Vacuum tunneling of gauge theory in Minkowski space

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We study vacuum tunneling of non-Abelian gauge theory directly in Minkowski space. We do this by constructing a family of field configurations which connects vacuums of different winding numbers and which satisfies conditions necessary and sufficient to produce maximum tunneling amplitude. Using these explicit configurations we obtain explicitly the potential-energy barrier in winding-number space, through which tunneling occurs. We finally discuss the possible connections among the tunneling solution, the classical solution, and the full quantum-mechanical solution to the field equations.

I. INTRODUCTION

Recently, much progress has been made in understanding the nature of the ground state for a non-Abelian gauge theory. The discovery of Euclidean solutions,[1-5] the pseudoparticles, and their possible interpretation as tunnelings among vacuums with different winding numbers[6-10] stimulate many speculations and excitement. In our opinion, there are many features concerning these pseudoparticle solutions which still need to be understood. In particular, the detailed mechanism of the vacuum tunneling in Minkowski space[10] and its relation to the pseudoparticle solution deserve a thorough investigation. We address ourselves in this paper to the following questions:

(1) What field configurations provide the vacuum tunneling, and what are the conditions for a maximal tunneling amplitude?
(2) In what space does vacuum tunneling take place, and what is the potential barrier in this space?
(3) To what extent can vacuum tunneling be understood as a solution to the field equation?

Our results provide partial answers to these questions. It is our hope that our work will stimulate further research on this topic.

In Sec. II of this paper, we formulate the vacuum tunneling as a quantum-mechanical transformation of field configurations from one vacuum to another. In Sec. III, we work out the necessary and sufficient condition for a family of field configurations to achieve the maximal tunneling rate. The proof is simple in Minkowski space, and the physical picture is clear. We work out an explicit example in Sec. IV for a $Q=1$ vacuum tunneling which gives rise to the maximal tunneling rate. We then explore in Sec. V the mass parameter and the potential barrier associated with vacuum tunneling in the winding-number space. We work out the potential barriers explicitly, and also plot them in graphs. In Sec. VI, we discuss the validity of the method and the possible interpretation of the vacuum tunneling as a simplified quantum-mechanical solution to the full field equations. We discuss and speculate on various topics in the last sections. We also include an Appendix A to illustrate several interesting features of vacuum tunnelings in a simple ϕ^4 theory, and an Appendix B (added in proof) to clarify the relation between the tunneling and the Euclidean solutions.

II. FORMULATION OF THE PROBLEM

We consider a non-Abelian (Yang-Mills) gauge theory[11] described by

$$\mathcal{L} = -\tfrac{1}{4} f^{(a)}_{\mu\nu} f^{(a)\mu\nu}, \tag{2.1}$$

$$f^{(a)}_{\mu\nu} = \partial_\mu a^{(a)}_\nu - \partial_\nu a^{(a)}_\mu + g\epsilon_{abc} a^{(b)}_\mu a^{(c)}_\nu, \tag{2.2}$$

with a as an isospin index, and μ,ν as Lorentz indices. In terms of matrix notation,

$$F_{\mu\nu} = g f^{(a)}_{\mu\nu} \tfrac{1}{2}\tau^a, \tag{2.3}$$

$$A_\mu = g a^{(a)}_\mu \tfrac{1}{2}\tau^a, \tag{2.4}$$

we have

$$\mathcal{L} = -\frac{1}{2g^2}\mathrm{Tr}(F_{\mu\nu})^2 \tag{2.5}$$

and

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \frac{1}{i}[A_\mu, A_\nu]. \tag{2.6}$$

In (2.3), $\tau^{(a)}$ $(a=1,2,3)$ are Pauli matrices. It is often more convenient to introduce

$$E_i = F_{0i}, \quad B_i = \tfrac{1}{2}\epsilon_{ijk}F_{jk}, \tag{2.7}$$

where $\vec{E}$ and $\vec{B}$ are generalizations of the electric and magnetic fields. In terms of $\vec{E}$ and $\vec{B}$, we have the Lagrangian density

$$\mathcal{L} = \frac{1}{g^2}\mathrm{Tr}(\vec{E}^2 - \vec{B}^2), \tag{2.8}$$

and the Hamiltonian density

$$\mathcal{H} = \frac{1}{g^2}\mathrm{Tr}(\vec{E}^2 + \vec{B}^2) \geq 0. \tag{2.9}$$

One can define the winding-number difference between two spacelike surfaces σ_1, σ_2 as[1]

$$Q = \frac{1}{4\pi^2}\mathrm{Tr}\int_{\sigma_1}^{\sigma_2} d^4x\,\vec{E}\cdot\vec{B}. \tag{2.10}$$

If $F_{\mu\nu}=0$ on σ_1, σ_2, then Q is an integer. Equation (2.10) also tells us that if the vacuums at σ_1 and σ_2 give rise to a nonvanishing Q, there must exist some regions between σ_1 and σ_2 such that $F_{\mu\nu}\neq 0$. Since the energy density $\mathcal{H}$ [see Eq. (2.9)] is positive-definite and vanishes only if $F_{\mu\nu}=0$, the existence of such a region (region with $F_{\mu\nu}\neq 0$) can only occur during a tunneling process. For a sourceless classical theory, $F_{\mu\nu}=0$ on σ_1 (or σ_2) implies $F_{\mu\nu}=0$ everywhere. Thus, it is impossible to change the winding number of a classical ground state. However, as a quantum system, the winding number of the ground state may change due to quantum fluctuations. This phenomenon can be understood as a tunneling in some collective mode in the field configuration. Recently, Gervais and Sakita studied the vacuum tunneling and its quantum corrections by means of the collective-coordinate method.[10] The readers are referred to that paper for details.

In this paper, we concentrate on the vacuum tunneling by considering only a single collective mode. We shall write down the rules for obtaining the maximal tunneling amplitude. In Sec. VI and the Appendix A, we demonstrate in a simple example that the vacuum tunneling is consistent with finding a first-quantized solution to the field equations.

For simplicity, we choose σ_1 and σ_2 as surfaces at equal time (t_1 and t_2). We work in the Coulomb gauge, $A^0=0$. We denote the vector potentials on σ_1 and σ_2 as $\vec{A}^{(1)}$ and $\vec{A}^{(2)}$. We require that $A_\mu^{(1)}$ and $A_\mu^{(2)}$ are associated with pure gauge transformations such that $F_{\mu\nu}=0$ on both σ_1 and σ_2. The winding-number difference Q can be written as the difference of two surface integrals[1]

$$\begin{aligned} Q &= \frac{i}{12\pi^2}\left(\int_{\sigma_2} - \int_{\sigma_1}\right) d\sigma_\mu \epsilon_{\mu\nu\lambda\sigma}\mathrm{Tr}(A_\nu A_\lambda A_\sigma) \\ &= \frac{i}{12\pi^2}\left(\int_{t_2} - \int_{t_1}\right) d^3x \epsilon_{ijk}\mathrm{Tr}(A_i A_j A_k). \end{aligned} \tag{2.11}$$

Thus, Q is independent of the intermediate field configurations as long as A_μ is regular inside the region.

