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Hypermomentum in hadron dynamics and in gravitation

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The infinite unitary irreducible spinor representations of the SL$(3,R)$ algebra of hadron excitations are embedded in a global GA$(4,R)$ with intrinsic dilation, shear, and spin pieces in its hypermomentum current (i.e., the affine generalization of angular momentum). When gauged over a space with a local Minkowski metric, GA$(4,R)$ reproduces the metric-affine theory of gravity, in which the intrinsic hypermomentum is coupled to the connection, and the energy-momentum to the tetrad.

I. THE SL$(3,R)$ ALGEBRA OF HADRON EXCITATIONS

It was suggested by Dothan *et al.*[1] (whom we refer to as DGN) that if "$\cdots$ long sequences of fairly well-defined levels should emerge from an experimental study of baryons and mesons, one might very well wish to describe them by means of a noncompact algebra," and that the excitations involved might be related to stresses causing deformations in an extended structure. The rotational bands in deformed nuclei with $\Delta J=2$ were cited as analogous: The appropriate algebra here is the Lie algebra of SL$(3,R)$, generated by the three orbital angular momentum operators and by the five time derivatives of the energy quadrupole operators, which generate shearlike deformations. It was suggested that a $\Delta J=2$ relation for the Regge trajectories[2] could arise from a similar mechanism. In the study of extended structures the notion of infinite trajectories generated by noncompact spectrum-generating algebras (SGA) has since been further exploited in other directions, for example in dual models and strings.[3]

The model presented in DGN will be briefly summarized: The generators of SL$(3,R)$ consist of three angular momentum operators $\vec{I}$ generating the compact SO(3) subgroup, and five noncompact generators N, which transform under SO(3) as an $I=2$ representation. Thus, the N connect different SO(3) representations at $\Delta J=2$ intervals. In DGN, $\vec{I}$ was taken to be $\vec{L}$, the "intrinsic" quark field *orbital* angular momentum,[4] defined by $\vec{L}=\vec{J}-\vec{S}$ where $\vec{J}$ is the total rest-frame angular momentum of a hadron and S the total quark spin $\int d^3x\, q^\dagger \vec{\sigma} q$ (the operators $q^\dagger\sigma_\alpha q = \bar{q}\gamma_\alpha\gamma_5 q$ are in principle observable,[5] since their matrix elements occur in Gamow-Teller weak transitions). The infinite-dimensional unitary representations $\mathfrak{D}_0(L=0,2,4,\dots)$ and $\mathfrak{D}_1(L=1,3,5,\dots)$ of SL$(3,R)$ (obtained by the action of the components of N as ladder operators[1]) were interpreted as bands of L excitations superimposed on the total quark spin, thus somewhat resembling the observed structure of the Regge trajectories.

Some further physical understanding of the orbital N operators is provided by their nuclear applications.[6] These involve a computational approximation in which one assumes that the spatial charge distribution is the same as that of mass. The band structure appears to fit observations roughly, but the commutator is far from saturated by the lower states, a fact which is possibly due to the approximations.[7] For hadrons[8] the algebra reproduces the Chew-Frautschi plot[2] $L=\alpha+\beta E^2$ asymptotically (i.e., for large L), and using the same approximations as in the nuclear case, yields plausible values for the electric radii.

The above scheme involves an "orbital" interpretation of the generators of SL$(3,R)$. The N generate (volume-preserving) shear strains. The operators $\vec{L}$ and N correspond to *orbital hypermomentum* charges [see Eq. (3.10) below]. Some of the experimental evidence seems, however, to call for direct J excitations. If one plots the most recent mass-squared values of hadrons[9] against their spins, then in the corresponding Regge trajectories of given parity there seems indeed to be a $\Delta J=2$ rule at work.[10] The reason for the orbital interpretation of the SL$(3,R)$ generators in DGN was chiefly that no half-integer representations of SL$(3,R)$ were known at that time. It is easy to see that SL$(3,R)$ has no finite-dimensional half-integer representations, since the fundamental triplet representation has $I=1$.

