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Spinor fields in an Einstein Universe. The vacuum-averaged stress-energy tensor

J. S. Dowker and Basil M. Altaie
Department of Theoretical Physics, The University, Manchester, United Kingdom
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The spin-1/2 Green's function in an Einstein universe is computed explicitly and used to evaluate the vacuum-averaged stress-energy tensor. In the massless limit we find Ford's value $\langle T_{00}\rangle = 17(1920\pi^2 a^4)^{-1}$.

I. INTRODUCTION

The importance of the vacuum average of the stress-energy tensor $\langle T_{\mu\nu}\rangle$ and the difficulty of its renormalization in a curved space-time are evidenced by numerous papers on the subject.[1]

In earlier work (Dowker and Critchley[2,3]) an expression was derived for $\langle T_{\mu\nu}\rangle$ in the case of a scalar field in an Einstein universe. It is our intention now to extend this calculation to spin-$\frac{1}{2}$ fields (both massive and massless). Perhaps a few general comments are in order at this point.

Ideally one would like to have a renormalization procedure that is valid in an arbitrary Riemannian space-time. However, there seems to be no general agreement on what this should be, and so it seems reasonable to look at those spaces that are sufficiently simple (this usually means those that have plenty of symmetry) such that a renormalization scheme presents itself naturally and/or so that the various standard renormalization methods can easily be implemented and compared. Such a space-time is the Einstein universe.

In our previous work the renormalization consisted of dropping the "direct" term in the scalar Feynman Green's function that occurs in the limiting expressions for the averaged energy-momentum tensor, etc. Such a procedure extends to the spinor case and will be detailed here.

II. SPINORS IN THE EINSTEIN UNIVERSE

Spinors can be introduced in the general and well-known fashion given first by Fock and Ivanenko and developed by Schrödinger, Bargmann, Infeld and Van der Waerden, and many others. This approach has been used by Schrödinger[4] and by Taub[5] in early work on wave equations in cosmological spaces.

For the Einstein universe there is another, equivalent, method that makes use of the symmetry group $G = SU(2)\otimes SU(2)$ of S^3, the three-dimensional spherical spatial section, in the same way that the Euclidean group is used in flat space. The Killing vectors of the left and right groups are taken as the local dreibein fields and this leads naturally onto the Cartan calculus. Since this approach has been described elsewhere[6] very little of the basic theory will be given.

If the Killing vectors of the left SU(2) group are chosen as the dreibein fields the corresponding spinors are called "left" spinors. Together with the "right" fields, they have the advantage that they transform linearly under the action of the symmetry group G. (They do, however, have the disadvantage that the discrete parity transformation P is effected nonlinearly. This can be seen easily from the facts that P takes the left Killing fields into the right ones and that the relation between these two fields is a position-dependent rotation.)

The intrinsic covariant derivative of ψ, $\nabla_\mu\psi$, is derived to be[6]

$$\nabla_0\psi = \partial_0\psi \quad\text{and}\quad \vec{\nabla}\psi = (\vec{X}+\vec{\Gamma})\psi\,,$$

where the $\vec{X}$ are the left infinitesimal operators of SU(2) and $\vec{\Gamma}$ is the spin affine connection given by

$$\vec{\Gamma} = \frac{1}{4a}\vec{\gamma}\times\vec{\gamma} \quad (a = \text{radius of } S^3) \tag{1}$$

in terms of the standard, constant Dirac matrices $\vec{\gamma} = \{\gamma^i\}$. We should emphasize that we are now using the Cartan frame representation.

It is assumed that ψ satisfies the minimal Dirac equation,

$$\left[i\gamma^0\frac{\partial}{\partial t} + i\vec{\gamma}\cdot(\vec{X}+\vec{\Gamma}) - m\right]\psi = 0\,. \tag{2}$$

The energy-momentum tensor is

$$T_{\mu\nu} = \tfrac{1}{4}i\{[\bar{\psi},\gamma_{(\mu}\nabla_{\nu)}\psi] - [\nabla_{(\nu}\bar{\psi}\gamma_{\mu)},\psi]\}\,,$$

where the square brackets and comma indicate antisymmetrization when ψ is interpreted as a second-quantized operator.