The procedures for obtaining the maximal single-mode tunneling amplitudes are as follows:

(1) We introduce a family of intermediate field configurations as

$$A_0 = 0, \tag{2.12}$$

$$\vec{A}(x,t) = \vec{f}(x, \lambda(t)), \tag{2.13}$$

where λ is a parameter describing the field configuration within the family. We require that

$$\vec{f} = \vec{A}^{(1)} \quad \text{at } \lambda = \lambda_1, \tag{2.14}$$

$$\vec{f} = \vec{A}^{(2)} \quad \text{at } \lambda = \lambda_2, \tag{2.15}$$

and that $\vec{f}$ varies continuously from $\vec{A}^{(1)}$ to $\vec{A}^{(2)}$ as λ varies from λ_1 to λ_2. Given (2.12) and (2.13), we obtain

$$E_i = \frac{\partial f_i}{\partial\lambda}\dot{\lambda}, \tag{2.16}$$

$$B_i = \tfrac{1}{2}\epsilon_{ijk}\left(\partial_j f_k - \partial_k f_j + \frac{1}{i}[f_j, f_k]\right). \tag{2.17}$$

The field variables $\vec{E}$ and $\vec{B}$ for intermediate values of λ usually do not vanish.

(2) From $\vec{E}$ and $\vec{B}$, we construct a single-parameter Lagrangian and Hamiltonian as

$$L = \tfrac{1}{2}m(\lambda)\dot{\lambda}^2 - V(\lambda) \tag{2.18}$$

and

$$H = \frac{p_\lambda^2}{2m(\lambda)} + V(\lambda), \tag{2.19}$$

where

$$m(\lambda) = \frac{2}{g^2}\int d^3x\,\mathrm{Tr}\left(\frac{\partial\vec{f}}{\partial\lambda}\right)^2, \tag{2.20}$$

$$V(\lambda) = \frac{2}{g^2}\int d^3x\,\mathrm{Tr}\vec{B}^2, \tag{2.21}$$

$$p_\lambda \equiv \frac{\partial L}{\partial\dot{\lambda}} = m(\lambda)\dot{\lambda}. \tag{2.22}$$

A typical potential $V(\lambda)$ is shown in Fig. 1.

To obtain the tunneling amplitude, we treat (2.19) as a quantum-mechanical Hamiltonian. With appropriate orderings of the operators, we can make H Hermitian. We can now obtain the tunneling amplitude by solving an ordinary Schrödinger equation. At the weak-coupling limit in which the tunneling amplitude is small, we can compute the tunneling amplitude by the WKB method as e^{-R} with

$$R = \int_{\lambda_1}^{\lambda_2} d\lambda[2m(\lambda)V(\lambda)]^{1/2}. \tag{2.23}$$

(3) By varying the field configurations $f(x, \lambda(t))$,

we can find the conditions for obtaining the maximal tunneling amplitude. In Sec. III, we shall obtain the necessary and sufficient conditions for the maximal amplitude. In Appendix B (added in proof), we clarify the relation between the maximal tunneling configuration and the Euclidean solution.

III. VACUUM TUNNELING IN MINKOWSKI SPACE

Now let us compute the WKB vacuum-tunneling amplitude due to a particular family of field configurations $[A_0=0, \vec{A}=\vec{f}(x,\lambda(t))]$. The total Hamiltonian H associated with $\{\vec{f}\}$ is

$$H=\frac{p_\lambda^2}{2m(\lambda)}+V(\lambda), \tag{3.1}$$

with $m(\lambda)$, $V(\lambda)$ as given in Eqs. (2.20) and (2.21). The first term of H denotes the kinetic energy with a λ-dependent mass $m(\lambda)$, and the second term denotes the potential energy. It is easy to see that

$$V(\lambda_1)=V(\lambda_2)=0 \tag{3.2}$$

and

$$V(\lambda)>0 \text{ whenever } \vec{B}\neq 0. \tag{3.3}$$

The WKB tunneling amplitude is $P=e^{-R}$ with

$$\begin{aligned} R&=\int_{\lambda_1}^{\lambda_2} d\lambda[2m(\lambda)V(\lambda)]^{1/2}\\ &=\frac{2}{g^2}\int_{\lambda_1}^{\lambda_2} d\lambda\left[\operatorname{Tr}\int d^3x\left(\frac{\partial\vec{f}}{\partial\lambda}\right)^2 \operatorname{Tr}\int d^3x\,\vec{B}^2\right]^{1/2}\\ &=\frac{2}{g^2}\int_{t_1}^{t_2} dt\left(\operatorname{Tr}\int d^3x\,\vec{E}^2\operatorname{Tr}\int d^3x\,\vec{B}^2\right)^{1/2}. \end{aligned} \tag{3.4}$$

We can interpret

$$\operatorname{Tr}\int d^3x\,\vec{a}(x)\cdot\vec{b}(x)\equiv a\cdot b$$

as a scalar product with a positive norm, i.e.,

$$|a^2|\equiv a\cdot a>0,$$

for every $a\neq 0$. This implies that the Schwarz inequality holds for every a and b,

$$(|a^2|\,|b^2|)^{1/2}\geqslant|a\cdot b|.$$

The Schwarz inequality implies in the present case

$$R\geqslant\frac{2}{g^2}\left|\operatorname{Tr}\int d^4x\,\vec{E}\cdot\vec{B}\right|=\frac{8\pi^2}{g^2}|Q|. \tag{3.5}$$

Thus, for a given winding-number difference Q, R is bounded from below by $(8\pi^2/g^2)|Q|$, and hence, the WKB tunneling amplitude is bounded from above by

$$P\equiv e^{-R}\leqslant\exp\left(-\frac{8\pi^2}{g^2}|Q|\right). \tag{3.6}$$

In addition, R reaches its minimal value $R=(8\pi^2/g^2)|Q|$ (or P reaches its maximal value) if and only if $\vec{E}$ is proportional to $\vec{B}$ for all $\vec{x}$ and t with an x-independent proportionality constant. The problem of finding the fastest WKB tunneling rate is equivalent to finding a set of field configurations which gives rise to parallel $\vec{E}$ and $\vec{B}$. Note that even though $\vec{E}$ and $\vec{B}$ are gauge-dependent quantities, the relevant quantities such as L, Q, and the condition $\vec{E}\parallel\vec{B}$ are all gauge independent. Hence, the WKB tunneling amplitude is gauge invariant. In the following section we shall present field configurations which give rise to the maximal tunneling rate for $Q=1$, and motivate the construction of such configurations.

