$

The other eight eigenstates in the $(Q=0,\ Q_5=-1)$ sector lie higher in energy with eigenvalues $+3\Lambda$, $+\frac{3}{2}\Lambda$, $+\frac{3}{2}\Lambda$, 0, 0, 0, $-\frac{3}{2}\Lambda$, $-\frac{3}{2}\Lambda$. Since the state having $(Q=0,\ Q_5=-3)$ is unique, it must be an eigenstate of H_p and it is clearly a null eigenstate.

The $Q=1$ sector has the 15 basis states listed in Table III along with their corresponding P transforms (C takes us from the $Q=1$ to the $Q=-1$ sector and is not important in this context).

The lowest eigenstate of H_p in the $Q=1$ sector is formed from the six odd eigenstates of P in the $Q_5=0$ sector; this corresponds to the odd parity of the single-site eigenstate according to (2.7). It is given by[8]

TABLE III. Three-site basis states, $Q=1$ sector.

Q_5	State	P transform
-2	$\|+00\rangle$	$-\|\pm\pm+\rangle$
	$\|00+\rangle$	$-\|+\pm\pm\rangle$
	$\|0+0\rangle$	$-\|\pm+\pm\rangle$
0	$\|+0\pm\rangle$	$-\|0\pm+\rangle$
	$\|0\pm+\rangle$	$-\|+0\pm\rangle$
	$\|0+\pm\rangle$	$-\|0+\pm\rangle$
	$\|+\pm0\rangle$	$-\|\pm0+\rangle$
	$\|\pm0+\rangle$	$-\|+\pm0\rangle$
	$\|\pm+0\rangle$	$-\|\pm+0\rangle$
	$\|+-+\rangle$	$-\|+-+\rangle$
	$\|++-\rangle$	$-\|-++\rangle$
	$\|-++\rangle$	$-\|++-\rangle$
2	$\|+\pm\pm\rangle$	$-\|00+\rangle$
	$\|\pm\pm+\rangle$	$-\|+00\rangle$
	$\|\pm+\pm\rangle$	$-\|0+0\rangle$

$$\begin{aligned}|+_p\rangle &= \tfrac{1}{18}(4-3i)|\pm+0\rangle_p - \tfrac{1}{18}(4+3i)|0+\pm\rangle_p \\ &\quad + \tfrac{1}{9}(1-3i)[\,|+0\pm\rangle + |0\pm+\rangle]_p \\ &\quad - \tfrac{1}{9}(1+3i)[\,|\pm0+\rangle + |+\pm0\rangle]_p \\ &\quad + \tfrac{4}{9}[\,|+-+\rangle + \tfrac{5}{8}(|++-\rangle + |-++\rangle)]_p, \\ H_p|+_p\rangle &= -3\Lambda|+_p\rangle . \end{aligned} \tag{2.12}$$

The states specified in (2.11) and (2.12) plus their two charge-conjugate counterparts $|\pm_p\rangle$ and $|-_p\rangle$ are the states used as a basis in which to construct the truncated Hamiltonian. This same process is then repeated by combining 3 neighboring three-site blocks as illustrated in Fig. 1. In order to do this, however, we must express the terms in $H_{pp'}$ (that couple different blocks to one another) in terms of creation and annihilation operators $B_p^\dagger$, B_p and $D_p^\dagger$, D_p, defined by

$$\begin{aligned} &B_p|0_p\rangle = D_p|0_p\rangle = 0 , \\ &B_p^\dagger|0_p\rangle = |+_p\rangle , \\ &D_p^\dagger|0_p\rangle = |-_p\rangle , \\ &B_p^\dagger D_p^\dagger|0_p\rangle = |\pm_p\rangle , \\ &\{B_p^\dagger, B_{p'}\} = \delta_{pp'}, \quad \text{etc.} \end{aligned} \tag{2.13}$$

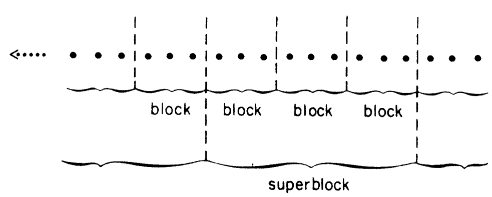

FIG. 1. Iteration of the blocking procedure.

To accomplish this let us define for site $\alpha=-1$, 0, 1 in the pth block,

$$\begin{aligned} \langle +_p|b_{3p+\alpha}^\dagger|0_p\rangle &\equiv u_\alpha , \\ \langle -_p|d_{3p+\alpha}^\dagger|0_p\rangle &\equiv v_\alpha . \end{aligned} \tag{2.14}$$

It then follows from the symmetries (2.4)–(2.6) that we can fix the remaining matrix elements: e.g., from (2.4) and (2.7) we see that

$$\begin{aligned} \langle \pm_p|b_{3p+\alpha}^\dagger|-_p\rangle &= \langle -_p|b_{3p+\alpha}|\pm_p\rangle^* \\ &= \langle +_p|Cb_{3p+\alpha}C|0_p\rangle^* \\ &= \langle +_p|b_{3p+\alpha}^\dagger|0_p\rangle^* \\ &= u_\alpha^* . \end{aligned} \tag{2.15}$$

From (2.5), we find that

$$u_\alpha = v_{-\alpha} \tag{2.16}$$

and from (2.6)

$$u_\alpha = u_{-\alpha}^* . \tag{2.17}$$

Combining these results, we see that we can write truncated operators

$$\begin{aligned} (b_{3p+\alpha}^\dagger)^{\rm TR} &\equiv B_p^\dagger(u_\alpha Q_{5p}{}^2 + u_\alpha^* Q_p{}^2) , \\ (d_{3p+\alpha}^\dagger)^{\rm TR} &\equiv D_p^\dagger(v_\alpha Q_{5p}{}^2 + v_\alpha^* Q_p{}^2) , \end{aligned} \tag{2.18}$$

where

$$Q_p \equiv B_p^\dagger B_p - D_p^\dagger D_p ,$$

$$Q_{5p} \equiv B_p^\dagger B_p + D_p^\dagger D_p - 1 .$$

Equations (2.18) summarize, in operator form, the contents of (2.13)–(2.17) and allow us to write the Hamiltonian restricted to the Hilbert space spanned by tensor products of the four basis vectors $|0_p\rangle$, $|\pm_p\rangle$, $|+_p\rangle$, and $|-_p\rangle$. Since in this subspace

$$Q_p{}^2 + Q_{5p}{}^2 = 1, \tag{2.19}$$

it follows that if u_α and v_α are real, then our truncation procedure amounts to making the simple substitutions [using (2.16) and (2.17)]

$$\begin{aligned} (b_{3p+\alpha}^\dagger)^{\rm TR} &= u_\alpha B_p^\dagger , \\ (d_{3p+\alpha}^\dagger)^{\rm TR} &= u_\alpha D_p^\dagger . \end{aligned} \tag{2.20}$$

