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Radiation and acceleration of a relativistic charged particle in an electromagnetic field

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This paper studies the characteristics of an ultrarelativistic charged particle in a general, large-scale, electromagnetic field, taking into consideration the effects of strong radiative damping. Formal solutions of the Lorentz-Dirac equation in an arbitrary external field are given. They are applied to cases of astrophysical interest. The final energy of the particle and the rate and characteristic frequency of the radiation are derived.

I. INTRODUCTION

Recently there has been a resurgence of interest in the electrodynamics of relativistic charged particles in intense external fields. This was prompted by a number of astronomical and laboratory developments such as the discovery of neutron stars with a surface magnetic field as high as 10^{13} G, and the generation of transit magnetic fields with strength up to a few MG by the flux-compression technique.

In an earlier article[1] we studied the characteristics of an ultrarelativistic electron in a uniform magnetic field, taking into consideration the effects of strong radiative damping and quantum corrections. In the present article we shall extend our previous calculation to a general, large-scale, electromagnetic field. Whereas the effort of Ref. 1 was aimed at obtaining results to test the validity of the Lorentz-Dirac equation at strong radiation damping in the laboratory, the practical purpose of this article, in addition to obtaining a formal solution to the Lorentz-Dirac equation, is to determine the effects of radiation damping on the acceleration and the radiation spectrum of electrons in cases of astrophysical interest.

In Sec. I we review the validity.condition of classical electrodynamics. Quantum corrections and radiation reactions are discussed. In Sec. II we present a mathematical analysis of the Lorentz-Dirac equation; formal solutions of the Lorentz-Dirac equation in an arbitrary external field including strong radiation damping are given. In Sec. III the formal solutions obtained in Sec. II are applied to a case of special astrophysical interest: relativistic electrons riding with the low-frequency wave emitted by a rotating neutron star. The final energy, the radiation rate, and the characteristic frequency of the electrons are obtained and compared with results obtained by previous authors, assuming weak or no radiation reaction.

In classical electrodynamics an accelerated charged particle radiates energy. In return, the radiation affects the motion of the particle. The radiation depends on the magnitude and direction of acceleration but not on the specific cause of acceleration. Thus analyses of the effect of radiation reaction on a relativistic charge in a general electromagnetic field is parallel to that in a pure magnetic field except to replace $\vec{H}$ by $\vec{H}^*$, where

$$H^* = \frac{\vec{\beta}\times\vec{E}}{\beta^2} + \vec{H} \tag{1}$$

is the effective field which results in acceleration perpendicular to the motion of the particle, which is the main cause of radiation. The dynamics of the particle's motion can be best summarized in Fig. 1. Link 1 indicates the effects of an external field on the particle; link 2 indicates the field produced by the particle; and link 3 indicates the feedback on the motion of the particle by the resulting field. Thus classical electrodynamics consists of two basic sets of equations. One specifies the motion of the particle (represented by link 1 and link 3), whereas the other describes the field generated by the particle (link 2). The latter are the Maxwell equations which have been firmly es-

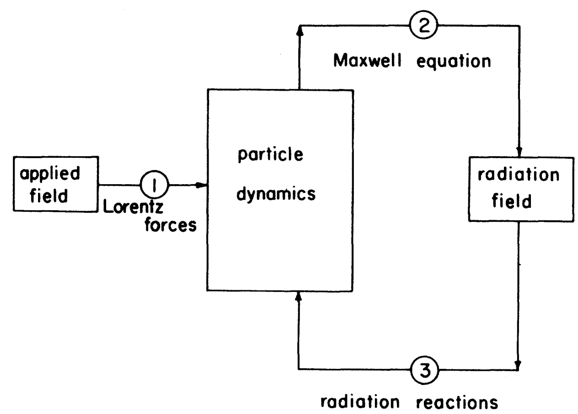

FIG. 1. Action and reaction on a classic point charge in an external field.

tablished both through experimental verification and mathematical deductions. The opinion on the former, the equations of motion, however, is much more controversial. Acceleration by the applied field (link 1) can, no doubt, be represented by the Lorentz-force equation, but expressions to describe the feedback from the radiation (link 3) often lead to difficulties, such as runaway solutions. There is the so-called Lorentz-Dirac equation, derived by Abraham, Lorentz, Dirac, and others,

$$\dot{u}_\mu = \omega_{\mu\nu}u^\nu + \omega_0^{-1}\left(\ddot{u}_\mu - \frac{1}{c^2}u_\mu \dot{u}^\nu \dot{u}_\nu\right), \tag{2}$$

where

$$\omega_{\mu\nu} = \frac{e}{mc}\begin{bmatrix} 0 & H_z & -H_y & E_x \\ -H_z & 0 & H_x & E_y \\ H_y & -H_x & 0 & E_z \\ -E_x & -E_y & -E_z & 0 \end{bmatrix} \tag{3}$$

is the electromagnetic field tensor in frequency units, $\omega_0 = 3mc^3/2e^2$, and u_μ is the four-velocity. Equation (2) has often been referred to as the equation of motion for a charged point particle, but the arguments leading to this equation are not without an *ad hoc* flavor. Dirac himself made the point perfectly clear; at a crucial point in deriving Eq. (2), he let a perfect differential $\dot{B}_\mu$ equal $k\dot{u}_\mu$ by the argument of simplicity and elegance.[2] Thus it is not surprising that over the years many new inventions have been claimed on this fundamental problem and several "new" equations have resulted. In an earlier article[3] we have shown that one of the proposed new equations of motion gives the same observable results as that of the Lorentz-Dirac equation within the realm of classical electrodynamics. We can now show that for any covariant equation of motion satisfying the general conservation law

$$\partial_\mu T^{\mu\nu} = 0, \tag{4}$$

where T is the total momentum flux tensor composed of a particle term and a field term, the discrepancy between the results derived from such an equation and the Lorentz-Dirac equation is smaller than the correction introduced by quantum effects by a factor of $\alpha = e^2/\hbar c = \frac{1}{137}$. Therefore we can now move one step further than we had done in Ref. 1 to claim that, within the realm of classical electrodynamics, the Lorentz-Dirac equation is "indeed" the exact equation of motion for a point charge. Together with the Maxwell equations, they form a rigorous base for classical electrodynamics, one of the most beautiful theories of natural sciences.

Therefore, application of the Lorentz-Dirac equation is restricted only by the requirement that quantum effects be negligible. For a radiating charge moving in an external field, there are, in general, two sorts of quantum effects. The first effect is on the nature of the particle; for equations of motion to be meaningful, the de Broglie wavelength of the charged particle must be much less than the characteristic wavelength of the system, so that the charge can be considered as a classical particle with definite trajectory. In a large-scale field, i.e., the scale length of the inhomogeneity is large in comparison with the radius of curvature of the particle's motion, this requirement can be expressed as

$$\frac{\gamma mc^2\beta}{eH^*} \gg \frac{\hbar}{\gamma mc\beta}. \tag{5}$$

For the highly relativistic case $\beta \sim 1$, Eq. (5) reduces to

$$\gamma^2 \gg H^*/H_q, \tag{6}$$

where

$$H_q = m^2c^3/e\hbar = 4.4\times 10^{13}\ \text{G}.$$

The second quantum effect is on the nature of the radiation; in classical electrodynamics an accelerated charged particle radiates continuously, whereas in quantum electrodynamics the radiation consists of discrete steps. A charge in an external field drops to a lower energy state by emitting a photon and it will stay in that state for a finite time before making another transition to a lower state by emitting a second photon. In order for the classical description of radiation to be valid, the effect of quantum recoil must be small, or $\langle\hbar\omega\rangle$, the average energy carried away by a photon, must be small compared with the energy of the particle itself,

$$\gamma mc^2 \gg \hbar\frac{\gamma^2 eH^*}{mc}$$

or

$$\gamma H^*/H_q \ll 1. \tag{7}$$

It is clear that for a highly relativistic particle, Eq. (7) is a more stringent condition than Eq. (6). Since in the classical Lorentz-Dirac equation the ratio of the reaction force F_R to the Lorentz force F_L is given by

$$R_c = F_R/F_L \cong \gamma^2\alpha H^*/H_q, \tag{8}$$

one sees that for a relativistic charge the particle can always be considered as a well-localized point charge. The radiation process may need to be treated quantum mechanically, but if $\gamma > 137$, it is also possible to have strong radiative damping with negligible interference from quantum effects. The significance of radiation reaction and quantum