The unitary infinite-dimensional irreducible representations of the principal series for SL(n,R) were described by Gelfand and Graev[11] in a functional form that is inappropriate in the present context. One of the authors enlisted the help of Joseph who proved[12] that there exists a *half-integer representation* $\mathfrak{D}_{1/2}$ ($I=\frac{1}{2},\frac{5}{2},\ldots$) but that there is no multiplicity-free $\mathfrak{D}_{3/2}$ representation. The theory was further developed by Biedenharn *et al.*[8] and by Ogievetskii and Sokachev[13], who supplied a refined construction of $\mathfrak{D}_{1/2}$. Recently, one of us[14] gave a detailed discussion of the bivalued representations of the group of general coordinate transformations also from a topological point of view.

Having thus reviewed the fundamental importance of the SL$(3,R)$ transformations for hadronic matter, we combine it with scale and Poincaré transformations, thereby arriving at the general affine group GA$(4,R)$.

II. THE GENERAL AFFINE GROUP GA(4,R)

This group is the semidirect product of the general linear group GL$(4,R)$ and the translations $\mathcal{T}$. Its Lie algebra is defined by the commutation relations

$$\begin{aligned}&[f_\alpha,f_\beta]=0\,,\\ &[f_\alpha{}^\beta,f_\gamma{}^\delta]=\delta_\alpha{}^\delta f_\gamma{}^\beta-\delta_\gamma{}^\beta f_\alpha{}^\delta\,,\\ &[f_\alpha,f_\gamma{}^\delta]=\delta_\alpha{}^\delta f_\gamma\,.\end{aligned}\tag{2.1}$$

The affine group GA(n,R) can be derived by contraction from the semisimple group GL$(n+1,R)$. The contracted group has $(n+1)^2$ generators, n^2+n of them generating GA(n,R). The remaining $n+1$ (which we denote by e^α and e), together with the translation generators f_α, generate an n-dimensional Heisenberg algebra $[f_\alpha,e^\beta]=\delta_\alpha{}^\beta e$, with e commuting with the entire contracted group.[15] This derivation of the affine group indicates that some of the Casimir operators and labeling characteristics of GL$(5,R)$ would be expected to be preserved in GA$(4,R)$.

We now consider the infinitesimal action

$$\delta\phi=\lambda^\alpha\partial_\alpha\phi+\lambda^\alpha{}_\beta(x^\beta\partial_\alpha+f_\alpha{}^\beta)\phi\tag{2.2}$$

of the group GA$(4,R)$ on fields ϕ in a space-time with a local Minkowski metric. The existence of the metric singles out the Poincaré subgroup $\mathcal{P}$, generated by $f_{[\alpha\beta]}$ and f_α. For a tensor field ϕ, the $f_\alpha{}^\beta$ are simply finite-dimensional matrices. To deal with spinor fields, we introduce the concept of a *polyfield*. This is an infinite-dimensional unitary representation of GL$(4,R)$ consisting of an infinite set of excited fermion fields

$$\psi=\psi_{1/2}\oplus\psi_{5/2}\oplus\cdots\,,$$

where ψ_s is a unitary spin-s representation of the Poincaré subgroup. The components of ψ_s can be characterized, for instance, as quantities $\psi_{\alpha\beta\cdots}$ with a completely symmetric set of $s-\frac{1}{2}$ tensor indices and a single Dirac index, satisfying certain identities coming from the subsidiary conditions of the field equations. In momentum space, ψ is obtained from the $\mathfrak{D}_{1/2}$ representation of the subgroup GL$(3,R)$ of GA$(4,R)$.[16] The compact generators $f_{[\mu\nu]}$ ($\mu,\nu=1,2,3$) will be direct sums of spin matrices. The dilation operator $f_\rho{}^\rho$ will also not connect different spins, while the noncompact generators $f_{(\mu\nu)}-\frac{1}{3}\delta_{\mu\nu}f_\rho{}^\rho$ will connect spin s with spin $s\pm2$. That is, we extend the little group SO(3) of $\mathcal{P}$, generated by $f_{[\mu\nu]}$, by introducing six extra generators. When dealing with lightlike momentum, GL$(3,R)$ would arise from a similar extension of the null-plane little group E_2 of $\mathcal{P}$.