For the present case of a free fermion model u_α and v_α are real.[9] However, this is not generally true, and the transformation (2.18) for the Thirring model takes the more general form as we shall see in the next section. Restricting to the free-field case and collecting our results, the truncated Hamiltonian (2.1) becomes

$$H_{(1)}^{\mathrm{TR}}=-(3\Lambda)\frac{\Lambda L}{3}+\sum_{p\neq q} i\left(\sum_{\alpha,\beta=-1}^{+1}\delta'(3(p-q)+\alpha-\beta)u_\alpha u_\beta\right)\times(B_p^\dagger B_q-D_p^\dagger D_q)\,, \tag{2.21}$$

where $H_{(1)}^{\mathrm{TR}}$ denotes the Hamiltonian obtained after the first iteration. Proceeding to the second iteration, we couple three adjacent "sites" of the new Hamiltonian (or blocks of the original Hamiltonian) and again retain the lowest eigenstate in each of the four sectors ($Q=\pm 1$, $Q_5=0$) and ($Q=0$, $Q_5=\pm 1$). Of course, this time the gradient is given in terms of a new function

$$\Delta_1(p-q)\equiv\sum_{\alpha,\beta=-1}^{+1}\delta'(3(p-q)+\alpha-\beta)u_\alpha u_\beta \tag{2.22}$$

which is readily calculated using (2.11)–(2.14) and the definition (2.1) of $\delta'(j)$.

Table IV compares the first few terms of $\delta'(j)$ to $\Delta_1(j)$. It is evident from the table that in a single iteration, the strength of the gradient is decreased by almost a factor of 2 and the second nearest-neighbor term is suppressed by $\approx 30\%$ relative to the nearest-neighbor one. Continuing this process, we find that, aside from the fact that the gradient's over-all scale keeps dropping by a factor $\approx\frac{8}{15}$, $\Delta_n(j)$ rapidly converges to a fixed form. One can approximately carry out the iteration procedure analytically by setting $\Delta_n(2)/\Delta_n(1)=-\frac{1}{2}=\delta'(2)/\delta'(1)$, in order to estimate the ground-state energy. One obtains

$$E\approx-3\Lambda\times\frac{\Lambda L}{3}-3\Lambda\left(\frac{8}{15}\right)\frac{\Lambda L}{3^2}-3\Lambda\left(\frac{8}{15}\right)^2\frac{\Lambda L}{3^3}+\cdots$$

$$=-\Lambda^2L\left(\tfrac{45}{37}\right)\simeq-1.216\Lambda^2L\;. \tag{2.23}$$

Carrying out the calculation exactly by means of a computer yields an almost identical result

$$E_0^{\mathrm{TR}}=-1.217\Lambda^2L\;. \tag{2.24}$$

The exact ground-state energy can, of course, be computed very simply from (2.1) by transforming to momentum space. This leads to the familiar expression

$$H=\sum_{k=-\pi\Lambda}^{+\pi\Lambda}k[n_b(k)-n_d(k)]\;, \tag{2.25}$$

where $n_b(k)$ and $n_d(k)$ are the number of particles and antiparticles, respectively, with momentum k. The ground state corresponds to filling all $k>0$ states with d quanta, $n_d(k)=\theta(+k)$, and all $k<0$ states with b quanta, $n_b(k)=\theta(-k)$. Actually the ground state is fourfold degenerate corresponding to whether $n_b(0)=0,1$ and $n_d(0)=0,1$ in the $k=0$ state (N.B., in the infinite volume limit it becomes infinitely degenerate). It follows that the ground-state energy for (2.25) is

$$E_0=-2\sum_0^{\pi\Lambda}k=-2\frac{L}{2\pi}\int_0^{\pi\Lambda}kdk=-\tfrac{1}{2}\pi\Lambda^2L\;. \tag{2.26}$$

Hence our simple truncation procedure leads to a ground-state energy that lies 22% above the exact answer. There are two ways in which this agreement can be improved. One is by keeping a larger subspace of states than the 4 of the 64 retained in the above calculation for each three-site block. This has proved very effective in the study of the Ising model. Alternatively we can determine which states to retain by a variational procedure as illustrated in Paper III.[10]

TABLE IV. A comparison of the free-field gradient functions before (δ') and after (Δ_1) the first iteration.

j	$\delta'(j)$	$\Delta_1(j)$
0	0	0
1	1	$\frac{8}{15}$
2	$-\frac{1}{2}$	$-\frac{4}{15}(1-\frac{9}{28})$

III. LATTICE THIRRING MODEL

Turning now to the Thirring model, we add a chirally invariant "potential term" to the Hamiltonian of the free-fermion theory. The lattice Hamiltonian is [see Eq. (2.15) and Sec. III A of Paper II]

$$H=\Lambda\left(\sum_{j_1,j_2}i\delta'(j_1-j_2)\chi_{j_1}^\dagger\alpha\chi_{j_2}-\frac{g}{2}\sum_j[(\bar\chi_j\chi_j)^2-(\bar\chi_j\gamma_5\chi_j)^2]\right)$$

$$=\Lambda\left(\sum_{j_1,j_2}i\delta'(j_1-j_2)(b_{j_1}^\dagger b_{j_2}-d_{j_1}^\dagger d_{j_2})-g\sum_j Q_{5j}{}^2\right)\,, \tag{3.1}$$

where Q_{5j} is defined by (2.3). Since the added potential term commutes with Q and Q_5, we again choose the site basis in Table I used for the massless free-fermion theory. The discrete symmetries C, P, and Θ of (2.4)–(2.7) are again useful for classifying states.

Proceeding as in Sec. II, we divide the lattice into blocks of three sites, and rewrite the Hamiltonian divided into single-block and block-block terms

$$H=\sum_p H_p+\sum_{p\neq p'} H_{pp'}$$
$$=\Lambda\sum_p\left[\sum_\alpha(-g)Q_{5p\alpha}{}^2+\sum_{\alpha\alpha'} i\delta'(\alpha-\alpha')(b^\dagger_{p\alpha}b_{p\alpha'}-d^\dagger_{p\alpha}d_{p\alpha'})\right]$$
$$+\Lambda\sum_{p\neq p'}\left(\sum_{\alpha\alpha'} i\delta'(3(p-p')+\alpha-\alpha')\times(b^\dagger_{p\alpha}b_{p'\alpha'}-d^\dagger_{p\alpha}d_{p'\alpha'})\right). \quad (3.2)$$

The plan is again to diagonalize the single-block terms exactly, to truncate the Hilbert space to the lowest few eigenstates, and to rewrite the block-block coupling term in terms of operators which are confined to the truncated space.

