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Lesson from a soluble model of quantum electrodynamics*

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We study the model which results from quantum electrodynamics if all photons are replaced by their longitudinal parts $-i\lambda k_\mu k_\nu/(k^2)^2$. The model is formulated in terms of a Lagrangian which features the higher-order or "grandfather" potential S which becomes a local field after quantization and from which the vector potential is derived by $A_\mu = \partial_\mu S$. The grandfather potential provides a convenient control over ultraviolet and infrared divergences, for its free propagator $-(1/2)i\lambda(\partial/\partial k^\mu)[k^\mu \ln(-a^2k^2 - i\epsilon)/(-k^2 - i\epsilon)^2]$ already contains the parameter a^{-1} with dimensions of mass which otherwise only appears in the theory, after regularization, as a renormalization mass. It produces much needed states, for we prove that there is no charged state satisfying the Gupta-Bleuler condition in $\bar{}(A,\psi)$, the closure of the space obtained from the vacuum by applying polynomials in A and ψ (the charged spinor field), but such states are easily found in $\bar{\mathfrak{D}}(S,\psi)$. As the lesson from the model, a new Lagrangian for quantum electrodynamics is exhibited, which features the grandfather or Hertz potential $\Pi^{\mu\nu} = -\Pi^{\nu\mu}$, from which the vector potential is derived by $A^\nu = \partial_\mu \Pi^{\mu\nu}$, and which is expected to yield corresponding advantages.

I. INTRODUCTION

We present a soluble four-dimensional model field theory which presents many formal similarities to quantum electrodynamics. It is renormalizable in the ultraviolet limit without being supernormalizable. It possesses similar infrared singularities in the Green's functions and the state space. It possesses a physical subspace provided by the Gupta-Bleuler condition whose quotient space by the vectors of zero norm is the space of free electrons and positrons. In fact, the model may be derived from quantum electrodynamics (QED) by replacing all photon propagators by their longitudinal parts $-i\lambda k_\mu k_\nu/(k^2)^2$. The interest of the model is not so much in solving it, which is easy enough, but rather in formulating it in such a way that the ultraviolet and infrared mechanisms become clear so that the lesson learned from the model may be applied in QED.

The characteristic feature of our formulation is that the vector potential A_μ is derived from a higher-order or "grandfather" potential S by the Lagrangian equation of motion $A_\mu = \partial_\mu S$. The advantage of the grandfather potential is that it introduces states into the theory from the start which otherwise only appear after renormalization of divergent quantities. The mechanism by which this occurs is that the free propagator of the field S, which is

$$-(16\pi^2)^{-1}\lambda \ln(-\mu^2x^2 + i\epsilon) \qquad (1.1)$$

in position space, already contains the parameter μ with dimensions of mass which otherwise only appears as a renormalization mass after regularization of divergent graphs. Thus, we expect use of this propagator to lead to a perturbation series which is finite in every order, as we have confirmed in low order. Similarly, the real infrared divergence is eliminated when the discontinuity of this propagator in moment space is used for the inner product. Among the states produced by the grandfather potential which are not otherwise available are all charged states which satisfy the Gupta-Bleuler condition. For we prove in our model[1,2] that there is no charged state satisfying the Gupta-Bleuler condition in $\bar{\mathfrak{D}}(A,\psi)$, the closure of the space obtained from the vacuum by applying polynomials in the vector potential A and the charged field ψ, but it is easily solved in $\bar{\mathfrak{D}}(S,\psi)$.

As the lesson learned from our model we exhibit an analogous Lagrangian for QED depending on the grandfather or Hertz[3] potential $\Pi^{\mu\nu} = -\Pi^{\nu\mu}$ and which yields as an equation of motion $A^\nu = \partial_\mu \Pi^{\mu\nu}$. Corresponding advantages in quantum electrodynamics are expected to accrue from this formulation.

The model presented here resembles the Schroer model[4] in that the vector potential is the gradient of a scalar field. However, in that model, the scalar field satisfies $\partial^2\varphi = 0$ and the coupling constant is dimensional, whereas in our model the coupling constant is dimensionless as in QED and the scalar field satisfies $\partial^2\partial^2 S = 0$, which describes a dipole ghost. In that respect it resembles the Froissart model[5] which contains a field that satisfies $(\partial^2 + m^2)^2 S = 0$, the massless case being more singular. The kinematics of a field satisfying $\partial^2\partial^2 S = 0$ in four dimensions recalls the kinematics of a field satisfying $\partial^2\varphi = 0$ in two dimensions, so our model bears some resemblance to two-dimensional soluble models.[6] It is closely related to a model introduced by Ferrari[7] in a study of the Higgs phenomenon, from which the present model is distinguished by coupling to the spinor field ψ, our interest being the analogy to QED.

The plan of the work is as follows. The remainder of this section is devoted to the Lagrangian which defines the model, and to the resulting equations of motion. In Sec. II the field commutators are found, and in Sec. III the Wightman and Green's functions in position space. In Sec. IV we obtain the momentum-space propagator corresponding to $-(16\pi^2)^{-1}\lambda\ln(-\mu^2x^2+i\epsilon)$, namely

$$-\tfrac{1}{2}i\lambda\,\frac{\partial}{\partial k^\mu}\left[\frac{k^\mu\ln(-a^2k^2-i\epsilon)}{(-k^2-i\epsilon)^2}\right]\ , \tag{1.2}$$

where $a=e^{\gamma-1/2}(2\mu)^{-1}$, γ is Euler's constant, and $\partial/\partial k^\mu$ is a weak derivative, and show how it yields an ultraviolet-convergent perturbation series. In Sec. V the representation space with indefinite metric is reconstructed from the Wightman functions. In the Appendix, a positive form is introduced which promotes the representation space into a Hilbert space. In Sec. VI the physical subspace is identified and defined by a Gupta-Bleuler condition. It is shown that there are no charged states satisfying the Gupta-Bleuler condition in $\mathfrak{D}(A,\psi)$ defined above and, in the Appendix, the proof is extended to the closure $\bar{\mathfrak{D}}(A,\psi)$ in the Hilbert-space topology. In the concluding section, a new Lagrangian for quantum electrodynamics is exhibited which embodies the wisdom learned from the model, namely that the vector potential should be derived from a higher-order potential.

Our model is defined by the Lagrangian density

$$L=-B\partial_\mu A^\mu+\tfrac{1}{2}\lambda B^2-I^\mu(A_\mu-\partial_\mu S)+\bar\psi(i\not\partial+e\not A-m)\psi\ , \tag{1.3}$$

which is invariant under the restricted gauge transformation of the second kind,

$$\psi\to\psi\exp(ie\Lambda),\quad S\to S+\Lambda\ ,\qquad A_\mu\to A_\mu+\partial_\mu\Lambda,\quad B\to B,\quad I_\mu\to I_\mu\ , \tag{1.4}$$

where $\Lambda(x)$ satisfies $\partial^2\Lambda=0$. For a physical interpretation, observable quantities may be defined as those that are invariant under this transformation. On varying, successively, with respect to I_μ, B, A_μ, S, and $\bar\psi$ we find

$$\partial_\mu S=A_\mu\ , \tag{1.5}$$

$$\partial\cdot A=\lambda B\ , \tag{1.6}$$

$$\partial_\mu B-I_\mu=-e\bar\psi\gamma_\mu\psi\ , \tag{1.7}$$

$$\partial_\mu I^\mu=0\ , \tag{1.8}$$

$$(i\not\partial+e\not A-m)\psi=0\ . \tag{1.9}$$

The equation of motion for ψ implies current conservation

$$\partial_\mu J^\mu=0,\quad J_\mu\equiv-e\bar\psi\gamma_\mu\psi\ . \tag{1.10}$$

The divergences of Eqs. (1.7) and (1.5) yield, respectively,

$$\partial^2B=0\ , \tag{1.11}$$

$$\partial^2S=\lambda B\ , \tag{1.12}$$

and thus

$$\partial^2\partial^2S=0\ . \tag{1.13}$$

We assume $\lambda\neq0$ since otherwise the relations become trivial.