Our aim in this paper is to evaluate the vacuum average of $\hat{T}_{\mu\nu}$, and we shall employ the Green's function method. Thus following Schwinger[7] and others, $\langle\hat{T}_{\mu\nu}\rangle$ defined by

$$\langle\hat{T}_{\mu\nu}\rangle = \langle 0_{\text{out}}|\,\hat{T}_{\mu\nu}\,|0_{\text{in}}\rangle/\langle 0_{\text{out}}|0_{\text{in}}\rangle$$

is written in terms of the Feynman Green's function

$$S_F(x,x') = -i\langle 0_{\rm out}|\, T\{\psi(x)\bar\psi(x')\}\,|0_{\rm in}\rangle / \langle 0_{\rm out}|0_{\rm in}\rangle$$

as

$$\langle \hat T_{\mu\nu}\rangle = \tfrac{1}{2}\lim_{x'\to x} {\rm tr}\,\gamma_{(\mu}(\nabla_{\nu)} - \nabla'_{\nu)})\,S_F(x,x')\,. \qquad (4)$$

For a static universe we can set $|0_{\rm in}\rangle = |0_{\rm out}\rangle = |0\rangle$.

In the present paper we do not wish to set up the full apparatus of Fock space. There is no particular problem, but we wish to move rapidly to the calculation of $\langle \hat T_{\mu\nu}\rangle$.

$S_F(x,x')$ satisfies

$$(i\gamma^\mu\nabla_\mu - m)\,S_F(x,x') = 1\delta(x,x')\,. \qquad (5)$$

Strictly speaking, since $S_F(x,x')$ diverges as x' tends to x, parallel propagators, spinor and tensor, should be incorporated into expression (4) in order to maintain covariance throughout the limit (cf. Deser and Boulware[8]). However, since the space-time is static no propagators will be needed if x and x' are separated only in the time direction. Further, our renormalization process is to subtract a term from S_F, giving an $S_{\rm sub}$ which is finite at $x=x'$, and then to use $S_{\rm sub}$ in place of S_F in (4). The limit is then unambiguous. The only question is whether there is any residual finite contribution from the term that has been dropped. We return to this question later.

III. SPINOR GREEN'S FUNCTIONS, $m \neq 0$

It is convenient to write Eq. (5) in terms of the two two-component parts of ψ. Thus we choose the Weyl forms

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix},$$

where the σ^i are the usual Pauli matrices. Correspondingly S_F is split,

$$S_F = \begin{pmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{pmatrix},$$

and then (5) reads

$$\begin{aligned} &i\left[\frac{\partial}{\partial t} + \vec\sigma\cdot(\vec{\rm X}+\vec\Gamma)\right]S_{21} - mS_{11} = 1\delta(x,x')\,,\\ &i\left[\frac{\partial}{\partial t} + \vec\sigma\cdot(\vec{\rm X}+\vec\Gamma)\right]S_{22} - mS_{12} = 0\,,\\ &i\left[\frac{\partial}{\partial t} - \vec\sigma\cdot(\vec{\rm X}+\vec\Gamma)\right]S_{12} - mS_{22} = 1\delta(x,x')\,,\\ &i\left[\frac{\partial}{\partial t} - \vec\sigma\cdot(\vec{\rm X}+\vec\Gamma)\right]S_{11} - mS_{21} = 0\,, \end{aligned} \qquad (6)$$

where now $\vec\Gamma = -(i/2a)\vec\sigma$.

If S_{12} and S_{21} are eliminated we find that both $-S_{11}/m$ and $-S_{22}/m$ satisfy the same second-order equation,

$$\left(\frac{\partial^2}{\partial t^2} - \vec{\rm X}^2 + \frac{i}{a}\vec\sigma\cdot\vec{\rm X} + \frac{9}{4a^2} + m^2\right)G(x,x') = 1\delta(x,x') \qquad (7)$$

(cf. Dowker[6]) and S_F is related to the 2×2 G by

$$S_F = -(i\gamma^\mu\nabla_\mu + m)\,G\oplus G\,. \qquad (8)$$

The 4×4 Green's function $G\oplus G$ corresponds to the $\mathfrak{G}$ of DeWitt.[9]

The solution of (7) can be effected by the general method outlined by Dowker,[6] where we gave the corresponding Pauli-Jordan commutator function. A few details of the calculation will be given since they may be of interest.

The δ function on the Einstein universe is

$$\delta(x,x') = \delta(t-t')\,\delta(q,q')\,,$$

where $\delta(q,q')$ is the δ function on S^3, $(q,q'\in S^2)$,

$$\delta(q,q') = -\frac{\pi}{|M|}\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sum_{n=-\infty}^{\infty}\delta(\theta+2\pi n)\,. \qquad (9)$$

$|M| = 2\pi^2 a^3$ is the volume of S^3 and $a\theta$ is the geodesic distance between the points q and q'.