The enumeration of block states and the block diagonalization of H_p according to Q and Q_5 go through as for the free field, and once more we choose the lowest eigenstates $|0_p\rangle$, $|+_p\rangle$, $|-_p\rangle$, and $|\pm_p\rangle$ and define B_p and D_p.

Equations (2.16) and (2.17) still hold, because they depend only on the same symmetries (2.4)–(2.6). Furthermore, (2.18) is unchanged; indeed, when we truncate our block Hilbert space, that expression for $(b^\dagger)^{\rm TR}$ is the most general operator on the subspace with selection rules $\Delta Q=+1$, $\Delta Q_5=+1$; and analogously for $(d^\dagger)^{\rm TR}$. However, we can no longer write (2.20), because u_j is no longer real. The reality condition depended on the symmetry of H under $Q\longleftrightarrow Q_5$, which does not hold for $g\neq 0$ in (3.1). Unlike the free-field case, now

$$\langle +_p|b^\dagger_j|0_p\rangle\neq\langle\pm_p|b^\dagger_j|-_p\rangle .$$

Thus (2.18) no longer turns into a simple scaling transformation for $b^\dagger$ and $d^\dagger$. Instead it expresses a nonlinear Bogoliubov transformation which forces these operators to take on their most general forms in the next iteration.

The single-site term H_p in (3.2) becomes the sum of two terms: a constant representing an over-all energy shift and a term proportional to $(Q_{5p})^2$ which splits the degenerate pairs $|+_p\rangle$ and $|-_p\rangle$ from $|0_p\rangle$ and $|\pm_p\rangle$; viz.,

$$\frac{(H_p)^{\rm TR}_{(1)}}{\Lambda}=E_1-g_1Q_{5p}{}^2 . \quad (3.3)$$

This is its most general form consistent with the symmetries. Correspondingly, the kinetic Hamiltonian turns into

$$\frac{(H_{p,p'})^{\rm TR}_{(1)}}{\Lambda}=B^\dagger_p[A(p-p')Q_{5p}{}^2Q_{5p'}{}^2+B(p-p')Q_p{}^2Q_{5p'}{}^2-B^*(p-p')Q_{5p}{}^2Q_{p'}{}^2-A^*(p-p')Q_p{}^2Q_{p'}{}^2]B_{p'}$$
$$-D^\dagger_p[-A^*(p-p')Q_{5p}{}^2Q_{5p'}{}^2+B(p-p')Q_p{}^2Q_{5p'}{}^2-B^*(p-p')Q_{5p}{}^2Q_{p'}{}^2+A(p-p')Q_p{}^2Q_{p'}{}^2]D_{p'} , \quad (3.4)$$

where

$$A(p-p')=\sum_{\alpha\alpha'} i\delta'(3(p-p')+\alpha-\alpha')u_\alpha u^*_{\alpha'} ,$$
$$B(p-p')=\sum_{\alpha\alpha'} i\delta'(3(p-p')+\alpha-\alpha')u^*_\alpha u^*_{\alpha'} . \quad (3.5)$$

Note that if $u_\alpha=u^*_\alpha$, then $A=B$ and (3.4) reduces to (2.21) when we use (2.19). Equation (3.4) is in fact the most general site-site (or block-block) coupling Hamiltonian which conserves Q, Q_5, C, and P, for any functions A and B subject to

$$A(-p)=A^*(p),\quad B(-p)=-B(p) . \quad (3.6)$$

We are now faced with the prospect of iterating (that is, blocking and truncating) this operator as a kinetic-energy term in the Hamiltonian. It is clear that the generality of (3.4) precludes any further change in form in the truncated subspace to which we have restricted ourselves.

In the second iteration, we treat our former blocks as sites and group them into (super) blocks as in the free-field theory. After diagonalizing, we will need matrix elements such as (letting α index a "site" within "block" p)

$$\langle +_p|B^\dagger_{3p+\alpha}Q_{3p+\alpha}{}^2|0_p\rangle\equiv r_\alpha$$
and
$$\langle +_p|B^\dagger_{3p+\alpha}Q_{5,3p+\alpha}{}^2|0_p\rangle\equiv w_\alpha . \quad (3.7)$$

Indeed, use of C and P symmetries shows these to be the only independent matrix elements. Then the newly truncated $H_{pp'}$ takes the form (3.4) with new $A(p)$ and $B(p)$ given by a generalization of (3.5):

$$A(p-p')\to\sum_{\alpha\alpha'}[A(x)r_\alpha r^*_{\alpha'}-A^*(x)w_\alpha w^*_{\alpha'}+B(x)w_\alpha r^*_{\alpha'}-B^*(x)r_\alpha w^*_{\alpha'}] ,$$
$$B(p-p')\to\sum_{\alpha\alpha'}[B(x)r^*_\alpha r^*_{\alpha'}-B^*(x)w^*_\alpha w^*_{\alpha'}+A(x)w^*_\alpha r^*_{\alpha'}-A^*(x)r^*_\alpha w^*_{\alpha'}] , \quad (3.8)$$

where

$$x=3(p-p')+\alpha-\alpha' .$$

This procedure can in fact be specialized to the first iteration if we start with $A(j)=B(j)=i\delta'(j)$.

It is convenient, in order to make later comparison with the free-field case more transparent, to rewrite $H_{(n)}$ (defined as $H^{\mathrm{TR}}_{(n)}/\Lambda$ in the nth iteration) as

$$H_{(n)}=\sum_{p_1,p_2} iX_n(p_1-p_2)(B^\dagger_{p_1}B_{p_2}-D^\dagger_{p_1}D_{p_2})-g_n\sum_p (B^\dagger_p B_p+D^\dagger_p D_p-1)^2+E_n$$
$$+\sum_{p_1,p_2} i[Z_n(p_1-p_2)(B^\dagger_{p_1}Q_{p_1}{}^2Q_{p_2}{}^2B_{p_2}-D^\dagger_{p_1}Q_{5p_1}{}^2Q_{5p_2}{}^2D_{p_2})+Z_n^*(p_1-p_2)(B^\dagger_{p_1}Q_{5p_1}{}^2Q_{5p_2}{}^2B_{p_2}-D^\dagger_{p_1}Q_{p_1}{}^2Q_{p_2}{}^2D_{p_2})] , \tag{3.9}$$

where $X_n(p)$ and $Z_n(p)$ are the appropriate linear combinations of A's and B's: viz., $iX_n(p)\equiv B_n(p)$ and $iZ_n(p)=B_n^*(p)-A_n^*(p)$.

In this way our Hamiltonian (3.2), with generalized kinetic term (3.4), takes on the same form when blocked and truncated, subject to an energy shift and a change in the value of g and in the form of "gradient functions" $A(j)$ and $B(j)$. We iterate this procedure until one of three limits is reached:

(a) X_n and $Z_n\to\infty$ relative to g_n. The resulting gradient functions must then be compared with those resulting from the free field (where $g=0$ to begin with). More on this later.