IV. AN EXAMPLE

An explicit family of field configurations which gives rise to the maximal vacuum-tunneling amplitude for $Q=1$ is

$$A_0=\frac{\vec{x}\cdot\vec{\tau}}{\vec{x}^2+\lambda^2+a^2}\dot{\lambda}, \tag{4.1}$$

$$\vec{A}=\frac{\lambda\vec{\tau}+\vec{x}\times\vec{\tau}}{\vec{x}^2+\lambda^2+a^2}. \tag{4.2}$$

The vector potential A_μ given in (4.1) and (4.2) is not in the Coulomb gauge. We can transform it into the Coulomb gauge $A'_\mu=(0,\vec{A}')$ by a gauge transformation U,

$$A'_\mu=U^{-1}[(\partial_\mu-iA_\mu)U], \tag{4.3}$$

with

$$U=\exp\left[\frac{i\vec{x}\cdot\vec{\tau}}{(\vec{x}^2+a^2)^{1/2}}\tan^{-1}\frac{\lambda}{(\vec{x}^2+a^2)^{1/2}}\right]. \tag{4.4}$$

The Coulomb gauge vector potential is quite complicated. Since both the Lagrangian L and the

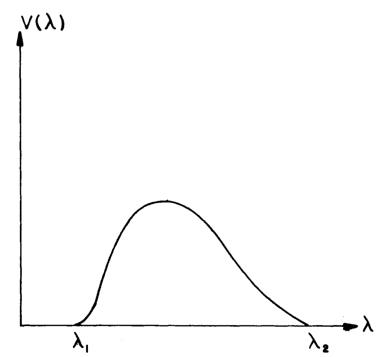

FIG. 1. A typical potential barrier associated with the vacuum tunneling. The parameter λ denotes the change of field configurations. Note that the potential $V(\lambda)$ vanishes at $\lambda=\lambda_1$ and λ_2, and that it is positive-semidefinite, $V(\lambda)\geq 0$, for $\lambda_1<\lambda<\lambda_2$.

winding number Q are gauge invariant, we can compute them in the original gauge specified by (4.1) and (4.2). We can evaluate the vacuum-tunneling amplitude once L and Q are known. Using (4.1) and (4.2), we find

$$\vec{E} = -\frac{2a^2\dot{\lambda}}{(\vec{x}^2+\lambda^2+a^2)^2}\vec{\tau}, \tag{4.5}$$

$$\vec{B} = -\frac{2a^2}{(\vec{x}^2+\lambda^2+a^2)^2}\vec{\tau}, \tag{4.6}$$

which are indeed parallel to each other. The effective one-parameter Lagrangian L and the Hamiltonian H are

$$\begin{aligned} L &= \frac{1}{g^2}\mathrm{Tr}\int d^3x(\vec{E}^2-\vec{B}^2) \\ &= \frac{3\pi^2a^4}{g^2(\lambda^2+a^2)^{5/2}}\dot{\lambda}^2 - \frac{3\pi^2a^4}{g^2(\lambda^2+a^2)^{5/2}} \\ &= \tfrac{1}{2}m(\lambda)\dot{\lambda}^2 - V(\lambda) \end{aligned} \tag{4.7}$$

and

$$H = \frac{p_\lambda^2}{2m(\lambda)} + V(\lambda), \tag{4.8}$$

respectively, with

$$m(\lambda) = \frac{6\pi^2a^4}{g^2(\lambda^2+a^2)^{5/2}}, \tag{4.9}$$

$$V(\lambda) = \frac{3\pi^2a^4}{g^2(\lambda^2+a^2)^{5/2}}, \tag{4.10}$$

$$p_\lambda = m(\lambda)\dot{\lambda}. \tag{4.11}$$

For an intermediate value of $\lambda(t)$, we define a winding-number variable $q(\lambda)$ as

$$q(\lambda) \equiv \frac{1}{4\pi^2}\mathrm{Tr}\int_{-\infty}^{t} dt\int d^3x\,\vec{E}\cdot\vec{B}. \tag{4.12}$$

For $\vec{E}$ and $\vec{B}$ given in (4.5) and (4.6), we have

$$\begin{aligned} q(\lambda) &= \frac{6a^4}{\pi^2}\int_{-\infty}^{t} dt\int d^3x\frac{\dot{\lambda}}{(\vec{x}^2+\lambda^2+a^2)^4} \\ &= \frac{6a^4}{\pi^2}\int_{-\infty}^{\lambda} d\lambda\int d^3x\frac{1}{(\vec{x}^2+\lambda^2+a^2)^4} \\ &= \tfrac{1}{2}+\tfrac{1}{4}\frac{1}{(\lambda^2+a^2)^{1/2}}\left(\frac{a^2}{\lambda^2+a^2}+2\right). \end{aligned} \tag{4.13}$$

According to the definition (4.12), we have

$$\lim_{\lambda\to-\infty} q(\lambda) = 0, \tag{4.14}$$

$$\lim_{\lambda\to\infty} q(\lambda) = Q \ (=1 \text{ here}). \tag{4.15}$$

We can use $q(\lambda)$ as a dynamical variable to describe the vacuum tunneling in the winding-number space.

We can obtain the vacuum-tunneling amplitude by solving the one-dimensional quantum-mechanical system described by (4.7)–(4.11). The WKB tunneling amplitude is $P = e^{-R}$ with

$$\begin{aligned} R &= \int_{-\infty}^{\infty} d\lambda[2m(\lambda)V(\lambda)]^{1/2} \\ &= \frac{2}{g^2}\int_{-\infty}^{\infty} d\lambda\,\frac{3\pi^2a^4}{(\lambda^2+a^2)^{5/2}} = \frac{8\pi^2}{g^2}, \end{aligned}$$

which is indeed the maximal tunneling amplitude associated with a $Q=1$ transition. Our result agrees with the action integral over a $Q=1$ Euclidean pseudoparticle solution, and provides a justification for the usual interpretations of pseudoparticles as vacuum tunnelings. By working out the vacuum-tunneling amplitude in the Minkowski space, we also obtain an explicit expression for the potential barrier V.

If one considers a set of gauge potentials $A_0(\vec{x},t)$ and $A_i(\vec{x},t)$, the formulas that give $E_i = F_{0i}$ and $B_i = \frac{1}{2}\epsilon_{ijk}F_{jk}$ are the same, irrespective of the metric, Minkowskian or Euclidean, of the space. The constraint $E_i = B_i$ defines a self-dual field configuration in Euclidean four-space, whereas the constraint $E_i = \alpha(t)B_i$ guarantees that the barrier for tunneling is minimal in Minkowski space. Thus the most probable escape path in Minkowski space may be obtained from a self-dual field configuration in Euclidean space by replacing t by $\lambda(t)$ and multiplying $A_0(\vec{x},\lambda(t))$ by $\dot{\lambda}(t)$. For then one automatically gets

$$\vec{E}(\vec{x},\lambda(t)) = \dot{\lambda}\vec{B}(\vec{x},\lambda(t)).$$

This explains the forms of Eqs. (4.1) and (4.2) above, which are obtained from the pseudoparticle solution of Ref. 1 by the use of this procedure. See also Appendix B (added in proof) for a general discussion.

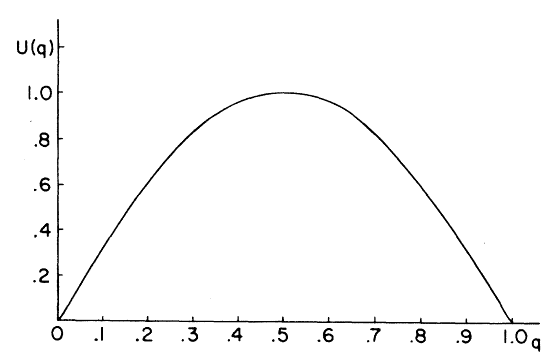

FIG. 2. The potential barrier as a function of the winding number q for a $Q=1$ vacuum tunneling. The winding number q is defined in Eqs. (4.12) and (4.13).