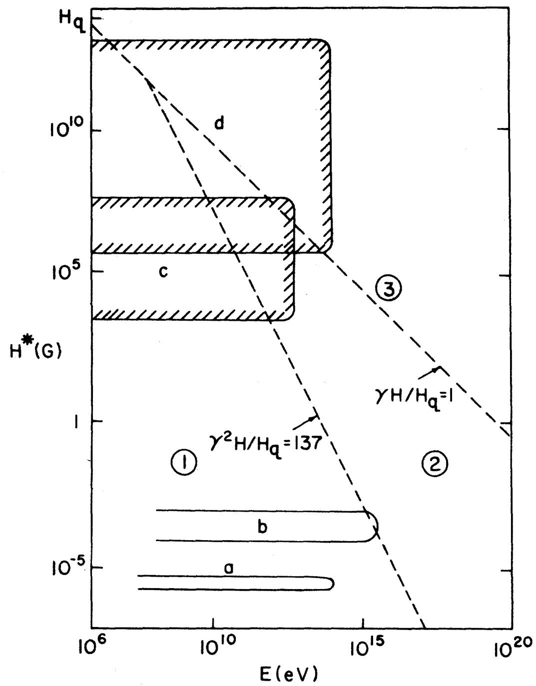

FIG. 2. Regions of applicability of different levels of electrodynamics: (a) classical electrodynamics with negligible radiation reaction, (2) classical electrodynamics with strong radiation reaction, (3) quantum electrodynamics. a, b, c, and d, represent four typical astrophysical situations: a represents the cosmic ray electrons in the galaxy; b represents relativistic electrons in the crab nebula; c represents the magnetosphere of a white dwarf; d represents the magnetosphere of a pulsar.

corrections on the motion and radiation of a charged particle can be best described by the energy–field-intensity diagram presented in Ref. 1, which also indicates the validity domain of classical and quantum electrodynamics, with or without radiation reaction. For reference purposes a simplified version of this diagram is reproduced in Fig. 2. In region 1 of Fig. 2, $\alpha\gamma^2H^*/H_q \ll 1$, classical electrodynamics is valid and the radiation reaction is small. The Maxwell equations and Lorentz equations adequately describe the physics. In region 2, $\gamma^2H^*\alpha\,|H_q>1$ and $\gamma H^*\,|H_q<1$, classical electrodynamics is still valid but the radiation-reaction force becomes stronger than the applied Lorentz force. The physical equations of this region are the Maxwell equations and the Lorentz-Dirac equation. In region 3, $\gamma H/H_q>1$, the energy carried away by the emitted photon becomes comparable to the energy of the emitting electron. Thus, although the electron itself is still a well-localized classical particle, and its motion can still be described by the Lorentz-Dirac equation, the radiation process must be calculated quantum mechanically, i.e., one has to calculate the spontaneous transition rate from the Dirac wave equation to determine the radiation rate and its spectrum.

II. FORMAL SOLUTION

It is not possible to solve the Lorentz-Dirac equation exactly, since it involves cross products of second-order differentials. However, within the reals of classical electrodynamics this equation can be expanded in powers of γ^{-1},

$$\dot{u}_\mu = \omega_{\mu\nu}u^\nu - Ku_\mu + O(\gamma^{-1}), \tag{9}$$

where

$$K = (\omega_{\mu\nu}u^\nu\omega^{\mu\lambda}u_\lambda)/\omega_0c^2 \tag{10}$$

is the radiation reaction.

The derivation of Eq. (9) is given in Appendix A. It is important to point out that in deducing Eq. (9) no approximation has been made on the magnitude of the damping forces. The only requirement is $\gamma H^*/H_q \ll 1$, which limits the validity of the Lorentz-Dirac equation. Since the ratio of the damping force Ku_μ to the Lorentz force $\omega_{\mu\nu}u^\nu$ is of the order of $R_c = \gamma^2\alpha H^*/H_q$, it is possible that R_c is greater than unity, even though $\gamma H^*/H_q \ll 1$. Referring to Fig. 2, Eq. (9) is applicable to all classical domains, including both region 1 (weak damping) and region 2 (strong damping).

It can be easily verified that if the four-vector f_μ is a solution of the reactionless Lorentz equation $\dot{f}_\mu = \omega_{\mu\nu}f^\nu$, then

$$u_\mu(\tau) = \eta(\tau)f_\mu(\tau) \tag{11}$$

is a solution of the Lorentz-Dirac equation in the form of Eq. (9), where

$$\eta(\tau) = \left[1 + 2(\omega_0c^2)^{-1}\int \omega_{kl}f^l\omega^{km}f_m d\tau\right]^{-1/2} \tag{12}$$

represents the damping effect.

Therefore, once the solution of the Lorentz equation is found, at least a formal expression (correct to the order of γ^{-1}) can be readily written down for the solution of the Lorentz-Dirac equation.

Let us consider the case of a constant electromagnetic field. Utilizing the fact that $\omega_{\mu\nu}$ is antisymmetric and $\dot{u}_\mu\dot{u}^\nu$ is symmetric, and the product of a symmetric tensor with an antisymmetric tensor gives identically zero, we find

$$\frac{d}{d\tau}(\omega_{kl}f^l\omega^{km}f_m) = 2\omega_{kl}f^l\omega^{km}\dot{f}_m = 2\omega^{km}\dot{f}_k\dot{f}_m = 0, \tag{13}$$

i.e., $\omega_{kl}f^l\omega^{km}f_m$ is a constant of time. Therefore, in a constant field

$$\eta(\tau) = [1 + 2(\omega_0c^2)^{-1}\omega_{kl}f^l\omega^{km}f_m\tau]^{-1/2} \tag{12a}$$

and u_μ can be explicitly expressed in terms of f_μ,

$$u_\mu = \frac{f_\mu}{[1+2(\omega_0 c^2)^{-1}\omega_{kl} f^l \omega^{km} f_m \tau]^{1/2}} \,. \tag{14}$$

The radiation effect $K(\tau)$, which is generally given by

$$K(\tau) = \eta^2 \omega^{\mu\lambda} f_\lambda / \omega_0 c^2 \,, \tag{15}$$

now reduces to

$$K(\tau) = \frac{K(0)}{1+2K(0)\tau} \,, \tag{16}$$

where $K(0)$ represents the radiation effect at $\tau = 0$.