(b) X_n and $Z_n\to 0$ relative to g_n. This, the "strong-coupling limit,"[11] is the Heisenberg antiferromagnet as discussed in Paper II, and the limiting form of the gradient functions before they go to zero must be examined to determine the degeneracy breaking of the ground state.

(c) X_n and Z_n approach a fixed form, and a fixed ratio with respect to g_n. In this case the dynamics at large scales remain complicated.

We have thus far applied our procedure of successively truncating the Hilbert space to compute the general form of the Hamiltonian. We may also apply this procedure to any other operator by computing its matrix elements between the states retained in each step of truncation. An example which will be useful for calculating the Schwinger term in the Appendix is the ground-state expectation value of the potential energy density [in units of $\Lambda=(2N+1)/L$]

$$V=\frac{1}{2N+1}\sum_j -gQ_{5j}{}^2$$
$$=\frac{3}{2N+1}\sum_p\left(\frac{1}{3}\sum_{\alpha=-1,0,1}(-g)Q_{5,3p+\alpha}{}^2\right)$$
$$\equiv\frac{3}{2N+1}\sum_p V_p . \tag{3.10}$$

Noting that $Q_{5j}{}^2$ is diagonal in our truncated basis and that it commutes with C, we let, in a three-site block basis,

$$a+b=-\frac{g}{3}\sum_\alpha\langle 0_p|Q_{5,3p+\alpha}{}^2|0_p\rangle$$
$$=\langle 0_p|V_p|0_p\rangle=\langle\pm_p|V_p|\pm_p\rangle ,$$
$$a=-\frac{g}{3}\sum_\alpha\langle +_p|Q_{5,3p+\alpha}{}^2|+_p\rangle=\langle +_p|V_p|+_p\rangle$$
$$=\langle -_p|V|-_p\rangle . \tag{3.11}$$

Then the matrix elements of V_p in the truncated basis are summarized by writing

$$V_p=a+bQ_{5p}{}^2 .$$

Since $\frac{1}{3}(2N+1)$ is just N_b, the number of blocks,

$$V=a+\frac{1}{N_b}\sum_p bQ_{5p}{}^2 . \tag{3.12}$$

This procedure may be repeated through following blockings, and the recursion formula is

$$a\to a+\frac{b}{3}\sum_\alpha\langle +_p|Q_{5,3p+\alpha}{}^2|+_p\rangle ,$$
$$b\to\frac{b}{3}\sum_\alpha[\langle 0_p|Q_{5,3p+\alpha}{}^2|0_p\rangle-\langle +_p|Q_{5,3p+\alpha}{}^2|+_p\rangle] ,$$
$$N_b\to\tfrac{1}{3}N_b. \tag{3.13}$$

Initial conditions are $a=0$, $b=-g$.

Thus we may calculate $\langle V\rangle$ in any of our final variational states. Since we already know what the Hamiltonian is doing in the iteration, we also determine the kinetic-energy density by this process. This will be of interest for determining the theory's Schwinger term.

In order to renormalize the field, we will need its amplitude to create a ($Q=1$, $Q_5=0$) state at rest from the vacuum, i.e.,

$$\sqrt{\Lambda}\,\langle 0|\chi(k=0)|+\rangle \equiv \left(\frac{\Lambda}{2N+1}\right)^{1/2}\left\langle 0\left|\sum_j \chi_j\right|+\right\rangle$$

$$\equiv \sqrt{\Lambda}\, Z_2{}^{1/2}\begin{pmatrix}1\\0\end{pmatrix} \quad . \tag{3.14}$$

The operator of interest in (3.14) is

$$\frac{1}{(2N+1)^{1/2}}\sum_j b_j = \left(\frac{3}{2N+1}\right)^{1/2}\sum_p\left(\frac{1}{\sqrt{3}}\sum_\alpha b_{3p+\alpha}\right) \tag{3.15}$$

after one blocking. The symmetries of the theory imply that in the truncated basis

$$\frac{1}{\sqrt{3}}\sum_\alpha (b_{3p+\alpha})^{\mathrm{TR}} = \zeta_1 B_p \ . \tag{3.16}$$

After n iterations, with $N_b = (2N+1)/3^n$, we have

$$\frac{1}{(2N+1)^{1/2}}\left(\sum_j b_j\right)^{\mathrm{TR}} = \frac{1}{\sqrt{N_b}}\,\zeta^{(n)}\sum_p B_p \ , \tag{3.17}$$

where

$$\zeta^{(n)} = \prod_{i=1}^{n} \zeta_i \ .$$

At this stage there is no order in the system on a scale greater than 3^n sites, so that the zero-momentum state is just

$$|+\rangle = \frac{1}{\sqrt{N_b}}\sum_p\left(|+_p\rangle\prod_{q\neq p}|0_q\rangle\right). \tag{3.18}$$

Therefore

$$\frac{1}{(2N+1)^{1/2}}\sum_j \langle 0|b_j|+\rangle = \frac{1}{N_b}\,\zeta^{(n)}\sum_p \langle 0_p|B_p|+_p\rangle$$

$$= \zeta^{(n)} \ . \tag{3.19}$$

Thus

$$Z_2{}^{1/2} = \lim_{n\to\infty}\zeta^{(n)} = \prod_{i=1}^{\infty}\zeta_i \ . \tag{3.20}$$

Using (3.16) and (3.7), we find that in iteration i

$$\zeta_i = \frac{1}{\sqrt{3}}\sum_{\alpha=-1}^{1}\left(r_\alpha^{(i)} + w_\alpha^{(i)}\right) \ . \tag{3.21}$$

IV. CONTINUUM THIRRING MODEL

Before discussing results obtained from our renormalization-group procedure, we review known features of the continuum solution with which we are interested in making contact. To begin, we recall that operator solutions to the Thirring model have been constructed in two ways. In the first,[3] the starting point is the formal equations of motion

$$i\partial\!\!\!/\,\psi = g_r\, j\!\!\!/\,\psi \ , \tag{4.1}$$

$$\partial^\mu j_\mu(x) = \partial^\mu(\bar\psi\gamma_\mu\psi) = 0 \ , \tag{4.2}$$

and

$$\partial^\mu \epsilon_{\mu\nu}\, j^\nu(x) = \partial^\mu j_{5\mu}(x) = 0 \ . \tag{4.3}$$

We refer to these as formal equations since if one begins with the fields $\psi(x)$ as fundamental, the operators $j_\mu(x)$ and $(j\!\!\!/\psi)(x)$ are singular. To give them meaning, one must adopt some prescription for rendering them finite. Johnson does this by introducing a point-separation prescription

$$j^\mu(x) = \lim_{\epsilon\to 0}\tfrac{1}{2}\langle[\bar\psi(x+\epsilon),\gamma^\mu\psi(x)]\rangle \ , \tag{4.4}$$

where the angular brackets indicate a suitable averaging process over directions of ϵ. Once both sides of Eqs. (4.1)–(4.3) are well defined, it is possible to compute all finite Green's functions.

