Self-dual Yang-Mills fields in Minkowski space-time

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We exhibit a complex, self-dual gauge field configuration in Minkowski space-time, with the property that its real part satisfies the Yang-Mills equations of motion.

The notion of self-duality has proved very fruitful in the study of Yang-Mills gauge theories. The dual $^*F_{\mu\nu}$ of the field strength $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu+[A_\mu,A_\nu]$ is defined as $^*F_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}$. We shall generally call self-dual a field configuration where $F_{\mu\nu}$ and $^*F_{\mu\nu}$ are proportional. Because of the Bianchi identity $D_{A\mu}{}^*F_\nu{}^\mu=0$, if $F_{\mu\nu}$ is self-dual, it satisfies the Yang-Mills equations of motion.

In Euclidean four-space, the eigenvalues of $\frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}$ are ±1 and the condition of self-duality is

$$F_{\mu\nu}=\pm\tfrac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma}\,. \tag{1}$$

This set of first-order nonlinear differential equations has been analyzed in great detail since the work of Belavin, Polyakov, Schwartz, and Tyupkin,[1] where the relevance of self-dual field configurations was first recognized. A large class of solutions to Eq. (1) has been constructed,[2] their properties and physical implications have been exposed,[3] and existence theorems for more solutions have been given.[4] In a very interesting recent development, moreover, it appears that the most general solution to Eq. (1) can be constructed by methods of algebraic geometry.[5]

In Minkowski space-time the eigenvalues of $\frac{1}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}$ are $\pm i$ and the condition of self-duality, $F_{\mu\nu}=\pm i\,{}^*F_{\mu\nu}$, can be satisfied only if the field is complex. A clarification of our conventions is in order here. We represent the gauge potential A_μ and field strength $F_{\mu\nu}$ as matrices in the space of infinitesimal group generators

$$A_\mu=\sum_a A_\mu^a\frac{\sigma_a}{2i}\,, \tag{2}$$

$$F_{\mu\nu}=\sum_a F_{\mu\nu}^a\frac{\sigma_a}{2i}\,. \tag{3}$$

We shall consider an SU(2) gauge group in this paper. σ_a, $a=1,2,3$, are then the three Pauli matrices. It is apparent that the equation

$$F_{\mu\nu}=\pm\frac{i}{2}\epsilon_{\mu\nu}{}^{\rho\sigma}F_{\rho\sigma} \tag{4}$$

can be implemented only if the potentials A_μ^a are complex, or, equivalently, if the gauge group is extended to a larger noncompact gauge group, in our case SL(2,C).

The fact that in physically interesting situations one usually deals with compact gauge groups seems to make self-dual solutions in Minkowski space-time scarcely relevant, but some procedure might exist to obtain from a complex self-dual gauge field a real field, which still solves the Yang-Mills equations of motion.[6] In the Abelian case, where self-dual field configurations describe photons of definite helicity, one can trivially extract a real solution from a complex self-dual one by taking its real part. In the theory of non-Abelian Yang-Mills fields, the nonlinearity of the equations of motion in general bars this simple procedure. The experience with self-dual fields in Euclidean four-space has shown, however, that after a suitable choice of superpotential functions an effective linearization of the equations of motion occurs. What we have in mind is the possibility that, after some analogous linearization, self-dual field configurations can be superimposed to obtain real solutions of the Yang-Mills equations. Given the simpler structure and geometrical meaning of the self-duality condition, this would imply a substantial step towards the solution of the classical theory.

The purpose of the present communication is to exhibit a self-dual field configuration of high symmetry in Minkowski space-time, with the remarkable property that its real part is also a solution of the Yang-Mills field equations. The real part is gauge equivalent to the solution recently discovered by de Alfaro, Fubini, and Furlan.[7] On the other hand, the full self-dual field configuration can be considered as the analytic continuation to real time of the single-pseudoparticle solution in the Euclidean domain presented in Ref. 1. This estabilishes a new, interesting relation between the pseudoparticle and the solution of Ref. 7.

The complex self-dual solution is given by the expression

$$\vec{A} = -i\frac{f(t,x^2)}{(1-x^2)^2+4t^2} \times [2\vec{x}\times\vec{\sigma}+\vec{\sigma}(1+x^2)+2\vec{x}(\vec{x}\cdot\vec{\sigma})],$$
$$A_4 = i\frac{f(t,x^2)}{(1-x^2)^2+4t^2}2\vec{x}\cdot\vec{\sigma}t, \quad (5)$$

with

$$x^2 = t^2 - \vec{x}^2,$$
$$f(t,x^2) = 1 - \frac{2it}{1-x^2}. \quad (6)$$

The real part [i.e., the anti-Hermitian part of A_μ, according to our conventions, see Eqs. (2) and (3)] is obtained replacing f with its real part, 1.