Two interesting physical implications can be readily drawn from Eq. (16):

(1) *Radiation from a relativistic electron decreases with time.*

The timelike component of the Lorentz-Dirac equation gives the radiation rate

$$\left(\frac{dW}{dt}\right)_{\text{Radiation}} = Kmc^2 . \tag{17}$$

Since $K = K_0/(1+2K_0\tau)$, one sees that the radiation rate is a monotonically decreasing function of time. This is true even for cases where the particle is being accelerated (in a cross field, for example). This is because, although the radiation rate is proportional to the square of the particle's energy, the field always orients the direction of the particle's motion fast enough to diminish the radiation rate.

(2) *Relative influence of the radiation damping force on a relativistic charge also decreases with time.*

Owing to the fast orientation of the particle's motion by the field, both the Lorentz force F_L and the damping force F_R decrease with time, but

$$F_L(\tau) = |\omega_{\mu\nu} u^\nu| = |\omega_{\mu\nu} f^\nu| / [1+2K(0)\tau)^{1/2}, \tag{18}$$

$$F_R(\tau) = |K u_\mu| = |K(0) f_\mu| / [1+2K(0)\tau]^{3/2}, \tag{19}$$

and their ratio is

$$\frac{F_R(\tau)}{F_L(\tau)} = \frac{F_R(0)}{F_L(0)} (1+2K_0\,\tau)^{-1} . \tag{20}$$

Thus the effect of radiative damping (as measured by the relative strength of the damping force to the Lorentz force) is also decreasing with time. Since within the realm of classical electrodynamics R_c/γ must be smaller than $\frac{1}{137}$, a particle which starts at rest in a field of strength less than 4.4 $\times 10^{13}$ G will never encounter strong damping (i.e., $F_R > F_L$), even though it later may be accelerated to a very high energy.

Although the remarks made above are derived for a constant electromagnetic field, they are also applicable to a slowly changing inhomogeneous field. In general, if $\mathring{\omega}_{\mu\nu}$ is small but not zero, one would have

$$\eta = \left[1+2(\omega_0 c^2)^{-1} (\omega_{kl} f^l \omega^{km} f_m)_{\tau=0} \times \left(1 + \left\langle \frac{f_m \mathring{\omega}^{km}}{f_m \omega^{km}} \right\rangle_{\text{ave}} \tau \right)\right]^{-1/2}, \tag{21}$$

where

$$\left\langle \frac{f_m \mathring{\omega}^{km}}{f_m \omega^{km}} \right\rangle_{\text{ave}} = \frac{1}{\tau} \int^\tau \frac{f_m \mathring{\omega}^{km}}{f_m \omega^{km}} d\tau = O\left(\frac{1}{\nu^*} \frac{\partial H^*}{\partial t} \Big/ H^*, r^* \frac{\partial H^*}{\partial x} \Big/ H^* \right) . \tag{22}$$

Thus in a slowly varying field, where the effective Larmor frequency is

$$\nu^* = \frac{eH^*}{\gamma mc} \gg \text{any other frequency},$$

and the effective Larmor radius is

$$r^* = \frac{\gamma mc^2}{eH^*} \ll \text{any other length},$$

the expression $K(\tau) = K(0)/[1+2K(0)\tau]$ still holds.

To compute u_μ, one needs to find f_μ first. For a charged particle in a constant electric and magnetic field this has become a standard textbook exercise.[4] The usual procedure is to solve the Lorentz equation in a reference frame moving with a velocity $\vec{\beta} = (1+\beta^2)[\vec{E} \times \vec{H}/(E^2+H^2)]$ relative to the given frame, in the new reference system the electric and magnetic fields are parallel and the solution can be easily obtained. Transforming back to the original frame gives the trajectory of the particle in parametric form. Not only is the process tedious in such an approach but the solutions also look rather clumsy. A more elegant method for obtaining formal solutions to the Lorentz equation is to utilize projection operators. In general, the Lorentz equation can be written as

$$f_\mu = e^{\omega_{\mu\nu}\tau} f_\mu(0), \tag{23}$$

where $\omega_{\mu\nu}$ is the matrix operator and $f_\mu(0)$ is the initial value of f_μ.

Equation (23) has the same form as the transformation of a vector $f_\mu(0)$ in the four-dimensional space, with $\omega_{\mu\nu}$ as the transformation operator. Following a proof first given by Rosen[5] for a Hermitian matrix, one can easily show (see Appendix B) that if the eigenvalues G_j of $\omega_{\mu\nu}$ obey the characteristic equation

$$\prod_{j=1}^{4} (\omega_{\mu\nu} - G_j I) = 0, \tag{24}$$

where I is the unit matrix, then one can express the transformed vector f_μ in terms of a complete set of projection operators and its appropriate eigenvalues:

$$f_\mu = e^{\omega_{\mu\nu}\tau} f_\mu(0) = \sum_{j=1}^{4} e^{G_j\tau} P_j f_\mu(0), \tag{25}$$

where

$$P_j = \prod_{l\neq j}^{4} \frac{\omega_{\mu\nu} - G_l I}{G_j - G_l} \tag{26}$$

are the projection operators.

Direct computation gives the following eigenvalues for the matrix $\omega_{\mu\nu}$:

$$G_j = \pm \frac{e}{mc}\left[\frac{A \pm (A^2 + 4B^2)^{1/2}}{2}\right]^{1/2}, \tag{27}$$

where $A = E^2 - H^2$ and $B = \vec{E}\cdot\vec{H}$ are the two invariants of the field tensor. Hence, unless A and B are both zero (in which case, $\vec{E}$ and $\vec{H}$ are equal and mutually perpendicular), all G_j's are distinct. Substituting the values of G_j from Eq. (27) into Eq. (24) shows that $\omega_{\mu\nu}$ satisfies the characteristic equation. Therefore, except for the case where $\vec{E}$ and $\vec{H}$ are perpendicular and equal to magnitude, solutions of the Lorentz equation are given by Eq. (25). In the exceptional case $\vec{E}$ perpendicular to $\vec{H}$ and $|\vec{E}| = |\vec{H}|$, the solutions of f_μ are particularly simple. Choosing the x axis along $\vec{E}$ and the y axis along $\vec{H}$, we have

$$\begin{aligned} f_1(\tau) &= f_1(0) + \alpha_0\omega_H\tau, \\ f_2(\tau) &= f_2(0), \\ f_3(\tau) &= f_3(0) + f_1(0)\omega_H\tau + \tfrac{1}{2}\alpha_0(\omega_H\tau)^2, \\ f_4(\tau) &= f_4(0) - f_1(0)\omega_H\tau - \tfrac{1}{2}\alpha_0(\omega_H\tau)^2, \end{aligned} \tag{28}$$

where $\alpha_0 = -[f_4(0) + f_3(0)]$ and $\omega_H = eH/mc = eE/mc$. [Equation (28) differs slightly from the solution given in Landau and Lifshitz, where the authors have implicitly set $f_1(0)$ equal to zero.]

Therefore, in a constant field the general solution of the Lorentz-Dirac equation, correct to the lowest order of γ^{-1}, is given by

$$u_\mu(\tau) = \sum_{j=1}^{4} e^{G_j\tau}\eta(\tau) P_j u_\mu(0), \tag{29}$$

where G_j, $\eta(\tau)$, and P_j are given by Eq. (27), Eq. (12a), and Eq. (26), respectively. The special case where E and H are mutually perpendicular and equal to the above expression is not applicable, and the correct solution is given by multiplying Eq. (28) with $\eta(\tau)$.