A second procedure[12] is to take the currents and their commutators as fundamental, and from these reconstruct the Thirring model and the fermion field. According to this approach, the theory is defined in terms of currents j_μ and $j_{5\mu}$ where $j_\mu(x)$ is assumed to satisfy the commutators

$$[j_0(x), j_0(y)]_{\mathrm{ET}} = 0 \ , \tag{4.5}$$

$$[j_0(x), j_1(y)]_{\mathrm{ET}} = ic\,\delta'(x-y) \ , \tag{4.6}$$

$$[j_1(x), j_1(y)]_{\mathrm{ET}} = 0 \tag{4.7}$$

(where the subscript ET denotes equal time) and also the conservation laws

$$\partial^\mu j_\mu(x) = \partial^\mu\epsilon_{\mu\nu}\, j^\nu(x) = 0 \ . \tag{4.8}$$

The fermionic degrees of freedom are introduced, after finding general solutions of the problem specified by (4.5)–(4.8), by requiring that there exist a field $\psi(x)$ such that

$$\begin{aligned}&[j_0(x), \psi(y)]_{\mathrm{ET}} = -a\psi(x)\delta(x-y) \ ,\\ &[j_1(x), \psi(y)]_{\mathrm{ET}} = -\bar a\psi(x)\delta(x-y) \ .\end{aligned} \tag{4.9}$$

A Hamiltonian is then constructed in terms of currents alone in such a way that $\psi(x)$ satisfies the equation

$$i\partial\!\!\!/\,\psi(x) = g_r :j\!\!\!/(x)\psi(x): \ , \tag{4.10}$$

where : : indicates an appropriately defined normal-ordering prescription for the composite operator.

In either approach, if one studies the full operator solution of the theory, one learns that only two of the four parameters $(a, \bar a, g_r, c)$ are independent and that the resulting theory has no mass gap. Moreover, the wave-function renormalization $Z_2(g_r)$ is found to vanish at a finite value $g_r = g_{\mathrm{cr}}$ if one adopts Johnson's specific point-separation procedure. The parameter c in (4.6) is singular at g_{cr} as $1/[1-(g_r/g_{\mathrm{cr}})^2]$, and is negative for

$g_r > g_{cr}$. Thus the Hamiltonian, which when written in terms of light-cone components of the Bose field $j_\mu(x)$ is[4]

$$H = \frac{1}{4c}\left(\int :j_+^2(u): du + \int :j_-^2(v): dv \right), \qquad (4.11)$$

cannot be constructed past this point.

V. CALCULATION

In this section, we present the calculational details necessary to understand how we numerically carry out and interpret the results of our iterative "renormalization-group" procedure for the lattice Thirring model. We find that the picture emerging from our truncation procedure and the resulting equations (3.3)–(3.9) is consistent with the continuum model in that there is a finite critical value $g_{cr} = 1.1087$ such for $g < g_{cr}$ the theory has no mass gap; for $g > g_{cr}$ the cut-off lattice theory cannot be multiplicatively renormalized in the usual fashion. The lattice theory still exists for $g > g_{cr}$; in fact, for this region its behavior seems entirely sensible and is precisely as described in Paper II–i.e., for $g > g_{cr}$ we are driven to the strong-coupling limit which corresponds to a Heisenberg antiferromagnetic chain *with nearest-neighbor interactions*. As was discussed in Paper II, this theory possesses a massless excitation spectrum as first proved 46 years ago by Bethe.[13] For $g > g_{cr}$, however, the single-particle operator $(1/\sqrt{L}) \int dx \psi^\dagger(x)$ fails to create any finite-energy states from the vacuum. In fact, the excitations of unit charge are found to lie an energy $\sim g\Lambda$ above the ground state. This result shows that for a certain region of the parameter g the particles and low-lying excitation spectra found in finite cutoff lattice theories are not simply related to the fundamental field introduced in the starting Lagrangian and Hamiltonian.

In order to describe more fully how we carry out the numerical solution, we recall that the generic form of the Hamiltonian obtained after n iterations will be as given in Eq. (3.9). The problem of storing and recomputing the infinite arrays $X_n(j)$ and $Z_n(j)$ is handled in our numerical iteration procedure by explicitly computing and storing the values of $X_n(j)$ and $Z_n(j)$ for $j = 1, \ldots, 100$. For $j > 100$ we parametrize $X_n(j)$ and $A_n(j)$ by

$$X_n^{as}(j) = (-1)^j \left(\frac{(A1)_n}{j} + \frac{(A2)_n}{j^3} \right), \qquad (5.1)$$

$$Z_n^{as}(j) = (-1)^j \left(\frac{(B1)_n}{j} + \frac{(B2)_n}{j^3} + \frac{i(C)_n}{j^2} \right), \qquad (5.2)$$

where as stands for asymptotic fit to large j, and compute the real coefficients $(A1)_n$, $(A2)_n$, $(B1)_n$, $(B2)_n$, and $(C)_n$.

Tables V through VII show the results of such a calculation for typical values of $g < g_{cr}$ and $g > g_{cr}$. The meaning of the various columns is as follows:

(i) The number of iterations performed.

(ii) The value of $X_n(1)$ since it proves convenient for numerical reasons to redefine $H_n = X_n(1) \times \mathcal{H}_n$, where the shortest range part of the gradient term in $\mathcal{H}_n$ is chosen to be normalized to the starting function $\delta'(j)$.

(iii) g_{eff} is the value of $g_n / |X_n(1)|$ and so, for example, $g_{eff} \to 0$ implies that up to an over-all scale factor the theory is being driven to a theory with no quartic single-site interactions.

(iv) $X_{norm}(j)$ gives the first five values of

$$(X_n(j) \times [(A1)_n + (A2)_n]) / (X_n(1) \times X_n^{as}(j)). \qquad (5.3)$$

The values of $Z_n(j)$ stay small for all initial values of g and iterate to zero relative to $X_n(j)$ so that we do not bother to display them here.

(v) $(A1)_n/X_n(1)$, $(A2)_n/X_n(1)$, etc., are self-explanatory.