If the fields were classical, the Dirac equation would be solved by

$$\psi=\psi_0\exp(ieS)\ , \tag{1.14}$$

where ψ_0 is a free Dirac field satisfying $(i\not\partial-m)\psi_0=0$. The solution of the quantum field theory is similar.

II. FIELD COMMUTATORS

The Lagrangian (1.3) is a first-order system of the form $\sum p\dot q-H(p,q)$. Thus we identify the canonical q variables as A^0 and S, which are conjugate, respectively, to the canonical p variables $-B$ and I^0. The remaining variables A^i and I^i, $i=1,2,3$, are determined by constraint relations of the form $\partial H/\partial q^r=0$ which are contained in the equations of motion (1.5) and (1.7). The canonical equal-time commutation relations are thus:

$$[-B(t,\vec x),A^0(t,\vec y)]=[I^0(t,\vec x),S(t,\vec y)]=-i\delta(\vec x-\vec y)\ , \tag{2.1}$$

$$[B(t,\vec x),S(t,\vec y)]=[A^0(t,\vec x),S(t,\vec y)]=0\ , \tag{2.2a}$$

$$[I^0(t,\vec x),A^0(t,\vec y)]=[I^0(t,\vec x),B(t,\vec y)]=0\ . \tag{2.2b}$$

The equal-time anticommutation relations of ψ and $\bar\psi$ are standard, and ψ and $\bar\psi$ commute at equal times with S, A^0, B, and I^0.

From the equations of motion $\dot B=J_0+I_0$ and $\partial^2B=0$, we find for all x and y

$$[B(x),S(y)]=-iD(x-y)\ . \tag{2.3}$$

Here $D(x)$ is the Pauli-Jordan function satisfying

$$\partial^2D(x)=0,\quad D(0,\vec x)=0,\quad \dot D(0,\vec x)=\delta^3(\vec x)\ , \tag{2.4}$$

$$D(x)=(2\pi)^{-1}\delta(x^2)\,\mathrm{sgn}(x^0)\ . \tag{2.5}$$

From the equation $\partial^2S=\lambda B$, we have

$$[B(x),B(y)]=0\ , \tag{2.6}$$

which shows that there will be an indefinite metric in the representation space. From the equation of motion $\dot S=A^0$ we find $[\dot S(t,\vec x),S(t,\vec y)]=0$, which together with $\partial^2 S = \lambda B$ gives, for all x and y,

$$[S(x), S(y)] = -i\lambda E(x-y) . \tag{2.7}$$

Here, $E(x)$ is the invariant function defined by

$$\partial^2 E(x) = D(x), \quad E(0,\vec{x}) = \dot{E}(0,\vec{x}) = 0 \tag{2.8a}$$

or by

$$\partial^2\partial^2 E(x) = 0 , \quad E(0,\vec{x}) = \dot{E}(0,\vec{x}) = \ddot{E}(0,\vec{x}) = 0, \quad \dddot{E}(0,\vec{x}) = \delta^3(\vec{x}) \tag{2.8b}$$

with solution

$$E(x) = 8\pi)^{-1}\theta(x^2)\,\mathrm{sgn}(x^0) . \tag{2.9}$$

The commutators of A_μ follow trivially from $A_\mu = \partial_\mu S$,

$$[A_\mu(x), S(y)] = -i\lambda\partial_\mu E(x-y) , \tag{2.10}$$

$$[A_\mu(x), B(y)] = -i\lambda\partial_\mu D(x-y) , \tag{2.11}$$

$$[A_\mu(x), A_\nu(y)] = i\lambda\partial_\mu\partial_\nu E(x-y) . \tag{2.12}$$

Finally, the commutator of these fields with ψ may also be found. From $\dot{B} = J^0 + I^0$ we find

$$[\dot{B}(t,\vec{x}), \psi(t,\vec{y})] = [J^0(t,\vec{x}), \psi(t,\vec{y})] = e\delta(\vec{x}-\vec{y})\psi(t,\vec{y}) , \tag{2.13}$$

which with $\partial^2 B = 0$ gives, for all x and y,

$$[B(x), \psi(y)] = eD(x-y)\psi(y) . \tag{2.14}$$

Similarly, from $\dot{S} = A^0$ and $\partial^2 S = \lambda B$, we find

$$[S(x), \psi(y)] = e\lambda E(x-y)\psi(y) \tag{2.15}$$

and hence also

$$[A_\mu(x), \psi(y)] = e\lambda\partial_\mu E(x-y)\psi(y) . \tag{2.16}$$

The anticommutators of ψ and ψ with each other turn out to be noncanonical, as the following section shows.

III. WIGHTMAN AND TIME-ORDERED FUNCTIONS

We wish to find the Wightman functions of a Hermitian scalar field $S(x)$ satisfying

$$\partial^2\partial^2 S(x) = 0 \tag{3.1}$$

with commutator

$$[S(x), S(y)] = -i\lambda E(x-y) , \tag{3.2}$$

$$E(x) = (8\pi)^{-1}\theta(x^2)\,\mathrm{sgn}(x^0) . \tag{3.3}$$

For this purpose $E(x)$ is decomposed into its positive- and negative-frequency parts, each of which is analytic in the past and future tubes, respectively,

$$E(x) = E^{(-)}(x) + E^{(+)}(x) , \tag{3.4}$$

$$E^{(-)}(x) = -(16\pi^2)^{-1} i \ln(-\mu^2 x^2 + i\epsilon x^0) , \tag{3.5a}$$

$$E^{(+)}(x) = [E^{(-)}(x)]^* = -E^{(-)}(-x) , \tag{3.5b}$$

with

$$\partial^2 E^{(\pm)}(x) = D^{(\pm)}(x) = \frac{\mp i}{4\pi^2}\,\frac{1}{-x^2 \mp i\epsilon x^0} , \tag{3.6}$$

$$\partial^2\partial^2 E^{(\pm)}(x) = \partial^2 D^{(\pm)}(x) = 0 . \tag{3.7}$$

The separation into positive- and negative-frequency parts is indeterminate to within a polynomial in x which is usually chosen for dimensional homogeneity. However, in the present case, no such argument precludes the presence of an arbitrary constant term in $E^{(-)}(x)$, reflected in the term $-(16\pi^2)^{-1}\ln\mu^2$. Here $\mu > 0$ is a constant with dimensions of mass, required for dimensional reasons. Its presence will be of utmost importance in the following. Corresponding to the decomposition of the commutator, the field $S(x)$ is similarly decomposed into negative-frequency (annihilation) and positive-frequency (creation) parts,

$$S(x) = S^{(-)}(x) + S^{(+)}(x) , \tag{3.8}$$

$$S^{(+)}(x) = [S^{(-)}(x)]^\dagger \tag{3.9}$$

with commutators

$$[S^{(-)}(x), S^{(+)}(y)] = -i\lambda E^{(-)}(x-y) , \tag{3.10a}$$

$$[S^{(-)}(x), S^{(-)}(y)] = [S^{(+)}(x), S^{(+)}(y)] = 0 . \tag{3.10b}$$

Let the vacuum vector Ω be defined as the vector, unique to within a phase, satisfying

$$S^{(-)}(x)\Omega = 0 , \tag{3.11}$$

$$\langle\Omega, \Omega\rangle = 1 . \tag{3.12}$$

The gauge invariance of the Lagrangian under the transformation $S \to S + \Lambda$, Eq. (1.4), is spontaneously broken by the existence of such a vacuum state.[8] In particular, a shift of S by a constant Λ corresponds to a change in the dimensional parameter μ, introduced in the last paragraph, as will be clear from an inspection of the Wightman functions.

In fact, the generic Wightman function $\langle\Omega, S(x_1)\cdots S(x_n)\Omega\rangle$ is easily calculated in the usual way by commuting all the $S^{(-)}(x)$ to the right and all the $S^{(+)}(x)$ to the left. The result is the Wightman functions of the generalized free field with 2-point function

$$W(x-y) \equiv \langle\Omega, S(x)S(y)\Omega\rangle = -i\lambda E^{(-)}(x-y) , \tag{3.13}$$

$$W(x) = -(16\pi^2)^{-1}\lambda \ln(-\mu^2 x^2 + i\epsilon x^0) \tag{3.14}$$

by Eqs. (3.10) and (3.5a).