Quantities of astrophysical interest, such as the radiation rate, the frequency spectrum, and the energy change of the particle, etc., can usually be deduced directly from u_μ without further integration. Since characteristics of the radiation depend only on the magnitude and direction of the velocity and acceleration but not on the specific cause of acceleration, and the four-acceleration of a charged particle in an electromagnetic field is

$$|\ddot{u}_\mu|^2 = \frac{e^2}{m^2c^2}[(\vec{u}\times\vec{H} + \vec{E})^2 - (\vec{E}\cdot\vec{u})^2],$$

formulas for quantities related to radiation can be readily obtained by replacing $|\vec{\beta}\times\vec{H}|$ with $[(\vec{\beta}\times\vec{H} + \vec{E})^2 - (\vec{E}\cdot\vec{\beta})^2]^{1/2}$ (or $\vec{H}$ by $\vec{H}^*$ if one neglects γ^{-1} terms) in formulas for synchrotron radiation derived in Ref. 1. These are summarized below:

1. *The radiation rate*. The radiation rate is given by

$$\begin{aligned} \left(\frac{dW}{dt}\right)_{\text{Radiation}} &= (m\omega_0)^{-1}\omega_{\mu\nu}u^\nu\omega^{\mu\lambda}u_\lambda \\ &= \frac{e^2c^2}{m\omega_0}\gamma^2[(\vec{\beta}\times\vec{H} + \vec{E})^2 - (\vec{E}\cdot\vec{\beta})^2]; \end{aligned} \tag{30}$$

it is not affected by radiation reaction. Physically this is because the forces exerted by radiation reaction are parallel (and opposite) to the direction of $\vec{\beta}$. The radiation generated by the force which is parallel to the velocity is a factor $1/\gamma^2$ smaller than the radiation generated by the force which is perpendicular to the velocity. Since within the realm of classical electrodynamics the ratio of the damping force F_R to the Lorentz force F_L is $R_c = (\gamma\alpha)(\gamma H^*/H_q) \ll \gamma\alpha$, the modification on the radiation rate by radiation reaction is in general smaller than $(\alpha\gamma)^{-1}$ unless the perpendicular acceleration is identically zero. In that case, both F_R and F_L are parallel to $\vec{\beta}$, but then the ratio $F_R/F_L \sim \omega_{\mu\nu}/\omega_0 \sim \alpha H^*/H_q \ll 1$. The effect of radiation reaction on the instantaneous radiation rate is again negligible.

2. *The radiation spectrum*. The standard formula for the radiation spectrum of an accelerated charged particle in units of energy radiated per solid angle per frequency interval is

$$I(\omega) = \frac{e^2}{4\pi^2c}\left|\int_{-\infty}^{\infty} \vec{n}\times(\vec{n}\times\vec{\beta})\exp[i\omega(t - \vec{n}\cdot\vec{r}/c)]dt\right|^2. \tag{31}$$

The integration over time covers the whole duration of particle's acceleration. For cases where particles lose energy rapidly, the characteristic of radiation may change significantly within the period of observation, then it is more meaningful to find out separately the instantaneous power spectrum and the cumulative spectrum. The instantaneous spectrum

$$I(\omega, t) = \frac{1}{\Delta T}\frac{e^2\omega^2}{\Delta\pi^2 c}\left|\int_{t-\Delta T/2}^{t+\Delta T/2}[\vec{n}\times(\vec{n}\times\vec{\beta})]\exp[i\omega(t' - \vec{n}\cdot\vec{r}/c)]dt'\right|^2, \tag{32}$$

where ΔT is chosen such that $\Delta T \gg \omega^{-1}$, but $|d\vec{\beta}/dt|\Delta T/\beta \ll 1$, a condition which requires the energy change of the particle within one period of the wave to be negligible. The cumulative spectrum integrates $I(\omega, t)$ over the observation time T (or the time the emitting particle stayed in the observation region if it is shorter than T),

$$I_c(\omega) = \frac{1}{T}\int_0^T I(\omega, t)dt, \tag{33}$$

where $I(\omega,t)$ represents the spectral distribution of radiation emitted at the instant t. Modifications by radiation reactions on $I(\omega,t)$ are only of the order γ^{-1}. $I_c(\omega)$ represents the observed spectrum of radiation. At strong radiative damping, $I(\omega,t)$ changes rapidly with time; then the cumulative spectrum $I_c(\omega)$ differs significantly from the spectrum when radiation reaction is neglected. (See detailed discussion given in Ref. 1.) Both $I(\omega,t)$ and $I_c(\omega)$ have been evaluated numerically for a constant magnetic field in Ref. 1. For practical application in astrophysics, it is usually good enough to know the critical frequency

$$\begin{aligned}\omega_c &= |\vec{u}\times\dot{\vec{u}}| \\ &= \gamma^2|\vec{\beta}\times(\vec{E}+\vec{\beta}\times\vec{H})|e/mc,\end{aligned} \tag{34}$$

which is given explicitly by u_μ. The instantaneous radiation spectrum is similar to the ordinary synchrotron radiation, which has a maximum at $\sim 0.3\omega_c$. Below $0.3\omega_c$ its intensity varies as $\omega^{1/3}$, where above $0.3\omega_c$ it drops exponentially as $\sim\omega^{1/2}\exp(-2\omega/3\omega_c)$.

In Sec. III we shall use the general formulas obtained in this section for practical applications. However, one must be cautious about the validity limit of those formulas. For example,

$$u_\mu u^\mu = f_\mu f^\mu \eta^2(\tau) = -c^2 n^2(\tau), \tag{35}$$

which apparently violates the requirement

$$u_\mu u^\mu = -c^2. \tag{36}$$

The explanation is that Eq. (36) is an exact result, where u_μ, given by Eq. (11) is derived from Eq. (9), which has neglected higher-order terms in γ^{-1}; thus u_μ is correct only to the lowest order of γ^{-1}. Since $u_\mu \sim \gamma c$, the lowest-order term of $u_\mu u^\mu$ should be proportional to $\gamma^2 c^2$, which vanishes as expected. The discrepancy between Eq. (35) and Eq. (36) is introduced by the neglected higher-order terms (although it appears as a constant). One should keep in mind that the u_μ derived in this article is correct only to terms proportional to γ.

III. APPLICATIONS

In Sec. II, solutions of the Lorentz-Dirac equation and quantities of physical interest are given in parametric form. In order to compare with observation one needs, however, to eliminate the parameter τ and express them in ordinary time. The procedure is straightforward but laborious, and it often leads to numerical instead of analytical results.