Exhibiting the solution is less informative than a clarification of its symmetry properties and of the method by which it has been obtained. The field configuration of Eq. (5) is characterized by being self-dual and invariant up to a gauge transformation under the O(4) subgroup of the O(4, 2) group of conformal transformations. A convenient formalism to exploit this invariance has been very recently introduced by Schechter[8] and, independently, by Lüscher.[9] We summarize here the aspects of the formalism necessary to our construction (we follow the notation of Ref. 8).

One introduces coordinates

$$\vec{r} = \frac{2\vec{x}}{\sqrt{\lambda}}, \quad r_0 = \frac{1+x^2}{\sqrt{\lambda}},$$
$$R_1 = \frac{2t}{\sqrt{\lambda}}, \quad R_0 = \frac{1-x^2}{\sqrt{\lambda}}, \quad (7)$$

with

$$\lambda = (1-x^2)^2 + 4t^2. \quad (8)$$

One verifies easily that $r_\mu r^\mu = r_0{}^2 + \vec{r}^2 = 1$, $R_a R^a = R_0{}^2 + R_1{}^2 = 1$, and that the whole Minkowski space-time is mapped, two-to-one, onto the hypertorus $r_\mu r^\mu = R_a R^a = 1$. The action of the O(4) × O(2) subgroup of conformal transformations is then represented by independent rotations of r_μ and R_a.

The ordinary derivatives $\partial/\partial x^\mu$ and gauge potentials A_μ are mapped into derivatives and gauge fields tangential to the surface of the torus. These are denoted by

$$\hat{\partial}_\mu = \frac{\partial}{\partial r^\mu} - r_\mu\left(r^\nu\frac{\partial}{\partial r^\nu}\right), \quad (9a)$$

$$\hat{\Delta}_a = \frac{\partial}{\partial R^a} - R_a\left(R^b\frac{\partial}{\partial R^b}\right), \quad (9b)$$

$$\hat{a}_\mu, \quad \hat{a}_\mu r^\mu = 0, \quad (9c)$$

$$\hat{A}_a, \quad \hat{A}_a R^a = 0. \quad (9d)$$

The relation between A_μ and $\hat{a}_\mu, \hat{A}_a$ is given by

$$A_i = (r_0 + R_0)\hat{a}_i + r_i(\hat{A}_0 - \hat{a}_0),$$
$$A_4 = (r_0 + R_0)\hat{A}_1 - R_1(\hat{A}_0 - a_0), \quad (10)$$

and is identical to the relation between ∂_μ and $\hat{\partial}_\mu$, $\hat{\Delta}_a$.

From $\hat{a}_\mu$ and $\hat{A}_a$ one constructs an "electric" field

$$\hat{E}_{a\mu} = \hat{\Delta}_a\hat{a}_\mu - \hat{\partial}_\mu\hat{A}_a + [\hat{A}_a, \hat{a}_\mu], \quad (11a)$$

and a "magnetic" field

$$\hat{H}_{\mu\nu} = \hat{\partial}_\mu\hat{a}_\nu - \hat{\partial}_\nu\hat{a}_\mu - r_\mu\hat{a}_\nu + r_\nu\hat{a}_\mu + [\hat{a}_\mu, \hat{a}_\nu]. \quad (11b)$$

Both are tangential, i.e., $r^\mu\hat{H}_{\mu\nu} = r^\mu\hat{E}_{a\mu} = R^a\hat{E}_{a\mu} = 0$, and the name is derived from the fact that near the origin $\hat{E}_{1i}$ and $\hat{H}_{ij}$ are proportional to the electric F_{4i} and magnetic F_{ij} components of $F_{\mu\nu}$.

O(4)-invariant field configurations are obtained setting[8]

$$\hat{a}_\mu = if(\theta)\sigma_{\mu\nu}r^\nu,$$
$$\hat{A}_a = 0, \quad (12)$$

with $R_0 = \cos\theta$, $R_1 = \sin\theta$. As shown in Refs. 8 and 9, the Yang-Mills equations of motion reduce then to

$$f'' + 2f(f+1)(f+2) = 0 \quad (13)$$

and the solution of Ref. 7 is gauge equivalent to the field configuration obtained with $f = -1$.