For the time-ordered 2-point function, or Green's function,

$$T(x-y) \equiv \langle\Omega, T[S(x)S(y)]\,\Omega\rangle \tag{3.15}$$

one has

$$T(x) = -i\lambda E_c(x) \ , \tag{3.16}$$

where $E_c(x)$ is the causal function

$$E_c(x) = \theta(x^0)E^{(-)}(x) + \theta(-x^0)E^{(+)}(x) \ , \tag{3.17a}$$

$$E_c(x) = -(16\pi^2)^{-1} i \ln(-\mu^2 x^2 + i\epsilon) \ , \tag{3.17b}$$

which satisfies

$$\partial^2 E_c(x) = D_c(x) \equiv \frac{i}{4\pi^2} \frac{1}{-x^2 + i\epsilon} \ , \tag{3.18}$$

$$\partial^2\partial^2 E_c(x) = \partial^2 D_c(x) = \delta^4(x) \ . \tag{3.19}$$

Thus we have

$$T(x) = -(16\pi^2)^{-1}\lambda \ln(-\mu^2 x^2 + i\epsilon) \ , \tag{3.20}$$

$$\partial^2\partial^2 T(x) = -i\lambda\delta^4(x) \ . \tag{3.21}$$

The Wightman and Green's functions for the fields A_μ and B are easily obtained from $A_\mu = \partial_\mu S$, $B = \lambda^{-1}\partial\cdot A = \lambda^{-1}\partial^2 S$.

The commutators of S, A_μ, and B with the charged field ψ, Eqs. (2.14)–(2.16), are obviously satisfied by posing

$$\psi(x) = \psi_0(x) \!:\! \exp[ieS(x)] \!:\ , \tag{3.22}$$

where ψ_0 is the canonical free Dirac field of mass m that commutes with S, and

$$:\!\exp[ieS(x)]\!: \ \equiv \exp[ieS^{(+)}(x)] \exp[ieS^{(-)}(x)] \ . \tag{3.23}$$

With this definition ψ commutes with itself and with S at spacelike separation, so it is a local field. It satisfies

$$(i\partial\!\!\!/ - m)\psi = -e\!:\!A\!\!\!/\psi\!: \ \equiv -e(A\!\!\!/^{(+)}\psi + \psi A\!\!\!/^{(-)}) \ , \tag{3.24}$$

which replaces the corresponding classical equation of motion.

All vacuum expectation values of the fields S, A_μ, and ψ may be calculated using formula (3.22). In particular for the 2-point functions, one has

$$\langle\Omega, \psi(x)\bar\psi(0)\Omega\rangle = (-\mu^2 x^2 + i\epsilon x^0)^{-(\alpha\lambda/4\pi)} \langle\Omega, \psi_0(x)\bar\psi_0(0)\Omega\rangle \ , \tag{3.25}$$

$$\langle\Omega, T[\psi(x)\bar\psi(0)]\,\Omega\rangle = (-\mu^2 x^2 + i\epsilon)^{-(\alpha\lambda/4\pi)} \langle\Omega, T[\psi_0(x)\bar\psi_0(0)]\Omega\rangle \ , \tag{3.26}$$

where $\alpha = e^2/4\pi$ is the fine-structure constant.

We see that the charged field ψ has picked up an anomalous dimension $\alpha\lambda/4\pi$. The engineering dimensions are maintained by the normalization constant $\mu^{\alpha\lambda/4\pi}$. Thus the dimensional constant μ which appears in the 2-point function of the free field S as an additive constant of the form $\ln\mu^2$ shows up as a multiplicative normalization constant of the field ψ. If the Green's functions had been obtained instead by summing renormalized perturbation theory, the dimensional constant would have arisen from a renormalization of the electron propagator, at some renormalization mass proportional to μ. This suggests that the ultraviolet divergences will be removed from perturbation theory provided that, in the expression for the photon propagator

$$D_{\mu\nu}(x) = -\partial_\mu\partial_\nu T(x) = (16\pi^2)^{-1}\lambda\partial_\mu\partial_\nu \ln(-\mu^2 x^2 + i\epsilon) \ , \tag{3.27}$$

the derivatives are understood to be weak derivatives. In other words they do not act on the logarithm considered as a function, for then the dependence on μ would disappear. Instead, a partial integration is understood. However, perturbation theory is more familiar in momentum space, to which we now turn.

IV. PROPAGATOR IN MOMENTUM SPACE AND FINITE PERTURBATION THEORY

The propagator $G(k)$ of the field S is introduced according to

$$T(x) = (2\pi)^{-4} \int e^{-ik\cdot x} G(k) d^4k \ . \tag{4.1}$$

Because $T(x)$ satisfies $\partial^2\partial^2 T(x) = -i\lambda\delta^4(x)$, $\partial^2 T(x) = -i\lambda D_c(x)$, we see that $G(k)$ is a solution of

$$k^2 G(k) = \frac{i\lambda}{-k^2 - i\epsilon} \ , \tag{4.2}$$

$$k^2 k^2 G(k) = -i\lambda \ . \tag{4.3}$$

The error which must at all costs be avoided is to conclude from this that $G(k)$ is given by

$$\frac{-i\lambda}{(-k^2 - i\epsilon)^2} \ . \tag{4.4}$$

This expression is not a distribution since $\int d^4k(-k^2 - i\epsilon)^{-2}$ is infrared divergent, and furthermore, the dimensional parameter μ which appears in $T(x)$ is absent. If one attempts to calculate $G(k)$ directly from the Fourier inversion formula

$$G(k) = -(16\pi^2)^{-1}\lambda \int e^{ik\cdot x} \ln(-\mu^2 x^2 + i\epsilon) \tag{4.5}$$

one encounters a divergence because, considered as a function, $\ln(-\mu^2 x^2 + i\epsilon)$ does not possess a Fourier transform, as will shortly be apparent. However, considered as a distribution, its Fourier transform is guaranteed to exist.

It is convenient to express $G(k)$ as a derivative of another distribution which is less singular at $k = 0$. For this purpose we write

$$T(x) = -x^2 \frac{T(x)}{-x^2 - i\epsilon}$$

so

$$G(k)=\partial_k{}^2H(k)\ , \tag{4.6}$$

$$H(k)=\frac{-\lambda}{16\pi^2}\int e^{ik\cdot x}\,\frac{\ln(-\mu^2x^2+i\epsilon)}{-x^2+i\epsilon}\,d^4x\ . \tag{4.7}$$

In evaluating this integral it is helpful to put

$$\frac{\ln(-\mu^2x^2+i\epsilon)}{-\mu^2x^2+i\epsilon}=\frac{\ln(\epsilon+i\mu^2x^2)+i\pi/2}{i(\epsilon+i\mu^2x^2)}$$

and to use the formula

$$\frac{\ln ab}{a}=-\int_0^\infty ds\,e^{-as}\ln(e^\gamma s/b) \tag{4.8}$$

for $a=\epsilon+i\mu^2x^2$, where $\gamma=-\int_0^\infty ds\,e^{-s}\ln s$ is Euler's constant. The integration over d^4x may then be effected by Gaussian quadrature with the result

$$H(k)=\frac{-i\lambda}{4}\,\frac{\ln[e^{2\gamma}(-k^2-i\epsilon)/4\mu^2]}{-k^2-i\epsilon}\ . \tag{4.9}$$

We thus find the desired propagator

$$G(k)=\frac{-i\lambda}{4}\,\partial_k{}^2\left\{\frac{\ln[e^{2\gamma}(-k^2-i\epsilon)/4\mu^2]}{-k^2-i\epsilon}\right\}\ , \tag{4.10}$$

$$G(k)=-\frac{1}{2}i\lambda\frac{\partial}{\partial k^\mu}\left[\frac{k^\mu\ln(-a^2k^2-i\epsilon)}{(-k^2-i\epsilon)^2}\right]\ . \tag{4.11a}$$

The constant

$$a=e^{\gamma-1/2}(2\mu)^{-1} \tag{4.11b}$$

has dimensions of length.