Let us choose a cross field as an example for detailed study,

$$\begin{aligned}\vec{E} &= H(z)(1,0,0), \\ \vec{H} &= H(z)(0,1,0),\end{aligned} \tag{37}$$

where $H(z)$ takes the form of $H_0(z/z_0)^n$, and H_0 and z_0 are constant. Mathematically simple, the case of the cross field offers the best possibility to illustrate the subtle points of strong radiation damping when a particle is being accelerated. It is also of practical interest since particles which enter a wave of large amplitude and low frequency often lock themselves up with the wave and see a nearly static cross field as it moves along in phase with the propagation of the wave.[6,7] Thus in the laboratory, as well as in cosmic space, there are many cases resembling a charged particle in a cross field. The characteristics of motion and radiation of the charged particle in the field are determined by the strength and the scale length of the field, which, for convenience, can be specified by the following two parameters in frequency units:

$$\omega_1 = \frac{eH_0}{mc},$$

$$\omega_2 = \frac{c}{z_0}.$$

Then the four-velocity is given by

$$u_1 = (u_{10} + \alpha_0 c\omega_1\chi)\eta^{-1}, \tag{38a}$$

$$u_2 = u_{20}\eta^{-1}, \tag{38b}$$

$$u_3 = \left(u_{30} + u_{10}\omega_1\chi + \frac{\alpha_0 c\omega_1^2\chi^2}{2}\right)\eta^{-1}, \tag{38c}$$

$$u_4 = -\frac{cdt}{d\tau} = \left(u_{40} - u_{10}\omega_1\chi - \frac{\alpha_0 c\omega_1^2\chi^2}{2}\right)\eta^{-1}, \tag{38d}$$

where

$$\chi = \int z^n d\tau, \quad Z = z/z_0, \tag{38e}$$

$$\eta = \left(1 + 2K_0\int z^{2n}d\tau\right)^{1/2}, \tag{38f}$$

$$\alpha_0 = \gamma_0(1-\beta_{30}), \tag{38g}$$

$$K_0 = \alpha_0^2 \omega_1^2 \omega_0^{-1}, \tag{38h}$$

and the subscript 0 denotes the initial values of u at $Z=1$ and $\tau = t = 0$. Since χ is an increasing function of time, the particle moves mainly along the z direction after a characteristic time of the order $\gamma\omega_1^{-1}$. $n=0$ corresponds to a constant cross field, which is a field the charge would see if it rides with a plane wave. In this case χ and τ, and the integration of Eq. (38) gives

$$\omega_1 t = a_1\eta^5 + a_2\eta^3 + a_3\eta^1 + a_4, \tag{39}$$

where

$$a_1 = \frac{\alpha_0\omega_1^3 c}{40K_0^3}, \tag{39a}$$

$$a_2 = \frac{\omega_1^2}{6K_0^5}\left(u_{10} - \frac{\alpha_0\omega_1}{2K_0}c\right), \tag{39b}$$

$$a_3 = \frac{\omega_1}{K_0}\left(-u_{40} - \frac{u_{10}\omega_1}{2K_0} + \frac{\alpha_0\omega_1^2 c}{8K_0^2}\right), \tag{39c}$$

$$a_4 = \frac{\omega_1}{K_0}\left(u_{40} - \frac{u_{10}\omega_1}{3K_0} - \frac{\alpha_0\omega_1^2 c}{15K_0^2}\right). \tag{39d}$$

With the help of Eq. (39), τ can be eliminated from u_μ and other relevant quantities. The asymptotic behavior of these quantities at short- and long-time limits are as follows:

(i) $\omega\tau \ll 1$. In this limit Eq. (39) reduces to

$$t = \gamma_0\tau\left[1 - \left(\frac{K_0}{2\omega} - \beta_{10}\right)\omega_1\tau + \cdots\right]. \tag{40}$$

We have the energy change

$$\gamma(t) = \gamma_0 + (\beta_{10}\omega_1 - K_0)t, \tag{41a}$$

the radiation rate

$$\left(\frac{dW}{dt}\right)_{\text{rad}} = Kmc^2 = K_0 mc^2(1 - \gamma_0^{-1}K_0 t), \tag{41b}$$

the critical frequency

$$\omega_c(t) = \alpha_0\gamma_0\omega_1(1 + \beta_{10}\gamma_0^{-1}\omega_1 t), \tag{41c}$$

the velocity parallel to $\vec{E}$

$$\beta_1 = \beta_{10} - (1 - \beta_{30} - \beta_{10}^2)\gamma_0^{-1}\omega_1 t, \tag{41d}$$

and the velocity perpendicular to both $\vec{E}$ and $\vec{H}$

$$\beta_3 = \beta_{30} + (\alpha_0\beta_{10}/\gamma_0)\gamma_0^{-1}\omega_1 t. \tag{41e}$$

Equation (41) indicates, as expected, that within the time $t < \gamma_0\omega^{-1}$ the energy change of the particle consists of the gain (or loss) from the work done by (or against) the electric field [the $\beta_{10}\omega_1 t$ term in Eq. (41a)] and the loss through radiation [$-K_0 t$ term in Eq. (41a)]. The radiation rate decreases with time where the critical frequency increases and the particle is being forced to move along the z direction.

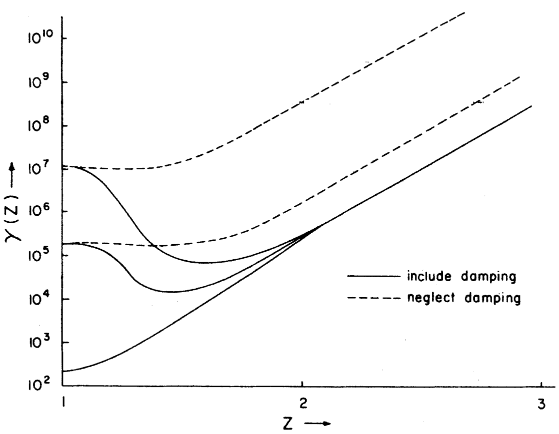

FIG. 3. Energy change of a charged particle in a constant cross field.

(ii) $\omega_1\tau \gg 1$. In this long-time limit Eq. (39) reduces to

$$t \cong \omega_1\omega_0^{1/2}\tau^{5/2}, \tag{42}$$

and we have the energy change

$$\gamma(t) \sim (\omega_0/\omega_1)^{1/5}(\omega_1 t)^{3/5}, \tag{43a}$$

the radiation rate

$$\left(\frac{dW}{dt}\right)_{\text{rad}} = (\omega_1^4\omega_0)^{1/5}(\omega_1 t)^{-2/5}mc^2, \tag{43b}$$

the critical frequency

$$\omega_c = (\omega_1\omega_0^4)^{1/5}(\omega_1 t)^{2/5}\omega_1, \tag{43c}$$

the parallel velocity,

$$\beta_1 = (\omega_0/\omega_1)^{1/5}(\omega_1 t)^{-2/5}, \tag{43d}$$

and the velocity perpendicular to the electric and magnetic field

$$\beta_3 = 1 - (\omega_0/\omega_1)^{2/5}(\omega_1 t)^{-4/5}. \tag{43e}$$

All numerical factors of order unity are omitted from Eq. (43).

Equation (43) indicates that after a time $t > (\omega_0\omega_1^{-3})^{1/2} = R_0^{-1}\gamma\omega_1^{-1}$, the characteristic of the motion and radiation of the particle are *independent of the initial conditions* but are determined by the competition between the radiation and the work done by the field. The energy of the particle keeps on increasing as $t^{3/5}$, where the radiation rate drops as $t^{-2/5}$. The trend of the variations is not much different from that when radiation damping is neglected; there the energy increases as

$$\gamma \sim \alpha_0^{1/3}(\omega_1 t)^{2/3}, \tag{44a}$$

the critical frequency increases as

$$\omega_c \sim \alpha_0^{4/3}(\omega_1 t)^{2/3}\omega_1\,, \tag{44b}$$

the radiation rate remains constant,

$$\left(\frac{dE}{dt}\right)_{\rm rad} = K_0 mc^2 = \text{const}, \tag{44c}$$

and the two velocity components are, respectively,

$$\beta_1 = \alpha_0^{1/3}(\omega_1 t)^{-1/3} \tag{44d}$$

and

$$\beta_3 = 1 - \alpha_0^{2/3}(\omega_1 t)^{-2/3}. \tag{44e}$$

One of the main distinctions is that, neglecting radiation reaction, the characteristics of the particle's motion are determined by the initial value of $\alpha_0 = \gamma_0(1-\beta_{30})$, where at strong damping the initial value plays a small role. (See Fig. 3.)