In the hypertoroidal formalism the self-duality constraint can be written

$$\hat{H}_{\mu\nu} = -i\epsilon_{\mu\nu\rho\sigma}\epsilon_{ab}R^a r^\rho\hat{E}^{b\sigma}. \quad (14)$$

This is checked by noticing that near the origin ($R_0 = r_0 = 1$, $R_1 = 0$, $\vec{r} = 0$) this equation reduces to

$$\hat{H}_{ij} = -i\epsilon_{ijk}\hat{E}_{1k}, \quad (15)$$

or

$$F_{ij} = i\epsilon_{ijk}F_{k4}, \quad (16)$$

in terms of the ordinary components of the field strength $F_{\mu\nu}$, and that it is manifestly invariant under O(4) × O(2) transformations.

With the ansatz of Eq. (12) the self-duality constraint reduces to the equation

$$\frac{df}{d\theta} = -i(2f + f^2), \quad (17)$$

which is solved by

$$f(\theta) = -\frac{e^{-i(\theta-\theta_0)}}{\cos(\theta-\theta_0)}. \quad (18)$$

The real part of $f(\theta)$ is -1, precisely the value that reproduces the solution of Ref. 7.

The expressions of $\vec{A}$ and A_4 in Eq. (5) are obtained projecting back to Minkowski space-time the field given by Eq. (12) and Eq.(18) with $\theta_0 = 0$. It is clear from Eq. (12), although not manifest in Eq. (5), that the field configuration is left invariant (up to a gauge transformation) by the O(4) subgroup of the conformal group. The real part of the solution has a larger group of invariance, $O(4)\times O(2)$, but this does not extend to the complex self-dual solution.

Notice that the self-duality constraint is invariant under the full noncompact $SL(2,C)$ gauge group, whereas the definition of the real part of the field is invariant only with respect to the compact SU(2) gauge transformations. The choice of the $SL(2,C)$ gauge is therefore relevant for our construction. The appropriate choice of gauge appears quite simple in the hypertoroidal formalism which puts into evidence the symmetry properties of the solutions, but is not immediate in the ordinary formalism. [Indeed, the solution of Ref. 7, as presented in the original communication, makes use of complex fields, and an $SL(2,C)$ gauge transformation is required to make it real.[10]]

Finally, a continuation to Euclidean four-space is performed by replacing time t and R_1 with it and iR_1. Euclidean four-space is then mapped onto the hyperbolic noncompact space defined by $r_0^2 + \vec{r}^2 = 1$ and $R_0^2 - R_1^2 = 1$. Setting $\theta = i\tau$, Eqs. (18) and (12) give

$$\hat{a}_\mu = -i\frac{e^{(\tau-\tau_0)}}{\cosh(\tau-\tau_0)}\sigma_{\mu\nu}r^\nu , \tag{19}$$

which now satisfies the self-duality condition in the Euclidean domain. The comparison with the standard expression of the pseudoparticle solution is most conveniently made by using the following equations to map the x_μ space onto the r_μ, R_a space:

$$\begin{aligned} r_\mu &= \frac{x_\mu}{\sqrt{x^2}} , \\ R_0 &= \cosh\tau , \\ R_1 &= \sinh\tau , \end{aligned} \tag{20}$$

with $\tau = \frac{1}{2}\ln x^2$, $x^2 = \vec{x}^2 + x_4^2$. [Equation (20) can be thought of as obtained from Eq. (7) by replacing first t, R, with ix_4, iR_1 and then performing a conformal transformation which maps the points $x_4 = \pm 1$, $\vec{x} = 0$ into the points at ∞ and the origin.] With this parametrization, the relation between the gauge fields becomes

$$A_\mu = \frac{\hat{a}_\mu}{\sqrt{x^2}} + \frac{x_\mu}{x^2}(\hat{A}_0 R_1 + \hat{A}_1 R_0) . \tag{21}$$

Substituting from Eqs. (19) and (20) gives

$$\begin{aligned} A_\mu &= -\frac{2i}{x^2+\lambda^2}\sigma_{\mu\nu}x^\nu , \\ \lambda^2 &= e^{2\tau_0} , \end{aligned} \tag{22}$$

where one recognizes the pseudoparticle solution of Ref. 1.

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