If $G(k)$ were a function, the differentiation could be effected with the result $-i\lambda(-k^2-i\epsilon)^{-2}$, which, as we have seen, is the wrong answer. Instead it is a distribution and $\partial/\partial k^\mu$ is a weak derivative which means that for a test function $f(k)$

$$\int G(k)f(k)d^4k\equiv\tfrac{1}{2}i\lambda\int d^4k\,\frac{\ln(-a^2k^2-i\epsilon)}{(-k^2-i\epsilon)^2}\times k^\mu\frac{\partial}{\partial k^\mu}f(k)\ . \tag{4.12}$$

The extra power of momentum which appears explicitly eliminates the divergence at $k=0$.

We may now understand how the ultraviolet divergences disappear from perturbation theory when the appropriate free photon propagator

$$D_{\mu\nu}(k)=k_\mu k_\nu G(k) \tag{4.13}$$

is used. It appears as a factor in the integrand of Feynman integrals,

$$I=\int d^4k\,D^{\mu\nu}(k)R_{\mu\nu}(k)\ . \tag{4.14}$$

If the remaining factor $R_{\mu\nu}(k)$ were in fact a test function, one would find by partial integration that $k_\mu k_\nu G(k)$ gets replaced by $-i\lambda k_\mu k_\nu(-k^2-i\epsilon)^{-2}$ because of the extra powers of k. However, in general, $R_{\mu\nu}(k)$ is not a test function, which is the basic reason why a renormalization prescription is conventionally required. Let it instead be left in the form

$$I=\tfrac{1}{2}i\lambda\int d^4k\,\frac{\ln(-a^2k^2-i\epsilon)}{(-k^2-i\epsilon)^2}\times k^\lambda\frac{\partial}{\partial k^\lambda}[k^\mu k^\nu R_{\mu\nu}(k)]\ . \tag{4.15}$$

Observe that the operator $k^\lambda\partial/\partial k^\lambda$ annihilates all terms of degree zero in k, which are the logarithmically divergent ones, whether in the ultraviolet or the infrared. This in fact eliminates all divergences which appear in calculating Green's functions because the photon propagator is longitudinal, so electron loop insertions and electron mass renormalization are automatically zero. We have verified this by explicit calculation to order e^2 and by superficial inspection to order e^4. Feynman integrals involving the propagator (4.15) are conveniently effected using the exponentiation formula (4.8) and others derived from it, after which integration over d^4k may be effected convariantly by Gaussian quadrature. Alternately, the logarithm may be represented by $\ln a=\lim_{\nu\to 0}(\partial/\partial\nu)a^\nu$, and integrating as in analytic regularization.

V. RECONSTRUCTION OF THE REPRESENTATION SPACE

The problem at hand is to reconstruct the representation space of the field S from its Wightman functions. Because the commutator $[S(x),S(y)]$ is a c number, the representation space $\mathfrak{s}_\gamma$ of the field S breaks up into orthogonal subspaces $\mathfrak{s}_\gamma^{(n)}$, labeled by "photon" number $n=0,1,2,\ldots$. The n-photon sector is spanned by vectors of the form $S^{(+)}(f_1)\cdots S^{(+)}(f_n)\Omega$, where $S^{(+)}(f)=\int S^{(+)}(x)f(x)d^4x$, and $f(x)$ is a test function $f\in\mathcal{S}(R^4)$. Thus, the problem reduces to the reconstruction of the one-photon subspace $\mathfrak{s}_\gamma^{(1)}$ consisting of vectors of the form $S^{(+)}(f)\Omega=S(f)\Omega$.

The inner product of two such vectors $\langle S(f)\Omega,S(g)\Omega\rangle$ defines a Hermitian symmetric form $\langle f,g\rangle=\langle g,f\rangle^*$, on the space of test functions whose kernel is the 2-point Wightman function

$$\langle f,g\rangle\equiv\langle S(f)\Omega,S(g)\Omega\rangle\ , \tag{5.1}$$

$$\langle f,g\rangle=\int d^4x\,d^4y\,f^*(x)W(x-y)g(y)\ . \tag{5.2}$$

When this form is non-negative, $\langle f,f\rangle\geq 0$ for all f, the reconstruction is well described by Streater and Wightman.[9] However, in the present case, the form is indefinite, namely for some f, $\langle f,f\rangle<0$, which calls for a slight modification of the reconstruction principle. Before proceeding to a concrete description in terms of wave functions, it is worthwhile to pause and consider the general method for reconstruction of an indefinite metric space from an indefinite form.[10] If the form were non-negative, it could be used to define a norm $\|f\| = (\langle f,f\rangle)^{1/2}$. In this case the vector designated by $S(f)\Omega$ is identified with the equivalence class of test functions $[f]$ which differ from f by a test function of zero norm. Observe that when the inner product is non-negative the Schwartz inequality holds, $\langle f,f\rangle\langle g,g\rangle \geq |\langle f,g\rangle|^2$, and f is of zero length if and only if it is orthogonal to every vector. (This is in fact the reason why the test functions of zero length form a linear space and why the inner product is nondegenerate on the space of equivalence classes.) We adopt this as the basic principle for reconstruction of the representation space when the Wightman functions define an indefinite form, namely, the vector $S(f)\Omega$ is identified with the equivalence class of test functions $[f]$ that differ from f by a test function which is orthogonal to every test function,

$$S(f)\Omega = [f]. \tag{5.3}$$

On the space of equivalence classes the form is nondegenerate by definition or, in other words, the form

$$\langle [f],[g]\rangle \equiv \langle f,g\rangle \tag{5.4}$$

vanishes for all $[g]$ if and only if $[f]$ is the zero-vector $[f]=[0]=0$. The class of test functions orthogonal to every test function forms a linear closed subspace $\mathcal{K}$ of the space of test functions $\mathcal{S}(R^4)$. The one-photon space $\mathcal{G}_\gamma^{(1)}$ is the quotient space

$$\mathcal{G}_\gamma^{(1)} = \mathcal{S}(R^4)/\mathcal{K}. \tag{5.5a}$$

This space has a natural topology induced by the projection $\mathcal{P}$ of $\mathcal{S}(R^4)$ onto $\mathcal{G}_\gamma^{(1)}$ defined by $f \to [f]$. Namely, let A be any set in $\mathcal{G}_\gamma^{(1)}$, let $B = \mathcal{P}^{-1}(A)$ be its inverse image in $\mathcal{S}(R^4)$ with closure $\bar{B}$, then the closure of A is defined by

$$\bar{A} = \mathcal{P}(\bar{B}) = \mathcal{P}(\overline{(\mathcal{P}^{-1}A)}). \tag{5.5b}$$

We call this the induced test-function topology. As a matter of mathematical convenience, the indefinite-metric space may be completed into a Hilbert space. This is done in the Appendix, where a positive form (f,g) is defined which bounds the indefinite form $\langle f,g\rangle$ according to

$$|\langle f,g\rangle| \leq \|f\|\,\|g\|, \quad \|f\| = (f,f)^{1/2}.$$

We shall now obtain a concrete description of the one-photon space in terms of on-shell wave functions. Because the Wightman function satisfies $\partial^2\partial^2 W(x)=0$, the structure of the inner product $\langle f,g\rangle = \int d^4x f^*(x) W(x-y) g(y) d^4y$ is apparent in momentum space. Put

$$W(x) = (2\pi)^{-4}\int \tilde{W}(k) e^{-ik\cdot x} d^4k, \tag{5.6}$$

$$\hat{f}(k) = (2\pi)^{-3/2}\int e^{ik\cdot x} f(x) d^4x, \tag{5.7}$$

which gives

$$\langle f,g\rangle = (2\pi)^{-1}\int d^4k\, \tilde{W}(k)\hat{f}^*(k)\hat{g}(k). \tag{5.8}$$