Equation (2.2) is interpreted as follows: While remaining within the context of special relativity, we have introduced an algebraic structure superimposed on Poincaré invariance, as in the cases of dilation invariance and spin independence. In all these cases, the algebraic structure has a dynamical origin (e.g., asymptotic freedom in quantum chromodynamics) and effectively enlarges the material (Hilbert space) Poincaré group without affecting the geometry.

III. THE CANONICAL HYPERMOMENTUM CURRENT–INTRINSIC AND ORBITAL

We now consider a simple special-relativistic Lagrangian model involving fields (or polyfields) ϕ, with Lagrangian density $\mathcal{L}(\phi,\partial_i\phi)$ and study the Noether currents associated with the transformations (2.1). The Noether current associated with the translation subgroup is the canonical energy-momentum

$$\Sigma_i{}^j\equiv\mathcal{L}\delta_i{}^j-\pi^j\partial_i\phi\quad(\pi^j\equiv\partial\mathcal{L}/\partial_j\phi)\tag{3.1}$$

and that associated with GL$(4,R)$ is the *hypermomentum current*[17,18]

$$\Upsilon_i{}^{jk}=\Lambda_i{}^{jk}+\Delta_i{}^{jk}\,,\tag{3.2}$$

which consists of an orbital piece

$$\Lambda_i{}^{jk}\equiv-x^j\Sigma_i{}^k\tag{3.3}$$

and an intrinsic piece

$$\Delta_i{}^{jk}\equiv-\pi^k f_i{}^j\phi\,.\tag{3.4}$$

The currents satisfy the conservation law

$$\partial_j\Sigma_i{}^j=0\tag{3.5}$$

and the quasiconservation law[17-19]

$$\partial_k\Upsilon_i{}^{jk}=-\sigma_i{}^j\quad(\text{i.e., }\Sigma_i{}^j-\sigma_i{}^j=\partial_k\Delta_i{}^{jk})\,,\tag{3.6}$$

where σ_{ij} is a *symmetric* tensor defined by the response of the Lagrangian density to strain:

$$\lambda^i{}_j\sigma_i{}^j = \delta\mathfrak{L} - \lambda^i{}_j(x^j\partial_i + \delta_i{}^j)\mathfrak{L}. \qquad (3.7)$$

The charges associated with the currents are the momentum

$$P_i \equiv \int d^3x\ \Sigma_i{}^0 \qquad (3.8)$$

and the total hypermomentum

$$\Upsilon_i{}^j \equiv \int d^3x\ \Upsilon_i{}^{j0} \qquad (3.9)$$

consisting of orbital hypermomentum

$$\Lambda_i{}^j \equiv \int d^3x\ \Lambda_i{}^{j0} = -\int d^3x\ x^j\Sigma_i{}^0 \qquad (3.10)$$

and intrinsic hypermomentum

$$\Delta_i{}^j \equiv \int d^3x\ \Delta_i{}^{j0} = -\int d^3x\ \pi^0 f_i{}^j\phi\,. \qquad (3.11)$$

[For an ordinary Dirac field, the quantities $\Lambda^{(\mu\nu)}$ are time derivatives of the energy quadrupoles. Such an interpretation is no longer possible for polyfields.]

Under the assumption of canonical equal-time commutation relations for ϕ, the intrinsic hypermomentum generates the intrinsic $\mathrm{GL}(4,R)_S$, and the three-space components of total hypermomentum and linear momentum generate the subgroup $\mathrm{GL}(3,R)$ consisting of dilations, shears, and rotations of the matter fields. In the spirit of current algebra, a reasonable hypothesis is that the hypermomentum and momentum of hadronic matter obey these same commutator algebras, and that the hadronic currents satisfy (3.5) and (3.6). $\mathrm{GL}(3,R)$ commutes with P_0, and can therefore be considered as an approximate rest symmetry—we have no trouble with "no go" theorems.[20] Note that it is the existence of the infinite-dimensional spinor representations of $\mathrm{GL}(3,R)$ that enable us to extend the concept of intrinsic spin to *intrinsic* hypermomentum, for fermionic matter.