It is interesting to see that the results obtained here take exception from a conclusion proposed by Pomeranchuk. As restated in Landau and Lifshitz,[4] Pomeranchuk observed that the damping force, being proportional to γ^2, becomes the main force acting on the particle in an electromagnetic field when the energy of the particle becomes sufficiently large. In this case the Lorentz force is negligible and the change of the particle's energy per unit length can be equated to the damping force alone. Therefore, there is a constant limit γ_c on the final energy of the particle after passing through a general electromagnetic field. The upper bound of the energy is given by

$$\gamma_c = \left[\int^{\infty} g(z)d(z)\right]^{-1}, \tag{45}$$

where

$$g(z) = \frac{1}{\omega_0 c}[(E_x - H_y)^2 + (E_x + H_y)^2]\left(\frac{e}{mc}\right)^2,$$

and the integration is performed along the path of the motion. The apparent discrepancy between our result and that of Pomeranchuk originates from the fact that the damping force not only depends on the square of the energy of the particle but also on the angle between the particle's motion and the field. Being forced by the acceleration and deceleration due to the Lorentz and the damping force, the particle tends to align itself to have minimum perpendicular acceleration. As shown in Eq. (20) of Sec. II, in a large-scale field the effect of the damping force (relative to the Lorentz force) is a monotonically decreasing function of time. Thus, even in a case where the radiation damping is indeed the main force when the particle first entered the field, the Lorentz force will eventually overtake the damping force. It is not justified to neglect the Lorentz force in the calculation of the final energy of the particle unless the passage of the particle through the field occurs in a time which is short compared to the "orientation time," i.e., the time needed to align the particle's motion along the direction of minimum perpendicular acceleration. In the present case, it is $(\omega_0/\omega_1^3)^{1/2}$.

It is revealing to consider the ratio of the radiation power Kmc^2 to the rate of the work done by the field $(e\vec{E}\cdot\vec{v})$. When both the damping and the Lorentz forces are included,

$$\frac{(dW/dt)_{\rm rad}}{(dW/dt)_{\rm work}} = \frac{\alpha\omega_1^2\omega_0^{-1}mc^2}{\omega_1\beta_1 mc^2} = \frac{1}{4}\,, \tag{46}$$

i.e., three quarters of the work done by the field is absorbed by the particle, whereas only one quarter of the work goes to radiation. When the effects of the Lorentz force on the particle's motion are neglected

$$\frac{(dW/dt)_{\rm rad}}{(dW/dt)_{\rm work}} = 1, \tag{47}$$

so all the work done by the field transforms to radiation with the energy of the particle remaining constant. On the other hand, if the damping effects of radiation are neglected,

$$\frac{(dW/dt)_{\rm rad}}{(dW/dt)_{\rm work}} \cong \alpha_0^{5/3}\omega_1\omega_0^{-1}(\omega_1 t)^{1/3}. \tag{48}$$

This is because the work done by the electric field diminishes as the cross field forces the particle to move in a direction perpendicular to $\vec{E}$ and $\vec{H}$, but the radiation rate remains constant because the effect due to the increase of the energy balances that due to the decrease of the perpendicular acceleration. Since Eq. (48) indicates that the radiation loss exceeds the work done, one would expect the particle to lose energy. The root of inconsistency between Eqs. (48) and (44a) originates from the fact that the energy is not conserved in the framework of the Lorentz equation and the Maxwell equation.

Next let us consider the case of $n=1$, which corresponds to the case of a charged particle carried by a spherical wave. The four-velocity is given by Eq. (38) with $\chi = \int Z^{-1}d\tau$. The short-time $(\omega_1\chi \ll 1)$ behavior of u_μ is similar to that of the $n=0$ case, for the particle hardly travels far enough to feel the gradient of the field. The long-time behavior, though, is quite different. The cumulative radiation-reaction effect on the motion of the particle is represented by the $2K_0\int Z^{-2}d\tau$ term in

$$\eta(\tau) = \left(1 + 2K_0\int Z^{-2}d\tau\right)^{1/2}. \tag{49}$$

Normally as $\omega_1\chi \gg 1$, $2K_0\int Z^{-2}d\tau$ becomes large and the unity term in η can be neglected (this is always true in the case of a constant cross field), but because of the inverse-square power of Z in the integral, for a certain initial condition [as we shall show, it is $\alpha_0 < (\omega_0^3\omega_{10}^{-4}\omega_2^{1})^{1/5}$] $2K_0 \int Z^{-2}d\tau$ can be small even at $\omega_1\chi \gg 1$. (Decrease of the field intensity along the Z axis coupled with the forced orientation of the particle's direction of motion limited the effect of damping.) Thus, even at $\omega_1\chi \gg 1$, one could have small cumulative damping,

$$\eta \cong 1 + K_0 \int Z^{-2}d\tau \approx 1. \tag{50}$$

It should be emphasized that large and/or small cumulative damping are not the same as strong and/or weak damping. The latter is determined by the ratio of the reaction force to the accelerating Lorentz force and specified by the parameter $R_c = \gamma^2\omega_H/\omega_0$, where the former is determined by the cumulative radiation effect over a distance l and specified by $\langle\gamma\omega_H^2\rangle(l/c\omega_0)$. One could have strong damping along a part of the passage but still come out with small cumulative damping; the converse is, of course, also possible.

For practical purposes it is usually more convenient to find u_μ and relevant quantities as a function of Z instead of t. Since no matter what the initial direction of motion is, the particle soon orients itself to moving along the Z direction, this purpose can be achieved simply by replacing t by z/c in the long-time limit. When the cumulative radiation-reaction effect is small, we obtain from Eq. (38) that at $Z \gg 1$ the energy changes to

$$\gamma(Z) = \gamma_1 - \alpha_0^2(\omega_1^2/\omega_0\omega_2), \tag{51a}$$

where

$$\gamma_1 = [\alpha_0(\omega_1/\omega_2)^2]^{1/3}; \tag{51b}$$

the radiation rate is

$$\left(\frac{dW}{dt}\right)_{\text{rad}} = -\alpha_0^2(\omega_1^2/\omega_0)Z^{-2}mc^2; \tag{51c}$$

and the critical frequency is

$$\omega_c = \alpha_0^{4/3}(\omega_1^5/\omega_2^2)^{1/3}Z^{-1}. \tag{51d}$$

On the other hand when the cumulative damping effect is large, we have, at large Z, the energy

$$\gamma = (\omega_1^2\omega_0/\omega_2^3)^{1/5}, \tag{52a}$$

the radiation rate

$$\left(\frac{dW}{dt}\right)_{\text{rad}} = -(\omega_1^2\omega_0^2\omega_2^2)^{1/5}Z^{-2}mc^2, \tag{52b}$$

and the critical frequency

$$\omega_c = (\omega_1^3\omega_0^4/\omega_2^2)^{1/5}Z^{-1}. \tag{52c}$$

From Eqs. (38) and (51), we see that whether the

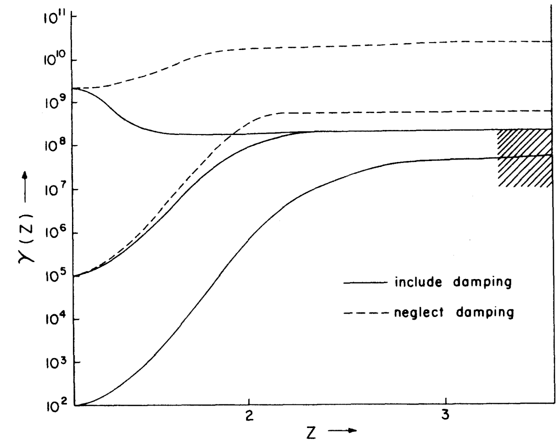

FIG. 4. Energy change of a charged particle in a cross field which varies as Z^{-1}.