V. WINDING-NUMBER SPACE

In Eq. (4.10), the potential barrier is described as a function of λ. Note that the winding number $q(\lambda)$ is also a function of λ. Combining Eqs. (4.10) and (4.13), we have a parametric definition of the potential energy V as a function of q. In Fig. 2, we plot the scaled potential,

$$U(q)\equiv\frac{g^2aV(q)}{3\pi^2}, \tag{5.1}$$

as a function of q. We can also express the kinetic energy term (E_K) into the standard form

$$E_K=\tfrac{1}{2}m'(q)\dot{q}^2, \tag{5.2}$$

with

$$m'=\frac{32\pi^2}{3g^2a^4}(\lambda^2+a^2)^{5/2}. \tag{5.3}$$

Equations (5.3) and (4.13) define m' as a function of q. In Fig. 3, we plot $3g^2m'/32\pi^2a$ as a function of q. We can view the vacuum-tunneling process as a particle of q-dependent mass going through the potential barrier $V(q)$.

We would like to point out that the expression q defined in (4.13) is *not* the only possible definition of the winding number. In fact, any function $q'(\lambda)$ which agrees with q on integer values is an equally acceptable definition of the winding numbers. We may write

$$q'=q\phi(q),$$

with

$$\phi(n)=1 \quad \text{for } n=\text{integer}.$$

A simple example of $\phi(n)$ is $\cos^2 n\pi$. Even though q and q' agree on integer values, their behaviors for noninteger values are entirely different. We can use this degree of freedom to define a new q' which gives rise to a q'-independent mass. Indeed, if we require

$$E_K=\frac{3\pi^2a^4}{g^2(\lambda^2+a^2)^{5/2}}\dot{\lambda}^2=\tfrac{1}{2}m''\dot{q}'^2, \tag{5.4}$$

with m'' being a constant, and

$$q'(-\infty)=0, \qquad q'(\infty)=1, \tag{5.5}$$

we obtain

$$m''=\frac{6\pi^2a}{g^2}\left(\frac{\Gamma(1/2)\Gamma(3/4)}{\Gamma(5/2)}\right)^2, \tag{5.6}$$

$$q'=\frac{\Gamma(5/2)}{\Gamma(1/2)\Gamma(3/4)}\int_{-\infty}^{\lambda/a}\frac{dt}{(t^2+1)^{5/4}}$$
$$=\tfrac{1}{2}+\frac{\text{sign}\lambda}{2B(1/2,3/4)}B_\xi(\tfrac{1}{2},\tfrac{3}{4}), \tag{5.7}$$

where

$$\xi=\frac{\lambda^2}{\lambda^2+a^2}, \tag{5.8}$$

and $B_\xi(a,b)$ is the incomplete beta function. In terms of the new winding number q', the potential barrier $U(q')\equiv g^2aV(q')/3\pi^2$ is a smoother function of q' near $q'=0$, and 1. We have plotted $U(q')$ as a function of q' in Fig. 4.

VI. FURTHER INTERPRETATIONS

In this section, we shall explore further the meaning of the vacuum tunneling. For notational simplicity, we denote the field variables as $A(x,t)$, and the initial and final field variables associated with the different vacuum modes are $A^{(1)}$ and $A^{(2)}$.

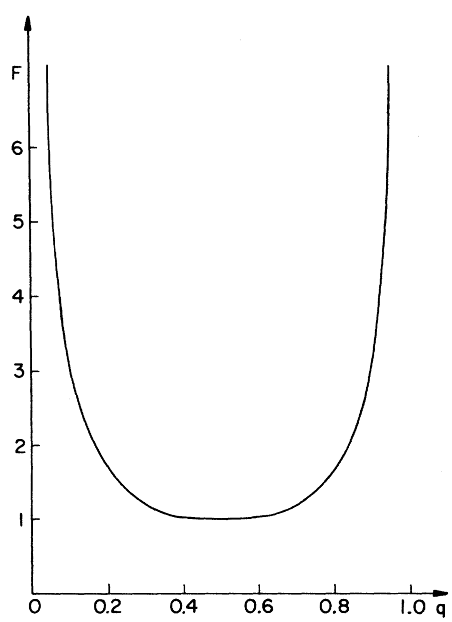

FIG. 3. The effective mass parameter as a function of the winding number q for a $Q=1$ vacuum transition. $F(q)$ is defined by $F(q)\equiv 3g^2m'(q)/32\pi^2a$.

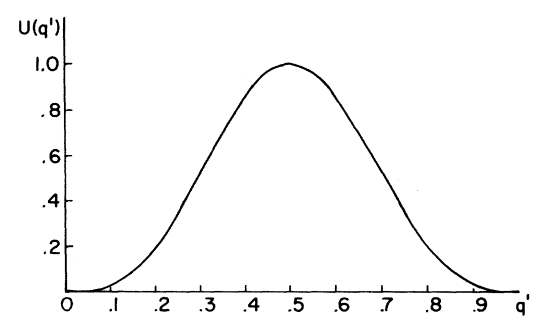

FIG. 4. The potential barrier as a function of a modified winding number q' The variable q' is defined in Eq. (5.7).

We can express the transition amplitude from $A^{(1)}$ to $A^{(2)}$ in the path-integral form as

$$\langle A^{(2)}|A^{(1)}\rangle = \int_{A^{(1)}}^{A^{(2)}} \mathfrak{D}A \exp\left[i\int L(A,\dot{A})dt\right], \quad (6.1)$$

where L is the Lagrangian. We have also suppressed a gauge-fixing δ-function term for notational simplicity.

For all paths connecting from $A^{(1)}$ to $A^{(2)}$, we can parametrize $A(x,t)$ by a set of variables $\{f(x,\lambda,(t)), a_1, a_2, \ldots\}$, where $f(x,\lambda(t))$ denotes a family of field configurations connecting $A^{(1)}$ to $A^{(2)}$, and a_i $(i=1,2,\ldots)$ represents the remaining variables describing the fluctuations around the path $f(x,\lambda(t))$. We can express the transition amplitude as

$$\langle A^{(2)}|A^{(1)}\rangle = \int \mathfrak{D}\lambda(\tau)\mathfrak{D}a_i J(\lambda,a_i) \times \exp\left[\int (iL_1+iL_2)\,dt\right], \quad (6.2)$$

where J is the Jacobian describing the transformation of A to $\{f(x,\lambda),a_i\}$,

$$L_1(\lambda,\dot{\lambda}) \equiv L(A,\dot{A})\big|_{A=f(x,\lambda(t))} \quad (6.3)$$

and

$$L_2(\lambda,\dot{\lambda},a_i) \equiv L - L_1. \quad (6.4)$$

Now we integrate out the variables a_i's, and obtain an effective action ΔS:

$$\int \mathfrak{D}a_i J(\lambda,a_i)\exp\left(i\int L_2 dt\right) = \exp[i\Delta S(\lambda)]. \quad (6.5)$$

We can write the transition amplitude as

$$\langle A^{(2)}|A^{(1)}\rangle = \int \mathfrak{D}\lambda(t)\exp\left[i\int (L_1(\lambda,\dot{\lambda}))dt + i\Delta S(\lambda)\right]. \quad (6.6)$$