We now have an alternative interpretation of the Regge trajectories, in which the quark $\mathfrak{D}_{1/2}$ is interpreted as the sequence ($J=\frac{1}{2}, \frac{5}{2}, \ldots$) of excitations of the *total* angular momentum and the meson and baryon trajectories are $q\bar{q}$ and qqq recombinations (including the trajectories $\frac{3}{2}$, $\frac{7}{2}, \ldots$). At the present state of our knowledge, the polyfield should be regarded as an intermediate description, presumably including the original quark field and some of the color-gluon effects (the excited levels may correspond to the action of a gluon pair with $J=2$ and no color). The volume-preserving stresses may actually correspond to the effects of confinement.

Note that the skewsymmetric part of (3.6) is simply the conservation of total angular momentum. The trace of the same equation shows that the dilation current ($\Upsilon_i \equiv \Upsilon_k{}^{ki}$) is not, in general, conserved. In the domain of asymptotic freedom, we would have an approximate scale invariance ($\sigma_k{}^k = 0$) which then leads to a conserved dilation current. The divergence of the intrinsic dilation current would then be the trace of the energy-momentum tensor $\partial_k\Delta^k = \Sigma_k{}^k$. Intrinsic $\mathrm{GL}(4,R)$ invariance, associated with the conservation of the intrinsic hypermomentum currents, may well be an approximate symmetry of the asymptotic freedom regime in quantum chromodynamics. We know that scaling arises as a logarithmic approximation, and a similar situation may describe spin independence [observed approximate SU(6)] and the $\Delta J = \pm 2$ excitation bands. We then have a unified description of these three phenomena[17,18]; they are manifestations of a single current, the hypermomentum current. This suggests a link with gravitation, since the intrinsic hypermomentum current is coupled to the linear connection of space-time in a very natural generalization of Einstein's theory. In the spirit of current algebra, this determines its matrix elements, just as the coupling to the metric field determines the matrix elements of the energy-momentum tensor. This generalization is the metric-affine theory of gravitation.[17-19] In the following section we show how the metric-affine theory arises as a gauge theory of $\mathrm{GA}(4,R)$.

IV. THE AFFINE GAUGE THEORY WITH LOCAL MINKOWSKIAN STRUCTURE

The metric-affine gravitational theory is based on a space (L_4, g) in which the components of the metric g_{ij} and the connection $\Gamma_{ij}{}^k$ (not necessarily symmetric) are regarded as 74 independent fields in a variational principle. The gravitational Lagrangian density is a scalar density $\mathfrak{V}$ constructed from these components and their derivatives. The derivatives of "matter fields" occurring in the rest of the Lagrangian density are covariant derivatives constructed from the connection $\Gamma_{ij}{}^k$. Thus we have a minimal coupling hypothesis that universally couples the connection to matter. Only gauge fields (electromagnetism, gluons, etc.) are not coupled to the connection.

Alternatively, the metric-affine theory can be arrived at by generalizing a global affine group $\mathrm{GA}(4,R)$ to a gauge group, over a metric space-time with a local Minkowskian structure. To establish the notation, consider first the usual Yang-Mills theory of an unspecified Lie group G, with generators f_A satisfying $[f_A, f_B] = c_{AB}{}^C f_C$. Consider the action of an infinitesimal element $\mu = \mu^A f_A$ of the gauge group G combined with an infinitesimal coordinate transformation $x^i \to x^{i\prime} = x^i - \xi^i$. For a set of polyfields or fields ϕ which transform among themselves linearly under the action of G, we have

$$\delta\phi \equiv \phi'(x) - \phi(x) = (\xi^i \partial_i + \mu)\phi\ . \tag{4.1}$$

Introduce the connection one-form for G ($\Gamma_i dx^i = \Gamma_i{}^A f_A dx^i$), the corresponding covariant derivative operator

$$d_i\phi \equiv \partial_i\phi + \Gamma_i\phi\ , \tag{4.2}$$

and the gauge fields

$$F_{ij} \equiv [d_i, d_j] = \partial_i\Gamma_j - \partial_j\Gamma_i + [\Gamma_i, \Gamma_j]. \tag{4.3}$$