cumulative damping is large or small depends on whether the initial value α_0 is larger or smaller than the critical value $\alpha_c = (\omega_0^3\omega_2/\omega_1^4)^{1/5}$. Similar to the case of constant cross field, when the damping is significant, the asymptotic behavior of these various quantities does not depend on the initial condition of the particle but is determined solely by the intensity and the gradient of the field. On the other hand, because of the weakening of the field as the particle moves along the Z axis, the final energy of the particle does approach a constant as it moves into a weaker and weaker field. (See Fig. 4.)

The above analysis can extend to cases of arbitrary n to obtain asymptotic solutions. From Eq. (38) we have at the long-time limit

$$\gamma(Z) \cong \frac{\alpha_0(\omega_1/\omega_2)^2(\int\gamma^{-1}Z^n dZ)^2}{2[1+2(K_0/\omega_2)\int\gamma^{-1}Z^{2n}dZ]^{1/2}}\quad Z \gg 1. \tag{53}$$

At negligible damping Eq. (53) gives

$$\gamma(Z) = \begin{cases} \gamma_s, & n < -1, \\ 9^{1/3}\gamma_s Z^{2(n+1)/3}, & n > -1, \end{cases} \tag{54}$$

where

$$\gamma_s = \left[\frac{\alpha_0}{2(n+1)^2}\left(\frac{\omega_1}{\omega_0}\right)^2\right]^{1/3}. \tag{55}$$

With significant damping Eq. (53) gives

$$\gamma(Z) = \begin{cases} \gamma_L f_1(n), & n < -1 \\ \gamma_L f_2(n) Z^{2(n+1)/3}, & -\frac{1}{4} > n > -1, \\ \gamma_L f_3(n) Z^{(2n+3)/5}, & n > -\frac{1}{4} \end{cases} \tag{56}$$

where $\gamma_L = (\omega_1^2\omega_0)^{1/2}$ and $f_1(n)$, $f_2(n)$, and $f_3(n)$ are functions of n with values of the order of unity. Therefore we arrive at the general conclusion that in a cross field of equal intensity, the energy of the particle approaches a constant if the field drops faster than Z^{-1}, and the energy of the particle increases indefinitely otherwise.

One of the more interesting applications is to apply the above analysis to the acceleration and radiation of electrons in the wave zone of a pulsar. A pulsar is a magnetized neutron star which rotates like a giant dipole generating very intense electromagnetic waves with frequencies equal to the rotational frequency of the star. Protons and electrons streaming out from the magnetosphere of the pulsar are swept up by the wave at the base of the wave zone. As the particles ride with the wave to the outer nebula they see a static corss field similar to that given by Eq. (37) with $n=-1$. Gunn and Ostriker[6] showed that if radiation from the accelerated charge can be neglected, a particle injected into the wave zone with an initial condition α_0 will be accelerated to $[\alpha_0(\omega_1/\omega_2)^2]^{1/3}$ with ω_2 as the frequency of the rotation. Later publications[9] have included radiation as perturbations. However, the radiative damping parameter $R_c = \gamma^2\omega_H/\omega_0$ can be very large for relativistic electrons, for example, ω_1 for an electron $\cong 5\times 10^{12}$ $\sec^{-1}$ at the base of the wave zone of the crab pulsar. Therefore it is essential to take strong radiation damping into consideration in the study of electron accelerations. It should also be noticed that $\omega_2 = 2\times 10^2 \sec^{-1}$ for the crab pulsar, hence $\omega_1/\omega_2 \approx 10^{10}$ and the $\chi = (\omega_1/\omega_2)\int(\gamma Z)^{-1}dZ \gg 1$ approximation is valid even for very-high-energy electrons.

Let us use $\omega_1 = 5\times 10^{12} \sec^{-1}$ and $\omega_2 = 2\times 10^2 \sec^{-1}$ in the formulas obtained in Sec. II to illustrate the physical picture expected in the wave zone of the crab pulsar.

If at injection the momentum of the particle is

$$\alpha_0 < \alpha_c = (\omega_0^3\omega_2^3/\omega_1^4)^{1/5} \approx 2\times 10^4, \tag{57}$$

and radiation damping is only a small perturbation, the particle will be accelerated to

$$\gamma = \gamma_1[1-(\alpha_0/\alpha_c)^{5/3}], \tag{58}$$

where

$$\gamma_1 = [\alpha_0(\omega_1/\omega_2)^2]^{1/3} \approx 10^7\alpha_0^{1/3}, \tag{59}$$

in Eq. (58) $-(\alpha_0/\alpha_c)^{5/3}$ represents the damping effect.

If $\alpha_0/\alpha_c > 1$, the damping force dominates the motion of the particle at first but is diminished to equalize the Lorentz force at an energy,

$$\gamma = \alpha_2 = (\omega_1^2\omega_0/\omega_2^3)^{1/5} = 2\times 10^8. \tag{60}$$

There are two other quantities of astrophysical interest, the total energy converted from the low-frequency dipole wave to the high-frequency radiation by an electron before it reaches the outer nebula, and the spectrum shape of this radiation. The radiation rate of the accelerating electron at r is given by

$$\left(\frac{dE}{dt}\right)_{\text{rad}} = \begin{cases} -\alpha_0^2\,\dfrac{\omega_1^2}{\omega_0}\left(\dfrac{r_0}{r}\right)^2 mc^2, & \alpha_0 \ll \alpha_c \\ -(\omega_1^2\omega_0\omega_2^2)^{1/5}\left(\dfrac{r_0}{r}\right)^2 mc^2, & \alpha_0 \gg \alpha_c. \end{cases} \tag{61}$$

Thus, the "up conversion" of the energy from the very-low-frequency wave to radiation well extended into the x-ray range by a single electron as the particle rides the wave out to the nebula, is

$$\Delta E = \int\left(\frac{dE}{dt}\right)_{\text{rad}} dt$$

$$= \begin{cases} (\alpha_0\omega_1/\omega_2\omega_0)mc^2, & \alpha_0 < \alpha_c \\ (\omega_1^2\omega_0/\omega_2^3)^{1/5}mc^2, & \alpha_0 > \alpha_c. \end{cases} \tag{62}$$

The frequency distribution of this radiation is given by $I(\omega) = \int I_i(\omega, r)dr$, where I_i is the instantaneous radiation spectrum at r. Because I_i peaked at a critical frequency ω_c, which is inversely proportional to γ, and because the intensity of I_i is inversely proportional to the square of γ, we find that

$$I(\omega) = (\alpha_0^2\omega_1/\omega_2\omega_0^3)^{1/3}mc^2$$

$$\text{and cut off at } \alpha_0^{4/3}(\omega_1^5/\omega_2^2)^{1/3} \text{ for } \alpha_0 < \alpha_c, \tag{63}$$

$$I(\omega) = \tfrac{1}{2}(\omega_2\omega_0^3\omega_1)^{-1/5}mc^2$$

$$\text{and cut off at } (\omega_1^3\omega_0^4/\omega_2^2)^{1/5} \text{ for } \alpha_0 > \alpha_c.$$