By symmetry the general form of $G(q,q',t,t')$ is

$$G(q,q',t,t') = 1G_0(\theta,t-t') + \vec\sigma\cdot\vec{\rm X}G_1(\theta,t-t')\,,$$

where the $G_i(\theta,t)$ are scalar functions, periodic in θ according to the form (9) of the δ function.

In order to avoid carrying through summation signs we shall replace (9) simply by the $n=0$ term, and the solution of the corresponding Eq. (7) will be denoted by

$$G^0 = 1G_0^0(\theta,t) + \vec\sigma\cdot\vec{\rm X}G_1^0(\theta,t) \qquad (10)$$

(we have set $t'=0$). The full solution is then regained by a "periodic" sum,

$$G_i(\theta,t) = \sum_{n=-\infty}^{\infty} G_i^0(\theta+2\pi n,t)\,. \qquad (11)$$

The following steps are now to be performed. Firstly the G_i^0 are Fourier-transformed with respect to t,

$$G_i^0(\theta,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} G_i^0(\theta,E)\,e^{iEt}dE\,,$$

and then the form (10) is substituted into (7) which separates after use is made of the identity

$$(\vec\sigma\cdot\vec{\rm X})^2 = \vec{\rm X}\cdot\vec{\rm X} + i\vec\sigma\cdot(\vec{\rm X}\times\vec{\rm X}) = X^2 + \frac{2i}{a}\vec\sigma\cdot\vec{\rm X}$$

to give the two coupled equations

$$\left(E^2-m^2-\frac{9}{4a^2}+X^2\right)G_0^0-\frac{i}{a}X^2G_1^0=(2\pi a^3\sin\theta)^{-1}\frac{\partial}{\partial\theta}\delta(\theta)$$

and

$$\left(E^2-m^2-\frac{1}{4a^2}+X^2\right)G_1^0=\frac{i}{a}G_0^0.$$

If G_1^0 is eliminated there results, after some rearrangement,

$$\left[E^2-m^2-\frac{1}{a^2}(p-\tfrac{1}{2})^2\right]\left[E^2-m^2-\frac{1}{a^2}(p+\tfrac{1}{2})^2\right]D_0^0(p,E)=-\frac{i}{2\pi a^3}p\left(E^2-m^2-\frac{p^2}{a^2}+\frac{3}{4a^2}\right),$$

where we have replaced X^2 by its radial part

$$(a^2\sin\theta)^{-1}\left(\frac{d^2}{d\theta^2}+1\right)\sin\theta,$$

since it now acts only on functions of the "radial distance" $a\theta$, and we have also Fourier-transformed $D_i^0(\theta,E)\equiv\sin\theta G_i^0(\theta,E)$ with respect to θ,

$$D_i^0(\theta,E)=\frac{1}{2\pi}\int_{-\infty}^{\infty}D_i^0(p,E)\,e^{-ip\theta}\,dp.$$

Similarly $D_1^0(p,E)$ is determined by

$$\left[E^2-m^2-\frac{1}{a^2}(p-\tfrac{1}{2})^2\right]\left[E^2-m^2-\frac{1}{a^2}(p+\tfrac{1}{2})^2\right]D_1^0(p,E)=\frac{1}{2\pi a^4}p.$$

Then, performing the back Fourier transform, we find after a little algebra

$$D_0^0(\theta,t)=-\frac{i}{16\pi^3a^2}\int dE\,dk\,\frac{(ka-\frac{1}{2})\exp(-i\frac{1}{2}\theta)+(ka+\frac{1}{2})\exp(i\frac{1}{2}\theta)}{E^2-k^2-m^2}e^{i(Et-ka\theta)}$$

$$=\frac{1}{16\pi^3a^2}\left(2\cos\tfrac{1}{2}\theta\frac{\partial}{\partial\theta}+\sin\tfrac{1}{2}\theta\right)I(\theta,t) \tag{12}$$

and

$$D_1^0(\theta,t)=\frac{i}{8\pi^3a}I(\theta,t)\sin\tfrac{1}{2}\theta, \tag{13}$$

where $I(\theta,t)$ is the standard two-dimensional integral,

$$I(\theta,t)=\int dk_0\,dk\frac{e^{i(k_0t-ka\theta)}}{k_0^2-k^2-m^2}.$$

For the Feynman Green's function we set $m^2\to m^2-i\epsilon$ and perform the integration to give the standard expression

$$I_F=-\pi^2H_0^{(2)}(m(\lambda-i\epsilon)^{1/2})=-\pi^2\left[\theta(\lambda)H_0^{(2)}(m(\lambda)^{1/2})+\frac{2i}{\pi}\theta(-\lambda)K_0(m(-\lambda)^{1/2})\right] \tag{14}$$

with $\lambda=t^2-a^2\theta^2$.