In terms of the parameters $\lambda \equiv \mu - \xi^i\Gamma_i$, we have the transformation laws

$$\begin{aligned} &\delta\phi = (\lambda + \xi^i d_i)\phi\ ,\\ &\delta\Gamma_i = -d_i\lambda + \xi^j F_{ij}\ . \end{aligned} \tag{4.4}$$

Let $\mathcal{L}$ be a Lagrangian density, dependent on ϕ, Γ_i, and a metric g_{ij} (and derivatives of these quantities). Invariance of $\mathcal{L}$ under coordinate transformations and space-time dependent G transformations leads to the identities

$$\begin{aligned} &\sqrt{-g}\,\nabla_j^{\{\}}\sigma_i{}^j - \mathcal{J}_A^j F_{ij}{}^A = 0\ ,\\ &d_i\mathcal{J}_A^i \equiv \partial_i\mathcal{J}_A^i - c_{AB}{}^C\Gamma_i{}^B\mathcal{J}_C{}^i = 0\ , \end{aligned} \tag{4.5}$$

where

$$\sigma^{ij} \equiv \frac{2}{\sqrt{-g}}\frac{\delta\mathcal{L}}{\delta g_{ij}}\ ,\quad \mathcal{J}_A^i \equiv \frac{\delta\mathcal{L}}{\delta\Gamma_i{}^A}. \tag{4.6}$$

We now simply take G to be the 20-parameter group GA(4,R) whose Lie algebra is defined by (2.1), and *identify* the translational part of the group with the operation of parallel transport in space-time. This means that the connection of GA(4,R) becomes a Cartan connection.[21] We obtain a tetrad $e_i{}^\alpha \equiv \Gamma_i{}^\alpha$ and an anholonomic linear connection $\Gamma_{i\alpha}{}^\beta$. Algebraically, the identification of the translations with parallel transport is expressed by $\xi^\alpha = -\lambda^\alpha$ (i.e., $\mu^\alpha = 0$). Since linear momentum, unlike angular momentum, has no intrinsic part, we also set $f_\alpha = 0$ for the field ϕ. Then the equations (4.4) become

$$\begin{aligned} &\delta\phi = (\lambda^\alpha_\beta f^\beta_\alpha + \xi^\alpha\nabla_\alpha)\phi\ ,\\ &\delta e_i{}^\alpha = e_i{}^\beta(\nabla_\beta\xi^\alpha + \lambda^\alpha_\beta + \xi^\gamma F_{\beta\gamma}{}^\alpha)\ ,\\ &\delta\Gamma_{i\alpha}{}^\beta = e_i{}^\gamma(-\nabla_\gamma\lambda^\beta_\alpha + \xi^\delta F_{\gamma\delta\alpha}{}^\beta)\ , \end{aligned} \tag{4.7}$$

where ∇_α is the covariant derivative operator associated with the homogeneous part of the group GL(4,R). We have precisely an affine generalization of the Poincaré gauge theory.[5,22] The Poincaré gauge theory, with a particular choice of the Lagrangian for the gauge fields, is identical with the U_4 gravitational theory of Sciama and Kibble.[23] In the same way, the affine gauge theory derived here is the metric-affine theory. Rewriting (4.7) in terms of the parameters $\mu_\alpha{}^\beta$, we find just the behavior of the anholonomic components of a field ϕ, of a tetrad, and of a connection, under coordinate transformations and space-time-dependent linear tetrad deformations. As shown by one of us,[19] the metric-affine gravitational theory can be formulated as a theory invariant under such tetrad deformations.

If the tetrad is chosen orthonormal, we find that

$$\begin{aligned} &\Sigma_\alpha{}^i = \frac{1}{\sqrt{-g}}\,\delta\mathcal{L}/\delta e_i{}^\alpha\ ,\\ &\Delta_\beta{}^{\alpha i} = -\frac{1}{\sqrt{-g}}\,\delta\mathcal{L}/\delta\Gamma_{i\alpha}{}^\beta \end{aligned} \tag{4.8}$$

are the canonical energy-momentum and the canonical intrinsic hypermomentum current of the field ϕ, which are now defined *dynamically* as the currents that couple to the gauge potentials of GA(4,R). In a Minkowski-space approximation with $e_i{}^\alpha = \delta_i{}^\alpha$, $\Gamma_{i\alpha}{}^\beta = 0$, the transformation law of ϕ in (4.7) becomes identical with (2.2) and the identities (4.5) reduce to the conservation law (3.5) and the quasi-conservation law (3.6).