Instead of calculating $\tilde{W}(k)$ directly by Fourier transform, we use

$$\tilde{W}(k) = G(k)_+ - G(k)_-, \tag{5.9}$$

where the discontinuity is evaluated along the right-hand axis of the k^0 plane. Insertion of the identity

$$\ln a = \lim_{\nu\to 0_+} \frac{\partial}{\partial\nu} a^\nu \tag{5.10}$$

into formula (4.10) yields

$$G(k) = \frac{-i\lambda}{4} a^2 e \frac{\partial}{\partial\nu} \partial_k{}^2(-a^2ek^2 - i\epsilon)^{-1+\nu}\Big|_{\nu=0_+}, \tag{5.11}$$

$$G(k) = \frac{i\lambda}{16}\frac{\partial}{\partial\nu}\partial_k{}^2\partial_k{}^2\left[\frac{1}{\nu(1+\nu)}(-a^2e^2k^2 - i\epsilon)^\nu\right]\Big|_{\nu=0_+}, \tag{5.12}$$

which gives

$$\tilde{W}(k) = \frac{\lambda}{8}\frac{\partial}{\partial\nu}\partial_k{}^2\partial_k{}^2\left[\frac{\sin\pi\nu}{\nu(1+\nu)}(a^2ek^2)^\nu\,\theta(k^0-\omega)\right]\Big|_{\nu=0_+}, \tag{5.13}$$

where $\omega = |\vec{k}|$, or

$$\tilde{W}(k) = 8^{-1}\pi\lambda\,\partial_k{}^2\partial_k{}^2[\theta(k^0-\omega)\ln a^2k^2]. \tag{5.14}$$

This is a well-defined Lorentz-invariant distribution, but the fact that it is a solution of

$$k^2k^2\tilde{W}(k) = 0 \tag{5.15}$$

and thus has its support on the light cone is not manifest. For this purpose some manipulations are required with the result that, for a test function $F(k)$,

$$\int d^4k\,\tilde{W}(k)F(k)$$
$$= -2\pi\lambda\int d^4k\frac{\delta(k^0-\omega)}{(2\omega)^2}\times\left[\ln(2\omega a)\left(\frac{\partial F}{\partial k^0}+\frac{\partial F}{\partial\omega}\right)+\frac{1}{2}\left(\frac{\partial F}{\partial k^0}-\frac{\partial F}{\partial\omega}\right)\right]. \tag{5.16}$$

The support of $\tilde{W}(k)$ is now clear. On putting $F = k^2G$, we verify

$$k^2\tilde{W}(k) = -2\pi\lambda(2\omega)^{-1}\delta(k^0-\omega). \tag{5.17}$$

The inner product, Eq. (5.8), is now expressible in terms of the on-shell quantities,

$$f^1(\vec{k}) \equiv \hat{f}(k^0, \vec{k})|_{k^0=\omega} , \qquad (5.18a)$$

$$f^0(\vec{k}) \equiv \left[\frac{\partial \hat{f}(k^0, \vec{k})}{\partial k^0} - \frac{\partial \hat{f}(k^0, \vec{k})}{\partial \omega}\right]\Bigg|_{k^0=\omega} , \qquad (5.18b)$$

by

$$\langle f, g\rangle = -\lambda \int \frac{d^3k}{(2\omega)^2}\left\{\ln 2\omega a \frac{\partial}{\partial \omega}[f^{1*}(\vec{k})g^1(\vec{k})] + \tfrac{1}{2}f^{1*}(\vec{k})g^0(\vec{k}) + \tfrac{1}{2}f^{0*}(\vec{k})g^1(\vec{k})\right\} . \qquad (5.19)$$

Here we have used the fact that $\partial/\partial k^0 + \partial/\partial\omega$ is the tangential derivative along the light cone given by

$$\left[\frac{\partial \hat{f}(k^0, \vec{k})}{\partial k^0} + \frac{\partial \hat{f}(k^0, \vec{k})}{\partial \omega}\right]_{k^0=\omega} = \frac{\partial f^1(\vec{k})}{\partial \omega} . \qquad (5.20)$$

A convenient form of the inner product is obtained by integrating the last pair of terms by parts on ω,

$$\langle f, g\rangle = -\tfrac{1}{2}\lambda \int \frac{d^3k}{(2\omega)^2}\ln(2a\omega) \times \frac{\partial}{\partial\omega}[f^{1*}(\vec{k})g^2(\vec{k}) + f^{2*}(\vec{k})g^1(\vec{k})] . \qquad (5.21)$$

Here we have introduced a convenient pair of wave functions

$$f^1(\vec{k}) = \hat{f}(k^0, \vec{k})|_{k^0=\omega} , \qquad (5.22a)$$

$$f^2(\vec{k}) = \left(-k^0\frac{\partial}{\partial k^0} + \sum_{i=1}^{3} k^i \frac{\partial}{\partial k^i} + 1\right)\hat{f}(k^0, \vec{k})\Bigg|_{k^0=\omega} . \qquad (5.22b)$$

It is easy to verify that the form is nondegenerate on this pair of wave functions, so we write, for the unique vector associated with the test function f,

$$S(f) = [f] = (f^1(\vec{k}), f^2(\vec{k})) . \qquad (5.23)$$

In other words, the equivalence class of test functions consists of those whose Fourier transforms coincide on the future light cone, together with a nontangential derivative there, and each equivalence class is conveniently represented by a pair of wave functions which are their common values there.

The off-diagonal form of the inner product is characteristic of dipole ghost states and occurs also in finite-dimensional indefinite-metric spaces[11] where the equation $A^2v = 0$ for symmetric A does not imply $Av = 0$.

The necessity for two complex wave functions arises because the generic classical solution to $\partial^2\partial^2 S = 0$ is determined by 4 real functions at $t = 0$: $S(0, \vec{x})$, $\dot{S}(0, \vec{x})$, $\ddot{S}(0, \vec{x})$, $\dddot{S}(0, \vec{x})$; whereas for the wave equation $\partial^2\varphi = 0$ the one complex wave function is determined by two real functions at $t = 0$: $\varphi(0, \vec{x})$, $\dot{\varphi}(0, \vec{x})$. The two wave functions represent the two nonphysical degrees of freedom of the photon.

The Poincaré transformation laws of $f^1(\vec{k})$ and $f^2(\vec{k})$ are easily obtained. In particular, for the translations we find

$$f_a^1(\vec{k}) = e^{-ik\cdot a} f^1(\vec{k}) , \qquad (5.24a)$$

$$f_a^2(\vec{k}) = e^{-ik\cdot a}[f^2(\vec{k}) + i(\omega a^0 + \vec{k}\cdot\vec{a})f^1(\vec{k})] , \qquad (5.24b)$$

$k^\mu = (\omega, \vec{k})$, with an analogous form for the homogeneous Lorentz transformations. This triangular representation possesses an invariant subspace defined by

$$f^1(\vec{k}) = 0 \qquad (5.25)$$

in which the inner product vanishes identically.