If all the pieces are collected we find for the exact Feynman Green's function, G_F, satisfying Eq. (7),

$$G_F(x,x')=\sum_{n=-\infty}^{\infty}e^{i(1/2)\vec{\sigma}\cdot\hat{q}\theta_n}\left\{\frac{\theta_n}{\sin\theta}\left[-\frac{m}{8\pi}\frac{H_1^{(2)}(m(\lambda_n)^{1/2})}{(\lambda_n)^{1/2}}\right]-\frac{1}{8\pi a^2\cos^2(\frac{1}{2}\theta)}\left[\tfrac{1}{4}H_0^{(2)}(m(\lambda_n)^{1/2})\right]\right\}, \tag{15}$$

where $\lambda_n=(t-t')^2-a^2\theta_n^2-i\epsilon$, $\theta_n=\theta+2\pi n$.

The exponential factor in (15) is the spinor parallel propagator between q and q', $\hat{q}$ being the unit tangent vector at q to the geodesic connecting q and q', and the quantities in square brackets are the ordinary flat-space scalar Green functions in four and in two dimensions, respectively.

Writing (15) in this form enables us to see immediately what the spinor quantum-mechanical propagator

$K(x, x', \tau)$ is. K is defined by

$$G_F(x, x') = i\int_0^\infty d\tau\, e^{-im^2\tau} K(x, x', \tau),$$

and, since we know what the ordinary scalar propagators are, (15) yields

$$K(x, x', \tau) = \frac{i}{(4\pi i\tau)^2} \sum_{n=-\infty}^{\infty} e^{i(1/2)\vec{\sigma}\cdot\hat{q}\theta_n} \frac{\theta_n}{\sin\theta} e^{-i\lambda_n/4\tau}\left(1 - \frac{i\tau}{a^2\theta_n}\tan\tfrac{1}{2}\theta\right). \tag{16}$$

The coincidence limit of this expression is

$$K(x, x, \tau) = \frac{i}{(4\pi i\tau)^2}\left(1 - \frac{1}{2a^2} i\tau\right)$$
$$+\text{terms exponentially small},$$

which agrees with the general results given by DeWitt (Ref. 9, problem 85) so far as these go. It can be seen that all the coefficients a_n, and therefore their coincidence limits, are zero for $n > 1$. The corresponding result in the scalar case is that all the a_n are zero except the first, a_0. In this case we have said that the WKB approximation is exact. One would like to have a similar statement for spinors. Of course the "naive" WKB approximation, the first term in parentheses in (16), cannot be exact because classical spinning particles do not travel along geodesics, but perhaps the definition of the spinor WKB approximation can be refined to make the statement true. Further discussion of this topic would take us beyond the scope of this paper and so we return to the main line.

The final step in the evaluation of S_F is to substitute (15) into (8), but for calculational purposes it is often best to leave (8) and (15) separated.

IV. THE VACUUM ENERGY

The object of interest to us is $\langle \hat{T}_{\mu\nu}\rangle$ given by (4). Because of symmetry, it is sufficient to concentrate on the vacuum energy $\langle \hat{T}_{00}\rangle$ and the averaged trace $\langle \hat{T}^\mu_\mu\rangle$. These are given by

$$\langle \hat{T}_{00}\rangle = \lim_{x'\to x} \operatorname{tr}\gamma_0\partial_0 S_F(x, x')$$
$$= \lim \operatorname{tr}\partial_0(S_{12} + S_{21})$$

and

$$\langle \hat{T}^\mu_\mu\rangle = -mi \lim_{x'\to x} \operatorname{tr} S_F(x, x'),$$

where we have used the equation of motion (5) and the fact that $\partial_0' S_F(x, x') = -\partial_0 S_F(x, x')$. It is better to rewrite the averages in terms of G_F using (8). Thus

$$\langle \hat{T}_{00}\rangle = -2i \lim_{x'\to x} \operatorname{tr}\partial_0\partial_0 G_F(x, x') \tag{17}$$

and

$$\langle \hat{T}^\mu_\mu\rangle = 2im^2 \lim_{x'\to x} \operatorname{tr} G_F, \tag{18}$$

where tr now stands for a 2×2 trace.