We see that if g_{eff} and $B1/X_n(1)$, $B2/X_n(1)$, and $C/X_n(1)$ go to zero as the number of iterations increases and if only $X_{norm}(j)$ takes a fixed nonzero form, then the Hamiltonian becomes a purely quadratic operator which can be diagonalized simply. Tables V–VII show that this is what happens for $g < g_{cr}$. On the other hand, for $g > g_{cr}$ we see that g_{eff} grows; in fact, $g_{eff} \times X_n(1)$ approaches a finite value. This means that the quartic terms of the theory dominate the gradient. Moreover, according to Table VII, only the first (i.e., nearest neighbor) part of $X_{norm}(j)$ survives an infinite number of iterations. Therefore, the Hamiltonian $\mathcal{H}_n \equiv H_n/X_n(1)$ is seen to approach the strong-coupling nearest-neighbor Thirring model—which was shown in Paper II to be equivalent to the Heisenberg antiferromagnet as studied by Bethe. The results shown are typical of the behavior for all values of $g > g_{cr}$. Since, as is shown in Table VI, the fixed form of $X_{norm}(j)$ is the same for all $g < g_{cr}$ and is, in fact, the same as for the free-field case $g = 0$, it is evident that, for $g < g_{cr}$, the large-distance behavior of the Thirring model will be that of a massless free-fermion theory. This is, of course, consistent with what is known about the continuum model.

In order to understand what the change in the behavior of the iterative solution at the point $g = g_{cr}$ signifies, we consider the wave-function renormalization constant $Z_2(g)$. Since the spectrum for $g < g_{cr}$ is always a free massless spectrum, there is no scale in the theory which is set by multiplicatively renormalizing $\psi(x)$ and so there is a great deal of arbitrariness in the definition of $Z_2(g)$. We therefore consider the ratio

TABLE V. Computational results for $g=0$ (free field).

Iteration (n)	$X_n(1)$	g_{eff}	$X_{\text{norm}}(j=1,\ldots,5)$	$(A1)_n/X_n(1)$	$(A2)_n/X_n(1)$	$(B1)_n/X_n(1)$	$(B2)_n/X_n(1)$	$C_n/X_n(1)$
1	−0.533 33	0	1.000 00 0.818 97 0.811 87 0.810 71 0.810 40	−0.625	−0.185 19	0	0	0
2	−0.313 76	0	1.000 00 0.608 75 0.602 04 0.601 00 0.600 72	−0.461 76	−0.138 78	0	0	0
3	−0.188 05	0	1.000 00 0.496 14 0.491 03 0.490 24 0.490 03	−0.380 47	−0.109 42	0	0	0
15	-2.9897×10^{-4}	0	1.000 00 0.312 56 0.309 77 0.309 34 0.309 22	−0.243 86	−0.065 28	0	0	0
20	-1.9376×10^{-5}	0	1.000 00 0.309 31 0.306 55 0.306 12 0.306 01	−0.241 38	−0.064 55	0	0	0

TABLE VI. Computational results for $g=0.1<g_{\text{cr}}$.

Iteration (n)	$X_n(1)$	g_{eff}	$X_{\text{norm}}(j=1,\ldots,5)$	$(A1)_n/X_n(1)$	$(A2)_n/X_n(1)$	$(B1)_n/X_n(1)$	$(B2)_n/X_n(1)$	$C_n/X_n(1)$
1	−0.532 65	0.076 589	1.000 00 0.818 96 0.811 87 0.810 72 0.810 40	−0.624 98	−0.185 21	0	2.7059×10^{-5}	−0.004 112 3
2	−0.313 13	0.058 668	1.000 00 0.608 72 0.602 01 0.600 98 0.600 70	−0.461 73	−0.138 78	0	1.3216×10^{-5}	−0.002 470 3
3	−0.187 60	0.046 113	1.000 00 0.496 11 0.491 00 0.490 21 0.490 01	−0.380 44	−0.109 42	0	4.5477×10^{-6}	−0.001 315 4
15	-2.9803×10^{-4}	0.005 397	1.000 00 0.312 56 0.309 77 0.309 34 0.309 22	−0.243 86	−0.065 28	0	8.1691×10^{-9}	-4.4633×10^{-5}
20	-1.9315×10^{-5}	0.002 389	1.000 00 0.309 31 0.306 55 0.306 12 0.306 01	−0.241 38	−0.064 55	0	1.5335×10^{-9}	-1.9239×10^{-5}

TABLE VII. Computational results for $g=2>g_{cr}$.

Iteration (n)	$X_n(1)$	g_{eff}	$X_{norm}(j=1,\ldots,5)$	$(A1)_n/X_n(1)$	$(A2)_n/X_n(1)$	$(B1)_n/X_n(1)$	$(B2)_n/X_n(1)$	$C_n/X_n(1)$
1	−0.374 52	2.914 7	1.000 00 0.817 34 0.811 11 0.810 34 0.810 22	−0.617 55	−0.192 82	0	0.007 23	−0.066 80
2	−0.128 98	5.488 5	1.000 00 0.589 01 0.583 19 0.582 55 0.582 44	−0.439 35	−0.143 30	0	0.009 09	−0.063 21
3	−0.032 991	16.712	1.000 00 0.442 20 0.438 14 0.437 78 0.437 78	−0.329 30	−0.108 71	0	0.010 48	−0.058 74
10	-1.6136×10^{-9}	-3.0816×10^{8}	1.000 00 0.001 19 0.001 20 0.001 23 0.001 24	-6.2370×10^{-4}	-6.4155×10^{-4}	0	2.3514×10^{-4}	3.8297×10^{-4}
15	4.1286×10^{-15}	$1.2044 \times 10^{+14}$	1.000 00 -2.0661×10^{-5} -2.7470×10^{-5} -3.0416×10^{-5} -3.1892×10^{-5}	1.6849×10^{-5}	1.7855×10^{-5}	0	-3.5695×10^{-5}	1.8895×10^{-7}

$$\left(\frac{Z_2(g)}{Z_2(0)}\right)^{1/2} \equiv \frac{\langle + \,|\int dx\psi^\dagger(x)|0\rangle_g}{\langle + \,|\int dx\psi^\dagger(x)|0\rangle_{g=0}} \tag{5.4}$$

with $Z_2(g)$ defined in (3.14) with $\psi=\sqrt{\Lambda}\,\chi$.

The result of our renormalization-group calculation of this ratio is shown in Fig. 2. Note that it vanishes for $g \geq g_{cr}$, the point dividing the two regions in which the Hamiltonian H_n iterates to very different forms. When $g<g_{cr}$, H_n converges to the same fixed form that the free $g=0$ Hamiltonian iterates to, whereas for $g>g_{cr}$ it iterates to the very different form in which the charge fermions move high up to mass $\sim g\Lambda$ and fail to propagate. This breakdown of multiplicative renormalization and the concomitant loss of the charged degrees of freedom from the finite mass spectrum is interesting in that it seems to occur in several types of lattice theories for appropriate values of coupling constant. Presumably if there is a qualitative difference between non-Abelian gauge theories, to which we look for an explanation of quark confinement, and these nongauge models, it will be that for the gauge theories whenever $g\neq 0$ we will lose the simple multiplicative renormalization procedure and, with it, propagating free fermion (quark) states.[14]

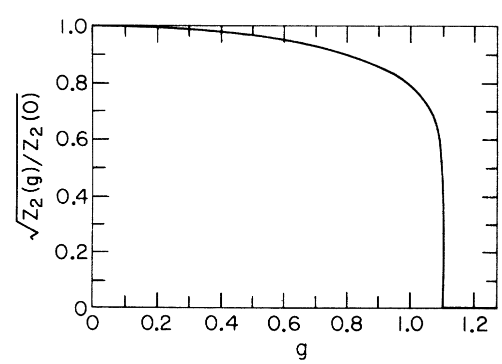

FIG. 2. Wave-function renormalization vs coupling constant g in the lattice Thirring model.