Note that our result (6.6) is still general. The result does *not* depend on the particular choice of the field configuration (collective coordinate) $f(x,\lambda(t))$. On the other hand, we can see easily that L_1 obtained here depends crucially on the choice of $f(x,\lambda(t))$. The difference in L_1 due to two choices of $f(x,\lambda)$ is compensated by the difference in the correction term ΔS. For some choices of f, ΔS may be small. We conjecture that the particular choice of $f(x,\lambda(t))$ which maximizes the transition amplitude

$$\langle A^{(2)}|A^{(1)}\rangle^{(0)} \equiv \int \mathfrak{D}\lambda(t)\exp\left[i\int L_1(\lambda,\dot{\lambda})dt\right] \quad (6.7)$$

gives rise to a small correction term ΔS. For this choice of $f(x,\lambda(t))$, we can ignore ΔS and write

$$\langle A^{(2)}|A^{(1)}\rangle \simeq \int \mathfrak{D}\lambda(t)\exp\left[i\int L_1(\lambda,\dot{\lambda})dt\right]. \quad (6.8)$$

The integration over the one-parameter function (path) $\lambda(t)$ leads to a one-dimensional quantum-mechanical transition amplitude. This result is equivalent to our calculation given in the previous section. We wish to emphasize that the smallness of ΔS is at the moment a conjecture. The above assumption is equivalent to the dominance of the "most probable escape path" (MPEP) in the function space.[12] We believe that this conjecture is a reasonable one.

Operationally, what we have done is to replace A by $A(x,t)=f(x,\lambda(t))$, and suppress all other variables. We would like to point out that the integration over $\lambda(t)$ for a given $A=f(x,\lambda(t))$ still includes many paths in the original (x,t) space. Namely, to each function $\lambda(t)$, it corresponds to a path in the (x,t) space. It is the integration over these paths which makes the quantum tunneling possible.

For transition between vacuum with different winding numbers, there is no classical path which minimizes the action. However, it is possible that there exists a two-dimensional surface (or more generally, an N-dimensional hypersurface) such that a path outside the surface gives rise to a larger action than that obtained from its projection in the surface. If such a surface does exist, the path integral will be dominated by the paths lying within the surface. If $\{a_i\}$ measures the variations of a path perpendicular to the surface, then the action is dominated by the surface $A=f(x,\lambda(t))$, $\{a_i\}=0$. Thus, we obtain naturally a semiclassical approximation by setting all $a_i=0$ (classical limit), but keeping the integration over $\lambda(t)$ exactly. This semiclassical approximation is precisely the method used in the previous section.

The above interpretation also suggests that, with a proper choice of $f(x,\lambda(t))$, $A(x,t)=f(x,\lambda(t))$ should obey the field equation $F_{\mu\nu;\nu}=0$. Now the relations among our tunneling solution, a classical solution, and a fully quantum-mechanical solution are transparent.[13] All three solutions obey the field equation $F_{\mu\nu;\nu}=0$. In a classical solution, the field variables are c numbers. In a fully quantum-mechanical solution, all field variables are q numbers and they obey the canonical quantization relations. In the tunneling solution, one pair of dynamical variables $(\lambda,\dot{\lambda})$ are treated as quantum operators, and the remaining variables (the a_i's) are treated as c numbers. At the moment, we have not yet found the proper parameterization which makes this interpretation manifest in the Yang-Mills theory. However, we are able to provide such a picture in a simple ϕ^4 theory. See Appendix A for detailed discussion of this model.

We can generalize this physical picture to multi-pseudoparticle solutions. In an N-pseudoparticle solution, the field equations $F_{\mu\nu;\nu}=0$ are still satisfied, but there are N pairs of dynamical variables being treated as quantum-mechanical operators.

VII. DISCUSSION

A. Relation to the pseudoparticle solutions

We have shown in Sec. V that the weak-coupling tunneling amplitude obtained from our solution agrees with the Euclidean action integral over a pseudoparticle solution. Thus, our solution provides a direct verification that the pseudoparticle describes the vacuum tunneling. The Euclidean description is valid if the tunneling is weak such that the WKB approximation is reliable. In the strong-coupling case, the WKB approximation is no longer reliable. We expect that the Euclidean version is not reliable either, and one has to solve the tunneling amplitude in the Minkowski space exactly. In the following, we shall examine the connection of the WKB and the Euclidean solution closely.

In the Minkowski solution, we do not have the self-dual condition $\vec{E}=i\vec{B}$ as given in the Euclidean solutions. We have instead a weaker condition $\vec{E}\parallel\vec{B}$ for all x and t. Combining the condition $\vec{E}\parallel\vec{B}$ with Eqs. (2.16), (2.20)–(2.22), we have

$$\begin{aligned}\vec{E}&=\vec{B}\left[\frac{m(\lambda)}{2V(\lambda)}\right]^{1/2}\dot{\lambda}\\&=\vec{B}\,\frac{p_\lambda}{[2m(\lambda)V(\lambda)]^{1/2}},\end{aligned}\tag{7.1}$$

which should be viewed as an operator equation. We can recover the self-dual condition and the Euclidean solution from (7.1) by noting that, in the WKB approximation, we may interpret formally the momentum p_λ during the tunneling to be imaginary,

$$p_\lambda=i(2mV)^{1/2}.\tag{7.2}$$

Note that this is only a formal, operationally convenient identification. The true quantum-mechanical operator p_λ is self-adjoint, and does not have any imaginary eigenvalue. Accepting this formal identification, and substituting (7.2) into (7.1), we obtain the self-dual condition

$$\vec{E}=i\vec{B}.\tag{7.3}$$

Indeed, if we apply this formal identification to our solution in Sec. IV, we obtain

$$\dot{\lambda}=i\tag{7.4}$$

and

$$\vec{E}=i\vec{B}=-\frac{i2a^2}{(\vec{x}^2+\lambda^2+a^2)^2}\,\vec{\tau},\tag{7.5}$$

which is precisely the analytic continuation of the pseudoparticle solution to the Minkowski space. See also Appendix B (added in proof).

B. Nonoptimal solution

In Secs. IV and V, we concentrate on the vacuum tunneling through an optimal path, i.e., the most probable escape path.[12] In reality, tunnelings through nonoptimal paths are also possible, but they give rise to slower rates. In the following, we shall provide an example for such a path.

Consider the tunneling through the field configuration

$$A_0=0,\tag{7.6}$$

$$\vec{A}=\frac{\lambda\vec{\tau}+\vec{x}\times\vec{\tau}}{\vec{x}^2+\lambda^2+a^2}\,.\tag{7.7}$$

The field configuration $\vec{A}$ in (7.7) is identical to that of (4.2). It is easy to see that the winding number changes by $1(Q=1)$ as λ varies from $-\infty$ to ∞. The $\vec{E}$ and $\vec{B}$ fields obtained from (7.6) and (7.7) are

$$\vec{E}=-\frac{(-\lambda^2+\vec{x}^2+a^2)\vec{\tau}-2\lambda(\vec{x}\times\vec{\tau})}{(\vec{x}^2+\lambda^2+a^2)^2}\,\dot{\lambda}\tag{7.8}$$

and

$$\vec{B}=-\frac{2a^2}{(\vec{x}^2+\lambda^2+a^2)^2}\,\vec{\tau},\tag{7.9}$$

respectively. Note that $\vec{E}$ and $\vec{B}$ are no longer parallel to each other. Indeed, when we compute the WKB tunneling amplitude, we find $P=e^{-R}$ with

$$R=15.8831\pi^2/g^2>8\pi^2/g^2.\tag{7.10}$$

At the small-g limit, P represents a much smaller tunneling amplitude than the maximal tunneling amplitude $e^{-8\pi^2/g^2}$.