From the one-photon space $\mathcal{G}_\gamma^{(1)}$, the Fock construction yields the many-photon space $\mathcal{G}_\gamma$. We introduce the notation

$$\int dk\, \varphi^*(\vec{k})\psi(\vec{k}) \equiv -\tfrac{1}{2}\lambda \int \frac{d^3k}{(2\omega)^2}\ln 2a\omega \frac{\partial}{\partial\omega}[\varphi^*(\vec{k})\psi(\vec{k})] \qquad (5.26)$$

to facilitate writing the inner product (5.21) which becomes simply

$$\langle f, g\rangle = \int dk [f^{1*}(\vec{k})g^2(\vec{k}) + f^{2*}(\vec{k})g^1(\vec{k})] . \qquad (5.27)$$

Creation and annihilation operators, $a_i^\dagger(\varphi)$ and $a_i(\varphi)$, $i = 1, 2$, are introduced which satisfy

$$[a_1(\varphi), a_1^\dagger(\psi)] = [a_2(\varphi), a_2^\dagger(\psi)] = 0 , \qquad (5.28a)$$

$$[a_1(\varphi), a_2^\dagger(\psi)] = [a_2(\varphi), a_1^\dagger(\psi)] = \int dk\, \varphi^*(\vec{k})\psi(\vec{k}) , \qquad (5.28b)$$

where the integral on the right-hand side has just been given the special meaning (5.26). Obviously, the commutator could be diagonalized by suitable linear combination, but the off-diagonal form will be kept because of the importance of the invariant subspace (5.25). The vacuum vector Ω is defined by

$$a_i(\varphi)\Omega = 0, \quad i = 1, 2 , \qquad (5.29)$$

and the many-photon space $\mathcal{G}_\gamma$ is spanned by applying creation operators to the vacuum.[12]

The field $S(x)$ is represented by

$$\int d^4x\, S(x) f(x) = S(f) = \sum_{i=1}^{2}[a_i^\dagger(f^i) + a_i(f^i)] , \qquad (5.30)$$

where the f^i are defined in Eq. (5.22). The action of the field B is also of interest. From $B=\lambda^{-1}\partial^2 S$ we find $B(g)=\lambda^{-1}S(\partial^2 g)$, so with $f=\lambda^{-1}\partial^2 g$ we have $\hat{f}(k)=-\lambda^{-1}k^2\hat{g}(k)$ and so we obtain, by Eqs. (5.22),

$$f^1(\vec{k})=0,\quad f^2(\vec{k})=\lambda^{-1}(2\omega)^2 g^1(\vec{k})\ , \tag{5.31}$$

$$B(g)=\lambda^{-1}[a_2^\dagger(4\omega^2 g^1)+a_2(4\omega^2 g^1)]\ . \tag{5.32}$$

The space $\mathcal{G}$, on which our model quantum electrodynamics is defined, is the product

$$\mathcal{G}=\mathcal{G}_\gamma\otimes\mathcal{L}_e \tag{5.33}$$

of the photon space $\mathcal{G}_\gamma$ just constructed with the space $\mathcal{L}_e$ of free electrons and positrons on which the free Dirac field ψ_0 is represented in a standard way. The charged field acts on $\mathcal{G}$ according to $\psi(x)=\psi_0(x)\colon\exp[ieS(x)]\colon$. Thus we have a representation space, with nondegenerate but indefinite inner product, on which all the fields of the theory are defined.

VI. PHYSICAL STATES

Physical subspaces are those whose vectors Φ are positive on all the observables θ

$$\langle\theta\Phi,\theta\Phi\rangle\geq 0\ . \tag{6.1}$$

We take observables to be the quantities which are invariant under the gauge transformation (1.4) whose generator is $G(\Lambda)=\int_{t=0}\Lambda(x)\overleftrightarrow{\partial}_0\dot{B}(x)d^3x$, as may be verified from the commutators of Sec. II. It follows that the observables satisfy[13,14]

$$[B(x),\theta]=0 \tag{6.2}$$

and hence, from Eq. (5.32) and the off-diagonal commutation relations, that they are independent of a_1 and $a_1^\dagger$,

$$\theta=\theta(a_2^\dagger,a_2)\ , \tag{6.3}$$

where dependence on electron-positron variables is suppressed.

We will verify that the Gupta-Bleuler subsidiary condition

$$B^{(-)}(x)\Phi=0 \tag{6.4}$$

provides a physical subspace $\mathcal{G}_0$, which, furthermore, is isomorphic, modulo vectors of zero norm, to the space of free electrons and positrons

$$\mathcal{G}_0\equiv\mathcal{L}_e\ . \tag{6.5}$$

The fact that photon variables disappear from this physical space is an expression of the triviality of gauge coupling. Note that, because $[B^{(-)}(x),\theta]=0$, the subspace $\mathcal{G}_0$ is invariant under the action of the observables, so it will be positive on the observables provided only that it is a subspace of non-negative norm. In terms of annihilation operators, the Gupta-Bleuler condition reads

$$a_2(\varphi)\Psi=0\ , \tag{6.6}$$

which must hold for all wave functions φ. Here a_2 plays the role of $k\cdot a$ in free-field QED. Because of the off-diagonal commutation relations (5.28), the generic solution to this condition is

$$\Psi=F(a_2^\dagger)\Omega\ , \tag{6.7}$$

where $F(a_2^\dagger)$ is a power series in $a_2^\dagger$ whose coefficients are power series in electron and positron creation operators. Again because $[a_2(\varphi),a_2^\dagger(x)]=0$, we have

$$\langle\Psi,\Psi\rangle=\langle F(0)\Omega,F(0)\Omega\rangle\geq 0\ , \tag{6.8}$$

where $F(0)$ is the zeroth term in the expansion of $F(a_2^\dagger)$. This expression is non-negative because $F(0)\Omega$ is a pure electron-positron state $F(0)\Omega\in\mathcal{L}_e$. Hence, $\mathcal{G}_0$ is a space of non-negative metric and, on taking its quotient by the vectors of zero norm, i.e., $F(a_2^\dagger)\Omega\to F(0)\Omega$, the electron-positron space $\mathcal{L}_e$ results.

An important open question in QED[1,2] is whether there exist solutions of the Gupta-Bleuler condition with nonzero value of the total electric charge $Q=-e\times$ (number of electrons − number of positrons). In QED as presently formulated, the analog of the grandfather potential S does not occur and the full representation space is the closure $\overline{\mathfrak{D}}(A,\psi)$ of the space obtained from the vacuum by applying polynomials in A and ψ smeared with test functions in $\mathcal{S}$. We shall show in the present model that there exist no states with definite electric charge $Q\neq 0$ in $\overline{\mathfrak{D}}(A,\psi)$ which satisfy the Gupta-Bleuler condition.

The annihilation operators $a_i(\vec{k})$, $i=1,2$, depending on a definite momentum $\vec{k}$ are defined by

$$a_1(\vec{k})\Omega=a_2(\vec{k})\Omega=0\ , \tag{6.9a}$$

$$[a_1(\vec{k}),a_2^\dagger(\varphi)]=[a_2(\vec{k}),a_1^\dagger(\varphi)]=\varphi(\vec{k})\ , \tag{6.9b}$$

the other commutators being zero. [The Hermitian-conjugate quantity $a_i^\dagger(\vec{k})$ is not an operator but merely an operator-valued distribution.] From

$$[a_2(\vec{k}),S(f)]=f^1(\vec{k})=\hat{f}(k)|_{k^0=\omega} \tag{6.10}$$

we find easily, using $A_\mu=\partial_\mu S$, $\psi=\psi_0\colon\exp(ieS)\colon$,

$$\left[a_2(\vec{k}),\ \int A_\mu(x)j^\mu(x)d^4x\right]=\frac{ik_\mu}{(2\pi)^{3/2}}\int d^4x\,e^{ik\cdot x}j^\mu(x)\ , \tag{6.11a}$$

$$\left[a_2(\vec{k}),\ \int\overline{\chi}(x)\psi(x)d^4x\right]=(2\pi)^{-3/2}ie\int d^4x\,e^{ik\cdot x}\,\overline{\chi}(x)\psi(x)\ , \tag{6.11b}$$

where j^μ, $\bar{\chi}$ are vector and spinor test functions. In particular, the zero-frequency operator satisfies

$$\left[a_2(0), \int A_\mu(x) j^\mu(x) d^4x\right] = 0 , \qquad (6.12a)$$

$$\left[a_2(0), \int \bar{\chi}(x)\psi(x) d^4x\right] = (2\pi)^{-3/2} ie \int d^4x \bar{\chi}(x)\psi(x) , \qquad (6.12b)$$

which, together with $a_2(0)\Omega = 0$, shows that

$$a_2(0)\Psi = (2\pi)^{-3/2} iQ\Psi, \quad \text{all } \Psi \in \mathfrak{D}(A,\psi) . \qquad (6.13)$$