The well-known difficulty now is how to allow for the divergences that arise in the coincidence limit. We shall adopt, for the time being, a straight ansatz of dropping certain divergent terms, as in our previous work.[2,3] Thus we shall simply omit the $n = 0$ term in the sum (15) for G_F, which is the only term that diverges as $t' \to t$ and $\theta \to 0$.

The evaluation of (17) and (18) is then straightforward and we find for the subtracted ("renormalized") quantities

$$\langle \hat{T}_{00}\rangle = \frac{m^2}{2\pi^2 a^2}\sum_1^\infty (-1)^{n+1}\left[12\frac{K_2(2\pi nma)}{(2\pi n)^2} + (1 + 4a^2m^2)\frac{K_1(2\pi nma)}{2\pi nma}\right] \tag{19}$$

and

$$\langle \hat{T}^\mu_\mu\rangle = \frac{m^2}{2\pi^2 a^2}\sum_{n=1}^\infty (-1)^n \{2a^2m^2[K_2(2\pi nma) + K_0(2\pi nma)] + K_0(2\pi nma)\}. \tag{20}$$

Symmetry allows the remaining averages to be deduced from those already found. Thus we must have

$$\langle \hat{T}_{ij}\rangle = \tfrac{1}{3} g_{ij}(\langle \hat{T}^\mu_\mu\rangle - \langle \hat{T}_{00}\rangle).$$

The massless limit of (19) is given by

$$\langle \hat{T}_{00}\rangle|_{m=0} = \frac{17}{960\pi^2 a^4}, \tag{21}$$

while $\langle \hat{T}^\mu_\mu\rangle$ is zero since the m^2 eliminates the ln divergence of the $H_0^{(2)}$ or K_0 terms.

The presence of this ln term prevents the calculation of S_F by the method of deriving G first when m is *strictly zero from the beginning*. There is no divergence in the S_F given by (8) as m tends to zero, but if m is identically zero this construction of S_F is not possible and one must start afresh from the massless spinor equations. Also, it must be remembered that the two-component neutrino is not identical to the general massless Dirac particle without extra conditions being imposed.

For these reasons we now wish to consider the two-component neutrino from the beginning. It will turn out that the Feynman propagator is not the massless limit of (8), which is to us a somewhat unexpected result.

V. TWO-COMPONENT NEUTRINO THEORY

The physical neutrino is left-handed and, in two-component theory, is governed by Weyl's equation,

$$i\left(\frac{\partial}{\partial t}-\vec{\sigma}\cdot(\vec{X}+\vec{\Gamma})\right)\varphi=0,$$

i.e., by a four-component function, $\psi=\binom{\varphi}{0}$ satisfying (2) for $m=0$ and with the Weyl form of the γ^{μ}.

In terms of the 4×4 Green's function S_F the restriction to the two-component theory is made by fixing S_{11}, S_{22}, and S_{21} to be zero. The remaining 2×2 block $S_{12}\equiv S_N$ describes the propagation of φ and satisfies the equation

$$i\left[\frac{\partial}{\partial t}-\vec{\sigma}\cdot(\vec{X}+\vec{\Gamma})\right]S_N(x,x')=1\delta(x,x'). \tag{22}$$

Then $\langle\hat{T}_{00}\rangle$ will be given by

$$\langle\hat{T}_{00}\rangle=\lim_{x'\to x}\operatorname{tr}\partial_0 S_N(x,x'). \tag{23}$$

Equation (22) is solved directly by setting

$$S_N(x,x')=1S_0(\theta,t-t')+\vec{\sigma}\cdot\vec{X}S_1(\theta,t-t')$$

and separating out two equations, as before. We simply give the resulting Fourier integrals

$$S_0^0(x,x')=-\frac{i}{8\pi^3 a\sin\theta}\times\int_{-\infty}^{\infty}p\,dp\,dE\,\frac{(E+1/a)e^{i[pa\theta-E(t-t')]}}{(E-1/2a)^2-p^2-i\epsilon}$$

and

$$S_1^0(x,x')=\frac{1}{8\pi^3 a\sin\theta}\times\int_{-\infty}^{\infty}p\,dp\,dE\,\frac{e^{i[pa\theta-E(t-t')]}}{(E-1/2a)^2-p^2-i\epsilon}.$$