C. The pseudoparticle density

In this paper, we consider the vacuum tunneling describing the change of field configurations around the origin ($x^\mu=0$). Evidently, vacuum tunneling can also appear at different space-time locations, and the tunnelings at large separations (distances $\gg a$) are dynamically independent. Also, tunneling processes involving small space-time regions can take place within the tunneling processes involving large space-time regions. Using simple scaling arguments, we expect that the total tunneling rate due to a single pseudoparticle (i.e., $Q=1$ transition) over the entire space-time region and sizes is

$$\Gamma\propto\frac{VT}{a^4}\,\frac{da}{a}\,e^{-16\pi^2/g^2}.\tag{7.11}$$

If the contributions due to different pseudoparticles are independent as in the dilute-gas approximation, we expect that the average densities of the pseudoparticle and of the anti-pseudoparticle are the same and are given by

$$dn\propto\frac{da}{a^5}\,e^{-16\pi^2/g^2}\tag{7.12}$$

(without quantum correction). The vacuum is thus filled with randomly distributed pseudoparticles and anti-pseudoparticles as in a dilute gas. For a fixed g, the density of small-size pseudoparticles varies as the fifth power of $1/a$. Presumably, the quantum corrections will modify the tunneling amplitude. It is reasonable to assume that the higher-order correction will change the coupling constant g in (7.12) to the running coupling constant $\bar{g}$, giving

$$dn \propto \frac{da}{a^5}\, e^{-16\pi^2/\bar{g}(a)^2} \tag{7.13}$$

(with quantum correction). According to the perturbative calculation,[14] one finds

$$\frac{8\pi^2}{\bar{g}(a)^2} = \frac{8\pi^2}{g^2} + \frac{22}{3}\ln\frac{a_0}{a}\,, \tag{7.14}$$

$$e^{-8\pi^2/\bar{g}^2} = e^{-8\pi^2/g^2}\left(\frac{a}{a_0}\right)^{22/3}, \tag{7.15}$$

where a_0 is the length associated with the renormalization point. This correction is sufficient to render the pseudoparticle density finite in the small-a region,

$$dn \propto \frac{da}{a^5}\left(\frac{a}{a_0}\right)^{44/3} e^{-16\pi^2/g^2} = \text{const}\times a^{29/3}da\,. \tag{7.16}$$

It is generally believed that at large a, $\bar{g}(a)$ becomes large too. Hence, the WKB tunneling rate is no longer reliable. If we take the formula (7.13) literally and study its large-a behavior, we find at large a,

$$dn \propto \frac{da}{a^5},$$

which gives rise to a finite density. This indicates that the total density of pseudoparticles over all sizes is finite.

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APPENDIX A: VACUUM TUNNELING IN A SIMPLE ϕ^4 THEORY

Consider a two-dimensional ϕ^4 theory described by the Lagrange function

$$\mathcal{L} = \tfrac{1}{2}\dot{\phi}^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \tfrac{1}{4}g(\phi^2 - c^2)^2 + B(c^2\phi - \tfrac{1}{3}\phi^3) + \tfrac{2}{3}Bc^3, \quad B>0. \tag{A1}$$

The field equation obtained from (A1) is

$$\ddot{\phi} - \frac{\partial^2\phi}{\partial x^2} + g(\phi^2 - c^2)\phi - B(c^2 - \phi^2) = 0. \tag{A2}$$

It is easy to see that for a classical field both $\phi = c$ and $\phi = -c$ are stable minima. Quantum mechanically, the $\phi = -c$ state represents an unstable vacuum, and the tunneling to the stable vacuum state $\phi = c$ will take place.[15,16] We shall consider the case of a small and positive B in which the vacuum tunneling is weak. The Hamiltonian of our system is

$$\mathcal{H} = \tfrac{1}{2}\dot{\phi}^2 + \tfrac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \tfrac{1}{4}g(\phi^2 - c^2)^2 - B(c^2\phi - \tfrac{1}{3}\phi^3) - \tfrac{2}{3}Bc^3. \tag{A3}$$

The energy density associated with the unstable vacuum state, $\phi = -c$, is chosen to be zero. The energy density associated with the true stable state, $\phi = c$, is $-\epsilon$ with

$$\epsilon \equiv \tfrac{4}{3}Bc^3 > 0. \tag{A4}$$

To understand the tunneling process, we note that there are kink-antikink states, which are degenerate with the $\phi = -c$ state. The exact form of the kink-antikink states was worked out by Katz in Ref. 16. For our purpose, we only need to consider an approximate form. The kink-antikink state centered at the origin is described approximately by

$$\phi = c\left[\tanh\left(\frac{\mu}{2}(x+\lambda_c)\right) - \tanh\left(\frac{\mu}{2}(x-\lambda_c)\right) - 1\right], \tag{A5}$$

with

$$\mu = (2gc^2)^{1/2}, \tag{A6}$$

$$\lambda_c = M/\epsilon, \tag{A7}$$

$$M = \frac{2\sqrt{2}}{3}\, g^{1/2}c^3. \tag{A8}$$

In (A6)–(A8), μ denotes the mass (or frequency) associated with small oscillations around $\phi = \pm c$, $2\lambda_c$ is the separation of the kink and the antikink, and M is the mass of the kink. The total energy associated with the field $\phi(x)$ given in (A5) is

$$E = 2M - 2\epsilon\lambda_c = 0,$$

as desired.

We shall work out the tunneling amplitude between the $\phi = -c$ unstable vacuum state and the state given by (A5). Once the field reaches the field configuration (A5), a real kink and antikink pair is produced. The kink and the antikink will move away with an acceleration $\propto O(B)$, and they leave behind them an increasing region of true ground state $\phi = c$. Thus, the tunneling between the $\phi=-c$ state and the (A5) state represents the vacuum tunneling that we are looking for.

The intermediate field configuration describing the vacuum tunneling is the kink-antikink configuration located at λ and $-\lambda$, and moving with velocity $\pm\dot{\lambda}$,

$$\phi=c\left[\tanh\left(\frac{\mu}{2}\frac{x+\lambda}{(1-\dot{\lambda}^2)^{1/2}}\right)-\tanh\left(\frac{\mu}{2}\frac{x-\lambda}{(1-\dot{\lambda}^2)^{1/2}}\right)-1\right]. \tag{A9}$$

The $1/(1-\dot{\lambda}^2)^{1/2}$ factor denotes the Lorentz contraction of the moving kinks. [In the tunneling process, $\dot{\lambda}$ is imaginary and the factor $1/(1-\dot{\lambda}^2)^{1/2}$ is smaller than one. It actually describes a Lorentz expansion. See Ref. 16.]