CONCLUSION

We have demonstrated that our iterative procedure of constructing an effective Hamiltonian on the lattice can be readily applied to fermion problems and is successful in reproducing known results for the Thirring model.

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APPENDIX: THE SCHWINGER TERM

In the preceding sections we described a technique for calculating the properties of a lattice version of the Thirring model and showed that at least in its most important features it corresponded to the continuum theory. If we wanted to construct a more complete correspondence with the continuum theory, we would have to give expressions for the operators $j_\mu(x)$ on a lattice with finite parts that converge to their continuum counterparts. The next step would be to fix the normalization of these operators by computing their matrix elements between an appropriately chosen set of low-lying states. As an illustration of such a general program we will discuss here the Schwinger term for this model. The principal purpose of this discussion is (i) to show how to define continuum operators "$j_\mu(x)$" for finite cutoff Λ and (ii) to show how to compute matrix elements of this operator and commutators of $j_0(x)$ with $j_1(x)$ between low-lying states.

To begin, we observe that our goal is to define a current $j_\mu(x)$ satisfying

$$\partial j_\mu/\partial x_\mu = 0 \tag{A1}$$

and to evaluate the commutator $[j_0(x,t), j_1(y,t)]$ which according to (4.6) must be nonvanishing. We can directly construct a conserved current $\rho(x)\equiv j_0(x)$ and $j_1(x)$ but it turns out technically to be simpler to use Eq. (A1) and

$$\mathring{\rho}(x) = i[H, \rho(x)] \tag{A2}$$

to rewrite (4.6)

$$[\rho(x), \mathring{\rho}(y)]_{\mathrm{ET}} = i[\rho(x), [H, \rho(y)]] = -ic\delta''(x-y)\ . \tag{A3}$$

There is no unique prescription for constructing a local charge density on the lattice. We adopt the straightforward procedure of defining the continuum field[15]

$$\begin{aligned}\psi(x) &= \sqrt{\Lambda}\, e^{iPx}\chi(j=0)e^{-iPx}\\ &= \left(\frac{\Lambda}{2N+1}\right)^{1/2}\sum_{k=-\pi\Lambda}^{\pi\Lambda}\chi(k)e^{ikx}\\ &= \sqrt{\Lambda}\sum_{j=-N}^{N}\chi_j\left(\sum_k \frac{e^{ik(x-j/\Lambda)}}{2N+1}\right)\\ &= \sqrt{\Lambda}\sum_j \chi_j \frac{1}{2N+1}\left(\frac{\sin\pi(\Lambda x-j)}{\sin[\pi(\Lambda x-j)/(2N+1)]}\right)\\ &\xrightarrow[N\to\infty]{} \sqrt{\Lambda}\sum_j \chi_j\frac{\sin\pi(\Lambda x-j)}{\pi(\Lambda x-j)}\end{aligned} \tag{A4}$$

and introducing

$$\begin{aligned}\rho_\Lambda(x) &= \psi^\dagger(x)\psi(x)\\ &= \Lambda\sum_{j,j'}\chi_j^\dagger\chi_{j'}\left(\frac{\sin\pi(\Lambda x-j)}{\pi(\Lambda x-j)}\right)\left(\frac{\sin\pi(\Lambda x-j')}{\pi(\Lambda x-j')}\right)\end{aligned} \tag{A5}$$

Next we construct $\partial\rho(x)/\partial t$ from the equation of motion

$$\begin{aligned}\mathring{\chi}_j &= i[H,\chi_j]\\ &= -i\Lambda\left(\sum_{j'}\{i\delta'(j-j')\alpha\chi_{j'} - g[\gamma_0\chi_j(\chi_j^\dagger\gamma_0\chi_j) + (\chi_j^\dagger\gamma_0\chi_j)\gamma_0\chi_j]\}\right)\ .\end{aligned} \tag{A6}$$

Using (A5), we find

$$\begin{aligned}\frac{d\rho_\Lambda}{dt} = -i\Lambda^2 &\sum_{j,j'}\left(\frac{\sin\pi(\Lambda x-j)}{\pi(\Lambda x-j)}\right)\left(\frac{\sin\pi(\Lambda x-j')}{\pi(\Lambda x-j')}\right)\\ &\times\left[\chi_j^\dagger\,\alpha\sum_{j''} i\delta'(j'-j'')\chi_{j''} + \sum_{j''}\chi_{j''}^\dagger\,\alpha i\delta'(j-j'')\chi_{j'} - 2g(\bar\chi_{j'}\chi_j)(\bar\chi_j\chi_j) + 2g(\bar\chi_j\chi_j)(\bar\chi_j\chi_{j'})\right]\ .\end{aligned} \tag{A7}$$

We will now evaluate the commutator (A2) and verify that the left-hand side of (A3) becomes a c number which is a function of the coupling g. This is easily accomplished once we recognize that at $k_{\max}$ we can simply pick off the coefficient c in the Schwinger term by averaging the commutator over a cell size $\sim 1/\Lambda$. In the lattice theory (A3) becomes

$$[\rho_\Lambda(x), \rho_\Lambda(y)]_{\mathrm{ET}} = -ic_\Lambda\sum_{k=-\pi\Lambda}^{\pi\Lambda}\frac{-k^2e^{ik(x-y)}}{2N+1}\ . \tag{A8}$$

An average over the lattice distance is performed by operating on both sides of (A8) with

$$l_x l_y \equiv \int_{-d/\Lambda}^{d/\Lambda} dx \int_{-d/\Lambda}^{d/\Lambda} dy \ , \tag{A9}$$

where $d \sim 1$. This gives

$$[l_x \rho_\Lambda(x), l_y \dot{\rho}_\Lambda(y)] = \frac{4\pi^2}{3} d^2 \Lambda(ic_\Lambda) \ . \tag{A10}$$