In contrast to the $m\neq 0$ case we have to translate the energy variable and we finally find

$$S_N(x,x')=-\frac{1}{4\pi^2}\left(\frac{\partial}{\partial t}+\vec{\sigma}\cdot\vec{X}-\frac{i}{2a}\right)\sum_{n=-\infty}^{\infty}\frac{\theta+2\pi n}{\sin\theta}e^{-(i/2a)(t-t')}[(t-t')^2-a^2(\theta+2\pi n)^2-i\epsilon]^{-1}. \tag{24}$$

In terms of the massless scalar Green's function, $D_F(x,x')$, used in our earlier work[2] S_N can be written

$$S_N(x,x')=-i\left(\frac{\partial}{\partial t}+\vec{\sigma}\cdot\vec{X}-\frac{i}{2a}\right)e^{-(i/2a)(t-t')}D_F(x,x').$$

This differs from the massless limit of (8), the relevant 2×2 block of which can be written

$$S_{12}(m=0)=-\frac{1}{4\pi^2}\sum_{n=-\infty}^{\infty}e^{i(1/2)\vec{\sigma}\cdot\hat{q}\theta_n}\left\{\left[\frac{\partial}{\partial t}+\frac{\vec{\sigma}\cdot\hat{q}}{a}\frac{\partial}{\partial\theta}-\frac{\tan\frac{1}{2}\theta}{2a}\left(\frac{t}{a\theta_n}+\hat{q}\cdot\vec{\sigma}\right)\right]\frac{\theta_n}{\sin\theta}\frac{1}{\lambda_n}\right\} \tag{25}$$

after some, not necessarily final, manipulation. In the flat-space limit ($a\to\infty$) the difference disappears, to give the standard neutrino Green's function.

The calculation of the averaged vacuum energy from (23) and (24) is now indicated. One would be inclined to make the same "renormalization" ansatz as before, that is, to drop the $n=0$ term in (24). If this is done, *ad hoc*, then we find

$$\langle\hat{T}_{00}\rangle=\frac{1}{60\pi^2 a^4}, \tag{26}$$

which differs from (21). [Actually we should compare twice (26) with (21) to allow for the two lower components of ψ.]

Two possible reasons for this difference are as follows: (a) The renormalization method of dropping the $n=0$ term is not correct if $m=0$. (b) It is due to the difference between (25) and (24) and is a genuine difference. In the next section we present some comments on this situation and discuss alternative approaches.

VI. DISCUSSION AND CONCLUSION

The traditional method of deriving the vacuum energy is the one employed by Casimir in his classic evaluation of the electromagnetic energy density between conducting plates. A mode expansion of the quantum field is substituted into $\langle\hat{T}_{00}\rangle$ and yields the vacuum energy as a sum of vacuum mode energies.

In the present context this method has been used by Ford,[10,11] Streeruwitz,[12] Mamaev *et al.*,[13] and others. Ford in Ref. 11 considers the massless spin-$\frac{1}{2}$ case and derives half the value (21) from the neutrino mode energies and degeneracies. This suggests that the value (26) derived by dropping the $n=0$ term in the sum over geodesics for the massless case is, in fact, incorrect.

To investigate this further consider the $n=0$ term in (24) and imagine taking the coincidence limit (23) in a timelike direction (i.e., $\theta=0$). Then it is easy to see that the time exponential in (24), when expanded in powers of $t-t'$, will give among other things a finite contribution proportional to $1/a^4$. It turns out that when this is added to (26) we simply regain the correct value, i.e., half of (21).

If it were desired to let x' tend to x from a spacelike direction, a parallel propagator would have to be inserted into the definition of $\langle \hat{T}_{00} \rangle$ in order to take care of the spinor transformation properties. This time an expansion of powers of θ will produce the extra term $\sim 1/a^4$.

Apparently all these problems can be avoided if one takes the massless limit of the massive theory, at least for spin $\frac{1}{2}$, despite the much simpler form of the massless Green's function (24).

Ford[11] also discusses the photon vacuum energy. If we were to extend the present approach to that case we would expect that the massless limit would be more complicated owing to gauge invariance, although it is possible to obtain Ford's results simply by adding in a mass to the massless mode energies in an *ad hoc* manner and then letting this mass tend to zero.[14] This method avoids introducing all but the ± 1 helicity states even for nonzero mass.

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