The metric-affine theory is complete when the Lagrangian density $\mathcal{V}/2k$ for the gravitational field variables g_{ij}, $e_i{}^\alpha$, and $\Gamma_{i\alpha}{}^\beta$ is specified. Choosing orthonormal tetrads, the field equations are

$$\begin{aligned} &\delta\mathcal{V}/\delta e_i{}^\alpha = -2k\sqrt{-g}\,\Sigma_\alpha{}^i\ ,\\ &\delta\mathcal{V}/\delta\Gamma_{i\alpha}{}^\beta = 2k\sqrt{-g}\,\Delta_\beta{}^{\alpha i}\ . \end{aligned} \tag{4.9}$$

The holonomic description is obtained by choosing $e_i{}^\alpha = \delta_i{}^\alpha$ and taking g_{ij} and $\Gamma_{ij}{}^k$ as the independent variables. Defining torsion and nonmetricity to be $S_{ij}{}^k \equiv \Gamma_{[ij]}{}^k$ and $Q_{ijk} \equiv -\nabla_i g_{jk}$, respectively, the connection can be written

$$\begin{aligned} &\Gamma_{ij}{}^k = \left\{{k \atop ij}\right\} - M_{ij}{}^k + \tfrac{1}{2}Q_{ij}{}^k\ ,\\ &M_{ijk} \equiv -S_{ijk} + S_{jki} - S_{kij} - Q_{[jk]i}\ . \end{aligned} \tag{4.10}$$

The tensor $M_{ijk} = -M_{ikj}$ is the contortion. The *spin current* and the *intrinsic dilation + shear current* are coupled to contortion and nonmetricity, respectively, in this formulation:

$$\begin{aligned} &\delta\mathcal{V}/\delta M_{kji} = -2k\sqrt{-g}\,\Delta^{[ij]k}\ ,\\ &\delta\mathcal{V}/\delta Q_{kji} = k\sqrt{-g}\,\Delta^{(ij)k}\ . \end{aligned} \tag{4.11}$$

(See Ref. 17 for details; compare also Ref. 24.) With the gravitational Lagrangian $\mathcal{V} = \sqrt{-g}(R + \beta Q_i Q^i)$ ($Q_i \equiv \frac{1}{4}Q_{ik}{}^k$), nonmetricity does not propagate outside matter, so that the comments of Hayashi[25] (reproducing the Einstein-Weyl dialogue) will not apply.

It is interesting to note that an affine-metric theory of f-gravity is possible, in which the spinor fields are nonlinear realizations of global GA$(4,R)$ rather than infinite-dimensional linear representations. The Goldstone bosons associated with the spontaneous breakdown of GA$(4,R)$ symmetry to Poincaré symmetry would have spin two and spin zero, and give rise to a metric. However, such a scheme is not consistent with the present approach: It is an alternative possibility for linking the metric-affine theory with particle physics in which Eq. (3.6) is interpreted as a "partial conservation of shear and dilation currents." There would be a formal resemblance to the work of Ogievetskii and Borisov,[26] except that the affine group has a different interpretation. Since their affine group is generated by the linear part of the infinite gauge algebra of the *coordinate transformations,* it could not have the dynamical role that they assign to it; there are no conserved Noether currents for such transformations. (Note that a symmetry with Goldstone-type spontaneous breakdown corresponds to a limit in which currents are conserved though the vacuum is not invariant.)

In the Sciama-Kibble theory, the Riemannian space-time of Einstein's theory is generalized to a U_4, so as to incorporate the spin-current dynamically as a source of torsion. There now appear to exist similar phenomenological arguments for a corresponding treatment of the intrinsic dilation and shear currents that give rise to nonmetricity. We hope that this note has clarified the theoretical and phenomenological consequences of this possibility, and shown how the metric-affine theory of gravitation with its (L_4, g) space-time would then provide an appropriate minimal coupling.

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