On the other hand, the Gupta-Bleuler subspace $\mathfrak{g}_0 \subset \mathfrak{g}$ is the eigenspace defined by $a_2(\vec{k})\Psi = 0$. Hence, no vector Ψ in $\mathfrak{D}(A,\psi)$ with definite charge $Q = Q' \neq 0$ satisfies the Gupta-Bleuler condition since the wave functions of $\mathfrak{D}(A,\psi)$ are all continuous. (A detailed calculation is presented at the end of the Appendix.) The same conclusions hold in the induced test-function topology defined in Eq. (5.5b), for in this topology all wave functions are continuous, being restrictions to the future light cone of test functions. More surprising is the fact that the conclusion still holds in $\bar{\mathfrak{D}}(A,\psi)$, the completion of $\mathfrak{D}(A,\psi)$ in a Hilbert-space topology, as is shown in the Appendix. The reason is that to ensure continuity of the inner product $\langle f, g\rangle$, the Hilbert-space norm $\|f\|$ is required to dominate it $\|f\|\,\|g\| \geq |\langle f, g\rangle|$. Because $\langle f, g\rangle$ is very singular at $\omega = 0$, the norm $\|f\|$ is correspondingly restrictive there, so no Cauchy sequence of vectors $\Psi^n \in \mathfrak{D}(A,\psi)$ satisfying $a_2(0)\Psi^n = (2\pi)^{-3/2} iQ'\Psi^n$, $Q' \neq 0$, converges to one satisfying $a_2(\varphi)\Psi = 0$ for all $\varphi \in \mathcal{K}_\gamma^{(1)}$.

It seems likely that this result holds also in QED, namely that there are no charged states in $\bar{\mathfrak{D}}(A,\psi)$ which satisfy the Gupta-Bleuler condition, although a proof has only been given for states obtained from the vacuum by local operators.[1] To get around this difficulty, the author recently proposed a modified Gupta-Bleuler condition,[2] which would correspond in our model to $a_2(\vec{k})\Psi = c(\vec{k})\Psi$. For arbitrary fixed $c(\vec{k})$ one can show that this condition defines a physical subspace $\mathfrak{g}[c]$, just as we have done above for $c = 0$. On the other hand, we have seen that in our model there is no difficulty in finding charged solutions to the Gupta-Bleuler condition. This is because we obtain other states by using the grandfather potential S whose analogy is lacking in QED as conventionally formulated. In the final section we will show how a grandfather potential may be introduced in QED.

VII. LESSON FOR QUANTUM ELECTRODYNAMICS

In a model with many formal similarities to quantum electrodynamics, we have found (1) a perturbation series which appears free of ultraviolet divergences in all orders (Sec. IV), (2) a representation space free of infrared divergences (Sec. V), and (3) a physical subspace including charged sectors provided by the Gupta-Bleuler condition (Sec. VI). These features result from the systematic use of the grandfather potential S, which is a potential for the vector potential $A_\mu = \partial_\mu S$. The field S has the propagator

$$G(k) = -\tfrac{1}{2} i\lambda \frac{\partial}{\partial k^\mu}\left[\frac{k^\mu \ln(-a^2k^2 - i\epsilon)}{(-k^2 - i\epsilon)^2}\right] , \qquad (7.1)$$

where the derivative is understood in the weak sense, whose discontinuity provides the inner product

$$\langle f, g\rangle = -\tfrac{1}{2}\lambda \int \frac{d^3k}{(2\omega)^2} \ln(2a\omega) \times \frac{\partial}{\partial\omega}[f^{1*}(\vec{k})g^2(\vec{k}) + f^{2*}(\vec{k})g^1(\vec{k})] . \qquad (7.2)$$

It is the appearance in the free propagator of the parameter a^{-1}, with the dimensions of mass which otherwise would appear in the theory as a renormalization mass, that leads to the attractive features (1), (2), and (3), and which distinguishes our model from QED as conventionally formulated.

We may expect to gain corresponding advantages for QED by introducing the analog of the grandfather potential S. This would be the Hertz potential[3] $\Pi^{\mu\nu} = -\Pi^{\nu\mu}$, from which the vector potential is obtained by $A^\nu = \partial_\mu \Pi^{\mu\nu}$, thereby making A^μ purely transverse instead of purely longitudinal. This may be achieved by adding to the Maxwell-Dirac Lagrangian density

$$\mathcal{L}_{\text{MD}} = -\tfrac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu - \partial_\nu A_\mu) + \tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(i\not{\partial} + e\not{A} - m)\psi \qquad (7.3)$$

the term

$$\mathcal{L}_1 = -I_\nu(A^\nu - \partial_\mu \Pi^{\mu\nu}) , \qquad (7.4)$$

where $\Pi_{\mu\nu}$ and the new field I_ν are to be varied independently. If this is done the equations of motion for $\Pi_{\mu\nu}$ are incomplete, the time derivative of Π_{ij} being undetermined. On adding the further term

$$\mathcal{L}_2 = \tfrac{1}{2}H^{\mu\nu}[\Pi_{\mu\nu} - (\partial_\mu U_\nu - \partial_\nu U_\mu)] + C\partial^\mu U_\mu , \qquad (7.5)$$

where U_μ, $H_{\mu\nu} = -H_{\nu\mu}$, and C are new fields that are varied independently, one obtains a complete system of equations of motion with a time derivative for every variable that is not determined by a constraint. The extra term is not a source of unwanted complication. On the contrary, the equations of motion which result from the Lagrangian density

$$\mathfrak{L} = \mathfrak{L}_{MD} + \mathfrak{L}_1 + \mathfrak{L}_2 \tag{7.6}$$

lead to

$$A_\mu = \partial^2 U_\mu \ , \tag{7.7}$$

which is more convenient to use than $A^\nu = \partial_\mu \Pi^{\mu\nu}$ because U_μ is a vector field. It is a potential which plays the role of a potential for the Hertz potential, $\Pi_{\mu\nu} = \partial_\mu U_\nu - \partial_\nu U_\mu$. An article is in preparation which describes the virtues of a formulation of quantum electrodynamics based on this Lagrangian density.

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APPENDIX

We will construct a positive form (f,g) which provides a norm $\|f\| = (f,f)^{1/2}$ that bounds the inner product $\langle f,g\rangle$ according to

$$|\langle f,g\rangle| \leq \|f\|\|g\| \ , \tag{A1a}$$

and which is nondegenerate and finite,

$$0 < \|f\| < \infty \tag{A1b}$$

for all $f \neq 0$, $f \in \mathcal{G}_\gamma^{(1)}$. For this purpose we rework the inner product $\langle f,g\rangle$, Eq. (5.21), to diagonal form.

We have

$$\langle f,g\rangle = \lambda \int \frac{d^3k}{(2\omega)^3}\, [f^{1*}(\vec{k})g^2(\vec{k}) + f^{2*}(\vec{k})g^1(\vec{k}) - 2f_0^* g_0 |\chi(\vec{k})|^2] \ , \tag{A2}$$

where f_0 is the value of the wave function at the origin,

$$f_0 = f^1(0) = f^2(0) = \hat{f}(0) \tag{A3}$$

by Eqs. (5.22). Here $\chi(\vec{k})$ is an on-shell test function

$$\chi(\vec{k}) = \hat{\chi}(k^0, \vec{k})|_{k^0=\omega}$$

that satisfies

$$\chi(0) = 1 \ , \tag{A4a}$$

$$\int \frac{d^3k}{(2\omega)^2} \ln(2a\omega) \frac{\partial}{\partial\omega} |\chi(\vec{k})|^2 = 0 \ , \tag{A4b}$$

in which case Eq. (A2) agrees with Eq. (5.21), as may be verified by partial integration on ω. A convenient choice is

$$\hat{\chi}(k) = \exp[-\tfrac{1}{4}b^2(k^{02} + \vec{k}^2)] \ , \tag{A5a}$$

$$b = 2ae^{-\gamma/2} \ , \tag{A5b}$$

where γ is Euler's constant, $\gamma = -\int_0^\infty ds\, e^{-s} \ln s$. Corresponding to this test function are the wave functions, Eq. (5.22),

$$\chi^1(\vec{k}) = \chi^2(\vec{k}) = \exp(-\tfrac{1}{2}b^2\omega^2) \ . \tag{A6}$$