After ignoring terms $O(B^2)$ or higher, we obtain the effective Lagrangian as

$$\begin{aligned}L&=\int dx\left[\tfrac{1}{2}\dot{\phi}^2-\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2-\tfrac{1}{4}g(\phi^2-c^2)^2+B(c^2\phi-\tfrac{1}{3}\phi^3+\tfrac{2}{3}c^3)\right]\\&=-2M(1-\dot{\lambda}^2)^{1/2}+2\epsilon\lambda.\end{aligned} \tag{A10}$$

The first term in (A10) denotes the Lagrangian of the kink and antikink moving with velocity $\pm\dot{\lambda}$, and the second term $2\epsilon\lambda$ describes the volume energy of the $\phi=c$ region between the kink and the antikink. The Hamiltonian associated with our system is

$$\begin{aligned}H&=\frac{\partial L}{\partial\dot{\lambda}}\dot{\lambda}-L\\&=(p^2+4M^2)^{1/2}-2\epsilon\lambda,\end{aligned} \tag{A11}$$

where

$$p\equiv\frac{\partial L}{\partial\dot{\lambda}}=\frac{2M}{(1-\dot{\lambda}^2)^{1/2}}\dot{\lambda}. \tag{A12}$$

We can identify the kinetic energy K and the potential energy V as

$$K=(p^2+4M^2)^{1/2}-2M, \tag{A13}$$

$$V=2M-2\epsilon\lambda. \tag{A14}$$

In Fig. 5, we plot V as a function of λ. As we vary λ from 0 to λ_c, $V(\lambda)$ first increases rapidly to $2M$ indicating the creation of a kink-antikink pair. $V(\lambda)$ decreases gradually to zero as λ increases such that the volume energy $-2\epsilon\lambda$ becomes important.

We now treat L and H as those of a one-dimensional quantum system, and obtain the WKB transition amplitude as $P=e^{-R}$ with

$$\begin{aligned}R&=\int_0^{\lambda_c}[4M^2-(2\epsilon\lambda)^2]^{1/2}d\lambda\\&=\pi M\lambda_c/2=\pi M^2/2\epsilon.\end{aligned} \tag{A15}$$

As a quantum-mechanical system, λ and p obey the following Heisenberg equations:

$$\dot{\lambda}=i[H,\lambda]=\frac{p}{(p^2+4M^2)^{1/2}} \tag{A16}$$

and

$$\dot{p}=i[H,p]=2\epsilon\,. \tag{A17}$$

Equation (A16) is identical to the definition of p appearing in (A12). We may express (A17) as the equation of motion for a relativistic particle moving under the influence of constant force ϵ,

$$\frac{d}{dt}\left[\frac{\dot{\lambda}}{(1-\dot{\lambda}^2)^{1/2}}\right]=\frac{\epsilon}{M}\left(=\frac{2Bc}{\mu}\right). \tag{A18}$$

The region $\lambda<\lambda_c$ is forbidden classically by energy conservation. However, vacuum tunneling can occur quantum mechanically, and it takes place precisely in the forbidden region $\lambda<\lambda_c$. The Heisenberg equations of motion (A16) and (A17) apply equally well to the tunneling process as well as to the real motions of the kink and antikink after tunneling.

We would like to show that, with a slight modification, ϕ given in (A9) obeys the full field equation (A2) if λ and $\dot{\lambda}$ obey (A16) and (A17). For simplicity, we shall treat λ and $\dot{\lambda}$ as c numbers. We expect that one can generalize the above result to accommodate the operator nature of λ and $\dot{\lambda}$.

To verify that the ϕ given in (A9) obeys the field equation, we note that the kink and the antikink solution in ϕ do not interfere with each other once they are separated by a distance $\lambda\gg 1/\mu$. Fortunately, the region $\lambda_c>\lambda\gg 1/\mu$ is exactly what we need to know for understanding the tunneling process. In this region, we can treat the kink and the antikink solution separately. We shall demonstrate in the following that both the kink and the antikink solution obey the full field equation (A2).

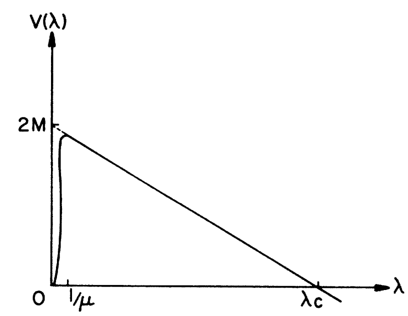

FIG. 5. The potential barrier as a function of the kink-antikink separation in a two-dimensional ϕ^4 theory. The variable λ denotes one-half the kink-antikink separation.

Consider the kink solution,

$$\phi_k = c\tanh\left[\frac{\mu}{2}\frac{x+\lambda}{(1-\dot{\lambda}^2)^{1/2}}\right]. \tag{A19}$$

We have

$$\dot{\phi}_k = \frac{c\mu}{2}\operatorname{sech}^2\left[\frac{\mu}{2}\frac{x+\lambda}{(1-\dot{\lambda}^2)^{1/2}}\right]\frac{\dot{\lambda}}{(1-\dot{\lambda}^2)^{1/2}}$$
$$+\frac{c\mu}{2}\operatorname{sech}^2\left[\frac{\mu}{2}\frac{x+\lambda}{(1-\dot{\lambda}^2)^{1/2}}\right]\frac{x+\lambda}{(1-\dot{\lambda}^2)^{3/2}}(-\dot{\lambda}\ddot{\lambda}). \tag{A20}$$

The first term in (A20) describes the c.m. motion of the kink. The second term in (A20) describes the change of shape of the kink as we accelerate it. We can remove the second term readily by including a small ($\sim O(B)$) shape-correction term in the argument of tanh in (A19). In the following, we shall ignore this term because its presence does not affect our final conclusion, but only complicates our discussion. With this simplification, we have

$$\dot{\phi}_k = \frac{c\mu}{2}\operatorname{sech}^2\left[\frac{\mu}{2}\frac{x+\lambda}{(1-\dot{\lambda}^2)^{1/2}}\right]\frac{\dot{\lambda}}{(1-\dot{\lambda}^2)^{1/2}}$$
$$+\text{terms which can be ignored}, \tag{A21}$$

$$\ddot{\phi}_k = \frac{c\mu}{2}\operatorname{sech}^2\left[\frac{\mu}{2}\frac{x+\lambda}{(1-\dot{\lambda}^2)^{1/2}}\right]\frac{d}{dt}\frac{\dot{\lambda}}{(1-\dot{\lambda}^2)^{1/2}}$$
$$-\frac{c\mu^2}{2}\operatorname{sech}^2\left[\frac{\mu}{2}\frac{x+\lambda}{(1-\dot{\lambda}^2)^{1/2}}\right]$$
$$\times\tanh\left[\frac{\mu}{2}\frac{x+\lambda}{(1-\dot{\lambda}^2)^{1/2}}\right]\frac{\dot{\lambda}^2}{(1-\dot{\lambda}^2)^2}$$
$$+\text{ terms which can be ignored}, \tag{A22}$$

where we have ignored shape-correction terms, and terms of $O(B^2)$ or higher. Hence, we have, for a single kink solution,

$$\ddot{\phi}_k - \frac{\partial^2\phi_k}{\partial x^2} + g(\phi_k{}^2 - c^2)\phi_k - B(c^2-\phi_k{}^2)$$
$$= \frac{c\mu}{2}\operatorname{sech}^2\left[\frac{\mu}{2}\frac{x+\lambda}{(1-\dot{\lambda}^2)^{1/2}}\right]\left(\frac{d}{dt}\frac{\dot{\lambda}}{(1-\dot{\lambda}^2)^{1/2}} - \frac{2Bc}{\mu}\right)$$
$$=0. \tag{A23}$$