In conclusion, the final energy γ, the radiation rate $(dE/dt)_{\text{rad}}$ and the characteristic frequency ω_c of a relativistic particle moving in a cross field varying as Z^{-n} are determined by three parameters: ω_0, ω_1, and ω_2, which characterize the fundamental frequency of a free electron, the strength of the field, and the gradient of the field, respectively. In the limit

$$\omega_0 = \frac{3mc^3}{2e^2} = 1.8\times 10^{23} \sec^{-1} \gg \omega_1 = \frac{eH_0}{mc} \gg \omega_2 = c/Z_0, \tag{64}$$

explicit expressions of γ, $(dE/dt)_{\text{rad}}$ and ω_c can be derived; they are given in this section. For astrophysical applications (a pulsar's magnetosphere, for example) Eq. (64) is usually satisfied.

ACKNOWLEDGMENT

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APPENDIX A

We want to show that within the realm of classical electrodynamics the Lorentz-Dirac equation for an ultrarelativistic particle can be expressed as

$$\dot{u}_\mu = \omega_{\mu\nu} u^\nu - K u_\mu + O(\delta) , \tag{A1}$$

where

$$K = \omega_0^{-1} \omega_{\mu\nu} u^\nu \omega^{\mu\,\lambda} u_\lambda \tag{A2}$$

is the radiation reaction and

$$\delta = \gamma \omega_{\mu\nu} / \omega_0 \tag{A3}$$

represents the ratio of the reaction force to the Lorentz force in the rest frame of the particle. For classical electrodynamics to be applicable, δ must be smaller than $e^2/\hbar c = \frac{1}{137}$.

Equation (A1) is not a new result. In Landau and Lifshitz it is derived by substituting

$$\dot{u}_\mu = \omega_{\mu\nu} u^\nu \tag{A4}$$

into the damping term $\omega_0^{-1} \dot{u}^\nu \dot{u}_\nu u_\mu$. Although later on the authors explain that this result is valid even at strong damping, the proof itself is not satisfactory at that limit, since when the damping force dominates the Lorentz force, Eq. (A4) is obviously incorrect.

A more rigorous derivation can be achieved by recalling that $\dot{u}^\nu \dot{u}_\nu$ is a four-scalar. Thus $\dot{u}^\nu \dot{u}_\nu$ measured in the laboratory must be equal to that measured in the instantaneous rest frame of the particle. Let us choose a Z axis along the instantaneous direction of motion; the fields in the rest frame are

$$\begin{aligned} &H_1' = \gamma(H_1 + \beta E_2) , \quad E_1' = \gamma(E_1 - \beta H_2) , \\ &H_2' = \gamma(H_2 - \beta E_1) , \quad E_2' = \gamma(E_2 + \beta H_1) , \\ &H_3' = H_3 , \quad E_3' = E_3 . \end{aligned} \tag{A5}$$

In the rest frame the four-accelerations are

$$\dot{u}_\mu' = \omega_{\mu\nu}' u'^\nu + O(\delta) , \tag{A6}$$

where

$$u'^\nu = (0,0,0,1) ;$$

hence

$$\begin{aligned} \dot{u}_\mu \dot{u}^\mu &= \dot{u}_\mu' \dot{u}'^\mu \\ &= \left(\frac{e}{mc}\right)^2 [\gamma^2 (E_1 - \beta H_2)^2 + \gamma^2 (E_2 + \beta H_1)^2 \\ &\qquad + E_3^2] + O(\delta) \\ &= \omega_{\mu\nu} u^\nu \omega^{\mu\,\lambda} u_\lambda + O(\delta) . \end{aligned} \tag{A7}$$

The time derivative of the four-accelerations is also a four-vector

$$\ddot{u}_\mu' = \omega_{\mu\nu}' \omega'^{\nu l} u_l' . \tag{A8}$$

Transforming back into the laboratory frame gives the Schött term

$$\omega_0^{-1} \ddot{u}_\mu = O(\delta/\gamma) . \tag{A9}$$

Therefore, the Lorentz-Dirac equation can be written as

$$\dot{u}_\mu = \omega_{\mu\nu} u^\nu - K u_\mu + O(\delta) + O(\delta/\gamma) .$$

The ratio of the damping force to the Lorentz force is given by $\gamma\delta \approx \gamma^2 \omega_{\mu\nu}/\omega_0$ except for cases where $\vec{\beta} \times (\vec{E} + \vec{\beta} \times \vec{H})$ vanishes [i.e., $(E_1 - \beta H_2) = (E_2 + \beta H_1) = 0$ in (A7), e.g., accelerated by a pure electric field along the field line]; then the damping force is proportional to $\omega_{\mu\nu}/\omega_0$.

APPENDIX B

We want to show that if G_j are eigenvalues of $\omega_{\mu\nu}$ then the solutions of the Lorentz equation are

$$\dot{f}_\mu = \omega_{\mu\nu} f^\nu . \tag{B1}$$

Equation (B1) can be expressed as

$$f_\mu = \sum_{j=1}^{4} e^{G_j \tau} P_j f_\nu (0) , \tag{B2}$$

where

$$P_j = \prod_{l \neq j}^{4} \frac{\omega_{\mu\nu} - G_l I}{G_j - G_l} \tag{B3}$$

are the projection operators. This follows directly from a proof first given by Rosen[5] for finite transformation in SU(3) space. Let

$$f = (f_1, f_2, \ldots, f_n) \tag{B4}$$

be an n-dimensional vector and $\omega_{\mu\nu}$ be an $n \times n$ matrix with eigenvalues $G_1, G_2, \ldots, G_n$, where

$$\prod_{j=1}^{n} (\omega_{\mu\nu} - G_j I) = 0 . \tag{B5}$$

When G_j are all distinct we can define a projection operator

$$P_j = \prod_{l \neq j}^{4} \frac{\omega_{\mu\nu} - G_l I}{G_j - G_l} . \tag{B6}$$

It follows from Eq. (B5) that

$$P_j \omega_{\mu\nu} = \omega_{\mu\nu} P_j = G_j P_j , \quad \text{no sum over } j \tag{B7}$$

hence

$$P_j P_l = P_l P_j = \delta_{jl} P_j . \tag{B8}$$

Also, by using the partial-fraction decomposition of the inverse of $\prod_{j=1}^{n} (\omega_{\mu\nu} - G_j)$ as a formal identity, one can prove that

$$\sum_{j=1}^{n} P_j = I . \tag{B9}$$

Thus P_j form a complete set of projection operators, and it follows immediately from Eqs. (B7) and (B9) that

$$e^{\omega_{\mu\nu}\tau} f_\mu = \sum_{j=1}^{n} e^{G_j\tau} P_j f_\mu . \qquad \text{(B10)}$$

Since in operator form the Lorentz equation

$$\dot{f}_\mu = \omega^{\mu\nu} f_\nu \qquad \text{(B11)}$$

can be written as

$$f_\mu = e^{\omega_{\mu\nu}\tau} f_\mu(0) . \qquad \text{(B12)}$$

Equation (B2) is the general formal solution of (B11).

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