We now shift the wave functions by subtracting out a part which is finite at the origin

$$f^3(\vec{k}) \equiv f^1(\vec{k}) - f_0\chi(\vec{k}) \ , \tag{A7a}$$

$$f^4(\vec{k}) = f^2(\vec{k}) - f_0\chi(\vec{k}) \ , \tag{A7b}$$

so the new wave functions $f^3(\vec{k})$ and $f^4(\vec{k})$ vanish like ω at the origin. The generic state $f \in \mathcal{G}_\gamma^{(1)}$ is represented by the triplet

$$f = (f_0, f^3(\vec{k}), f^4(\vec{k})) \tag{A8}$$

of a constant f_0 and the two wave functions $f^3(\vec{k})$ and $f^4(\vec{k})$. In terms of the new variables, the inner product reads

$$\langle f,g\rangle = \lambda \int \frac{d^3k}{(2\omega)^3} \{f^{3*}(\vec{k})g^4(\vec{k}) + f^{4*}(\vec{k})g^3(\vec{k}) + f_0^*\chi^*(\vec{k})[g^3(\vec{k}) + g^4] + [f^3(\vec{k}) + f^4(\vec{k})]^* g_0\chi(\vec{k})\} \ . \tag{A9}$$

The first two terms may be rewritten

$$\langle f,g\rangle_1 = \frac{\lambda}{2} \int \frac{d^3k}{(2\omega)^2} [(f^3+f^4)^*(g^3+g^4) - (f^3-f^4)^*(g^3-g^4)] \ , \tag{A10}$$

so a positive form $(f,g)_1$ which dominates it, $\|f\|_1\|g\|_1 \geq |\langle f,g\rangle_1|$ with $\|f\|_1 = (f,f)_1^{1/2}$, is obtained by changing λ to $|\lambda|$ and $-$ to $+$,

$$(f,g)_1 = \frac{|\lambda|}{2} \int \frac{d^3k}{(2\omega)^3}[(f^3+f^4)^*(g^3+g^4) + (f^3-f^4)^*(g^3-g^4)] \ . \tag{A11}$$

The last two terms of Eq. (A9) may be rewritten

$$\langle f,g\rangle_2=\frac{\lambda}{2}\int\frac{d^3k}{(2\omega)^3}\left\{\left[(2a\omega)^p f_0\chi+\frac{f^3+f^4}{(2a\omega)^p}\right]^*\left[(2a\omega)^p g_0\chi+\frac{g^3+g^4}{(2a\omega)^p}\right]\right.$$
$$\left.-\left[(2a\omega)^p f_0\chi-\frac{f^3+f^4}{(2a\omega)^p}\right]^*\left[(2a\omega)^p g_0\chi-\frac{g^3+f^4}{(2a\omega)^p}\right]\right\}\ . \tag{A12}$$

If p is a fixed real number satisfying

$$0<p<1\ , \tag{A13}$$

a finite positive form $(f,g)_2$ which dominates $\langle f,g\rangle_2$ is again provided by changing λ to $|\lambda|$ and $-$ to $+$. The cross terms cancel and one has

$$(f,g)_2=|\lambda|\int\frac{d^3k}{(2\omega)^3}\left[(2a\omega)^{2p}|\chi|^2 f_0^* g_0+\frac{(f^3+f^4)^*(g^3+g^4)}{(2a\omega)^{2p}}\right]\ . \tag{A14}$$

A positive form of the required type is provided by $(f,g)_1+(f,g)_2$ but the final expression is simpler if we add in

$$(f,g)_3=|\lambda|\int\frac{d^3k}{(2\omega)^3}\,\frac{(f^3-f^4)^*(g^3-g^4)}{(2a\omega)^{2p}}\ . \tag{A15}$$

Then with $(f,g)=(f,g)_1+(f,g)_2+(f,g)_3$ we have

$$(f,g)=|\lambda|\left\{\frac{\pi}{4}e^{p\gamma}\Gamma(p)f_0^* g_0+\int\frac{d^3k}{(2\omega)^3}\left[1+\frac{2}{(2a\omega)^{2p}}\right]\left[f^{3*}(\vec{k})g^3(\vec{k})+f^{4*}(\vec{k})g^4(\vec{k})\right]\right\}\ , \tag{A16}$$

$0<p<1$. This is a convenient form which by construction has the required properties (A1a) and (A1b).

Completion of $\mathcal{G}_\gamma^{(1)}$ in the norm $\|f\|=(f,f)^{1/2}$ yields the one-photon "large" Hilbert space $\mathcal{H}_\gamma^{(1)}$. Its elements are the triplets (A8), $(f_0, f^3(\vec{k}), f^4(\vec{k}))$, where $f^3(\vec{k})$ and $f^4(\vec{k})$ are square-integrable with respect to the measure $d^3k(2\omega)^{-3}[1+2(2a\omega)^{-2p}]$, instead of being restrictions of test functions, and f_0 represents the coefficient of the wave function $\chi(\vec{k})$ which is singular with respect to this measure. The Fock construction provides the many-photon Hilbert space $\mathcal{H}_\gamma$. The full Hilbert space of our model is the product $\mathcal{H}=\mathcal{H}_\gamma\otimes\mathcal{H}_e$, where $\mathcal{H}_e$ is the usual free electron-positron Hilbert space.

We may now answer the question whether there exist any solutions to the Gupta-Bleuler condition in $\overline{\mathfrak{D}}_q(A,\psi)$, which is the closure in the Hilbert-space topology of $\mathfrak{D}_q(A,\psi)$, the space obtained from the vacuum by applying polynomials, of definite electric charge q, in A and ψ smeared with test functions in $\mathcal{S}$. Expressed in terms of the old basis $f^1(\vec{k})=f_0\chi(\vec{k})+f^3(\vec{k})$, $f^2(\vec{k})=f_0\chi(\vec{k})+f^4(\vec{k})$, the Gupta-Bleuler subspace $\mathcal{H}_0\subset\mathcal{H}$ consists, by Eq. (6.7), in the n-photon sector, of wave functions $f^{i_1\cdots i_n}(\vec{k}_1\cdots\vec{k}_n)$ which vanish when any index $i_a=1,2$, $a=1\cdots n$, takes the value $i_a=1$. In particular, in the one-photon sector $f^1(\vec{k})=f_0\chi(\vec{k})+f^3(\vec{k})=0$, which means, in the topology of (A16), $f_0=0$, $f^3(\vec{k})=0$. Here, electron-positron varibles are suppressed. On the other hand, the elements Ψ of $\overline{\mathfrak{D}}_q(A,\psi)$ consist of Cauchy sequences of wave functions $\Psi^{(n)}$ of $\mathfrak{D}_q(A,\psi)$, which satisfy the coherence condition (6.13), $a_2(0)\Psi^{(n)}=(2\pi)^{-3/2}iq\Psi^{(n)}$. The corresponding wave functions in the one-photon sector, $f^{(n)}(\vec{k})=[f^{1(n)}(\vec{k}),f^{2(n)}(\vec{k})]$, therefore satisfy $f_0^{(n)}=f^{1(n)}(0)=(2\pi)^{-3/2}iqc^{(n)}$, where $c^{(n)}$ is the no-photon amplitude and again electron-positron variables are suppressed. The Cauchy sequences $f_0^{(n)}, c^{(n)}$ converge to complex numbers f_0 and c, characteristic of the generic state $\Psi\in\overline{\mathfrak{D}}_q(A,\psi)$, which satisfy $f_0=(2\pi)^{-3/2}iqc$. Hence in the charged sectors $q\neq 0$ we find $f_0\neq 0$ (if c were zero we would consider the leading nonvanishing amplitude.) Thus there are no solutions to the Gupta-Bleuler condition in the charged sectors of $\overline{\mathfrak{D}}(A,\psi)$.

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