The last relation follows from the equation of motion (A18). Of course, the result applies equally well to the antikink solution. We can now work out the field equation of ϕ associated with a kink and an antikink solution given in (A9) in a similar way, obtaining

$$\ddot{\phi} - \frac{\partial^2\phi}{\partial x^2} + g(\phi^2-c^2)\phi - B(c^2-\phi^2)$$
$$= \frac{c\mu}{2}\left\{\operatorname{sech}^2\left[\frac{\mu}{2}\frac{x+\lambda}{(1-\dot{\lambda}^2)^{1/2}}\right] + \operatorname{sech}^2\left[\frac{\mu}{2}\frac{x-\lambda}{(1-\dot{\lambda}^2)^{1/2}}\right]\right\}$$
$$\times\left[\frac{d}{dt}\frac{\dot{\lambda}}{(1-\dot{\lambda}^2)^{1/2}} - \frac{2Bc}{\mu}\right]$$
$$=0, \tag{A24}$$

provided that the Heisenberg equations of motion are satisfied. Since the Heisenberg equations of motion are satisfied for tunneling processes as well, we expect that the full field equation (A24) [or (A2)] is satisfied for the vacuum-tunneling solution.

APPENDIX B (ADDED IN PROOF): MOST PROBABLE ESCAPE PATHS AND EUCLIDEAN SOLUTIONS

As we have demonstrated in Sec. IV, a replacement of t in a Euclidean pseudoparticle solution by an arbitrary function $\lambda(t)$ gives rise to the field configuration corresponding to the maximal tunneling amplitude in Minkowski space. In other words, the most probable escape path (MPEP) $\vec{A} = \vec{A}(x, \tau(t))$ can be obtained from the Euclidean solution $\vec{A}(x,t)$ by this trivial substitution. We would like to show in this appendix that this is in fact a general result. We shall demonstrate this substitution rule in a scalar field theory. The result can be extended straightforwardly to a non-Abelian gauge theory in the $A^0=0$ gauge. Consider a scalar field theory described by

$$\mathcal{L} = \tfrac{1}{2}\dot{\phi}^2 - \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 - \mathcal{V}(\phi), \tag{B1}$$

where $\mathcal{V}(\phi)$ contains all nonderivative ϕ dependences. Let $\phi = \phi(x, \lambda(t))$ be a tunneling path, and and let E be the energy of the system. The Lagrange function along the path becomes

$$L = \tfrac{1}{2}m(\lambda)\dot{\lambda}^2 - V(\lambda) \tag{B2}$$

with

$$m = \int dx\left(\frac{\partial\phi}{\partial\lambda}\right)^2, \tag{B3}$$

$$V = \int dx\left[\frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \mathcal{V}(\phi)\right]. \tag{B4}$$

The WKB amplitude associated with (B2) is $P = e^{-R}$ with

$$R = \int d\lambda\{2m(\lambda)[V(\lambda)-E]\}^{1/2}. \tag{B5}$$

The condition for $\phi(x,\lambda(t))$ to be an MPEP is

$$\delta \int d\lambda [2m(V-E)]^{1/2} = 0\,. \tag{B6}$$

The following result holds:

Lemma. For any solution to (B6), there exists a parametrization $\tau = \tau(\lambda(t)) = \tau(t)$ such that $\phi(x,\tau)$ obeys the Euclidean solution

$$\frac{\partial^2\phi}{\partial\tau^2} + \frac{\partial^2\phi}{\partial x^2} - \frac{\partial \mathcal{V}}{\partial\phi} = 0\,. \tag{B7}$$

Conversely, if $\phi(x,\tau)$ obeys (B7), any parametrization $\tau = \tau(\lambda(t))$ gives rise to the variational result (B6).

The above conclusion is closely related to a result obtained by Coleman,[15] who obtained the above result by a formal substitution $\tau = it$ in the Euler-Lagrange equation. However, the result can be established directly without referring to any imaginary-time formulation as follows.

The variational principle implies from (B6)

$$\frac{\partial}{\partial\lambda}\left(\frac{\delta}{\delta\partial\phi/\partial\lambda}[2m(V-E)]^{1/2}\right) = \frac{\delta}{\delta\phi}[2m(V-E)]^{1/2}\,. \tag{B8}$$

Introducing a new parametrization $\tau(t) = \tau(\lambda(t))$ through

$$\frac{d\tau(\lambda)}{d\lambda} = \left(\frac{m(\lambda)}{2[V(\lambda)-E]}\right)^{1/2}, \tag{B9}$$

we have

$$\frac{\partial}{\partial\lambda}\left(\frac{\delta}{\delta\partial\phi/\partial\lambda}[2m(V-E)]^{1/2}\right) = \frac{\partial}{\partial\lambda}\left[\left(\frac{2(V-E)}{m}\right)^{1/2}\frac{\partial\phi}{\partial\lambda}\right] = \frac{\partial}{\partial\lambda}\left(\frac{\partial\phi}{\partial\tau}\right) = \frac{d\tau}{d\lambda}\,\frac{\partial^2\phi}{\partial\tau^2} \tag{B10}$$

and

$$\frac{\delta}{\delta\phi}[2m(V-E)]^{1/2} = \left(\frac{m}{2(V-E)}\right)^{1/2}\left(-\frac{\partial^2\phi}{\partial x^2} + \frac{\partial\mathcal{V}}{\partial\phi}\right) = \frac{d\tau}{d\lambda}\left(-\frac{\partial^2\phi}{\partial x^2} + \frac{\partial\mathcal{V}}{\partial\phi}\right). \tag{B11}$$

Equating (B10) and (B11), we obtain (B7).

To prove the converse, we multiply (B7) by $-\partial\phi/\partial\tau$ and integrate it over all x, obtaining

$$\frac{\partial}{\partial\tau}\int dx\left[-\frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \mathcal{V}(\phi)\right] = 0\,, \tag{B12}$$

or

$$\int dx\left[-\frac{1}{2}\left(\frac{\partial\phi}{\partial\tau}\right)^2 + \frac{1}{2}\left(\frac{\partial\phi}{\partial x}\right)^2 + \mathcal{V}(\phi)\right] \equiv -E\,, \quad \text{a constant.} \tag{B13}$$

Equation (B13) implies

$$\int dx\left(\frac{\partial\phi}{\partial\tau}\right)^2 = 2(V-E)\,. \tag{B14}$$

Thus, for an arbitrary parametrization $\tau = \tau(\lambda)$, we have

$$m(\lambda) = \left(\frac{d\tau}{d\lambda}\right)^2 \int dx\left(\frac{\partial\phi}{\partial\tau}\right)^2 = 2(V-E)\left(\frac{d\tau}{d\lambda}\right)^2,$$

or

$$\frac{d\tau}{d\lambda} = \left(\frac{m}{2(V-E)}\right)^{1/2},$$

which satisfies (B9) automatically. Now it is trivial to verify (B6).

We remark now that Eq. (B7) is identical in form to the Euclidean equation of motion for the field $\phi(x,\tau)$ with $\tau(t)$ replacing the Euclidean time. We emphasize here, however, that $\tau(t)$ [or $\lambda(t)$] is *not* Euclidean time or an analytic continuation thereof; it is a parameter which characterizes the tunneling in Minkowski space. Details of this work will appear in a separate publication.

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