The calculation of the left-hand side of (A10) is straightforward in the limit $L = (2N+1)/\Lambda \to \infty$ and we find

$$\frac{8d^2\Lambda}{2N+1} \sum_{j_1, j_2} \delta'(j_1 - j_2) \chi^\dagger_{j_1} \alpha \chi_{j_2} = \frac{4\pi^2 d^2}{3} \Lambda(ic_\Lambda)$$

or

$$c_\Lambda = \frac{-6}{\pi^2} \frac{T}{\Lambda(2N+1)} = \frac{-6T}{\pi^2 \Lambda^2 L} \ , \tag{A11}$$

where

$$T \equiv \Lambda \sum_{j_1 j_2} \chi^\dagger_{j_1} i\alpha \delta'(j_1 - j_2) \chi_{j_2} \tag{A12}$$

is the kinetic-energy term in the Hamiltonian. This result is independent of the specific form of the quartic interaction terms which are softer at short distances, i.e., the same result (A11) follows from prescriptions for defining the Thirring model on a lattice in terms of a quartic potential term free of umklapp processes or of a charge density defined by constructing $\rho(x)$ by the operation (A4) on the bilinear $\chi^\dagger_j \chi_j$. We must now verify that in the limit $L \to \infty$, (A12) becomes a c-number equation. We readily verify this by expressing T in a momentum basis and recognizing its content:

$$T = \sum_{k=-\pi\Lambda}^{\pi\Lambda} k(b^\dagger(k) b(k) - d^\dagger(k) d(k)) \ . \tag{A13}$$

The only term in (A13) proportional to volume L is the zero-point energy since all excitations are finite. Note, for the case $g = 0$, $H = T$ and since we know the ground state of the system, we find

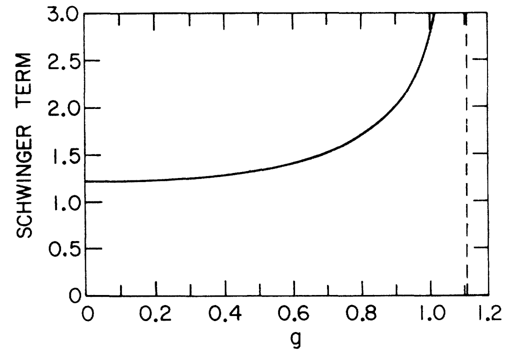

FIG. 3. Renormalized Schwinger term c vs coupling constant g in the lattice Thirring model.

$$\langle 0 | T | 0 \rangle = -2 \sum_{k=0}^{\pi\Lambda} k + (\text{finite excitations})$$
$$= -\tfrac{1}{2}\pi\Lambda^2 L[1 + O(1/L)] \ . \tag{A14}$$

Hence in the $L \to \infty$ limit the free-field Schwinger term is given by

$$c_\Lambda = 3/\pi \ . \tag{A15}$$

Finally, one must divide the c_Λ of (A11) by $(Z_2)^2$ in order to have the Schwinger term of the currents after field renormalization.[3] Since $Z_2 \to 0$ as $g \to g_{\rm cr}$, our c is indeed singular there. (See Fig. 3.)

Johnson has shown that the choice of a value of g_r and of a point-separation procedure are sufficient to determine the theory and its Schwinger term. Conversely, choice of values for g_r and c determine the theory[4] and every pair of values corresponds to a point-separation procedure.[5] We have given a renormalization-group construction of the operators determining c as a function of the "bare" coupling g. In order to complete the numerical correspondence to the continuum theory, we would have to similarly construct the operators $\partial\!\!\!/\psi$ and $j\!\!\!/\psi$, thus finding the "renormalized"[16] parameter g_r.

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[1]S. D. Drell, M. Weinstein, and S. Yankielowicz, Phys. Rev. D 14, 487 (1976); 14, 1627 (1976); 16, 1769 (1977). Hereinafter these are referred to as Papers I, II, and III, respectively. S. Yankielowicz, SLAC Report No. SLAC-PUB-1800, 1976 [lectures given at the 17th Scottish Universities Summer School in Physics, St. Andrews, Scotland, 1976 (unpublished)]; M. Weinstein, in Proceedings of the Summer Institute on Particle Physics [SLAC Report No. 198, 1976 (unpublished)].

[2]W. Thirring, Ann. Phys. (N.Y.) 3, 91 (1958).

[3]K. Johnson, Nuovo Cimento 20, 773 (1961).

[4]G. F. Dell'Antonio, Y. Frishman, and D. Zwanziger, Phys. Rev. D 6, 988 (1972).

[5]B. Klaiber, in *Lectures in Theoretical Physics*, lectures delivered at the Summer Institute for Theoretical Physics, University of Colorado, Boulder, 1967, edited by A. Barut and W. Brittin (Gordon and Breach, New York, 1968), Vol. X, part A.

[6]Paper II, Eqs. (1.3)–(1.5).

[7]Furthermore, we choose $\beta = \gamma_0 = \sigma_1$, $\gamma_1 = \beta\alpha = -i\sigma_2$.

[8]Recall that the phases of $|0_p\rangle$ and $|+_p\rangle$ are determined to preserve the single-site convention (2.7).

[9]This may be seen in a straightforward manner by diagonalizing the block Hamiltonian with a transformation to normal modes.

[10]A simple variational guess for u_α in (2.20) of $u_0=0$ and $u_\pm=1/\sqrt{2}$, thereby emphasizing the block-block coupling terms by maximizing the trial wave function at the block edges, cuts this error by almost a factor of 2.

[11]In the sense of large coupling constant g, as opposed to strong block-block coupling.

[12]This procedure for constructing operator solutions of the model, and not only the Green's functions, was carried out by Dell'Antonio, Frishman, and Zwanziger, Ref. 4.

[13]H. A. Bethe, Z. Phys. 71, 205 (1931).

[14]In this context, it is important to note what happens if we use the difference operator for the gradient, which couples only nearest-neighbor sites on the lattice, instead of our form $\delta'(j)=-(-1)^j/j$ for all $j\neq 0$. Recall that the difference operator doubles the number of fermionic degrees of freedom in the $g=0$ free particle theory. In this case, we find that using the same iteration-truncation procedure described in this paper we are led to a theory of the Heisenberg antiferromagnet for all $g\neq 0$. That is, we find $g_{cr}=0$. A. Luther [Phys. Rev. B 14, 2153 (1976)] has derived results for a lattice version of the *massive* Thirring model which uses the difference operator for the gradient and assigns particles and antiparticles to alternate lattice sites in order to avoid the doubling of degrees of freedom [in this connection see also J. B. Kogut, Cornell University Report No. PRINT-76-0865, 1976 (unpublished)]. He has found for $m\neq 0$ that there exists a $g_{cr}\neq 0$ such that for $g>g_{cr}$ the theory can only be defined by reinterpreting the ground state.

[15]These fields satisfy the anticommutation relations

$$\{\psi^\dagger(x),\ \psi(y)\}_{\mathrm{ET}}=\Lambda\frac{\sin\pi\Lambda(x-y)}{\pi\Lambda(x-y)}$$

with a nonlocality resulting from the $k_{\max}=\pi\Lambda$ cutoff.

[16]"Renormalized" in the definition (4.1) means *after* rendering the products finite, and *before* wave-function renormalization.