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Dynamics of Einstein's equation modified by a higher-order derivative term

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The search for a renormalized stress-energy operator of a quantum field in curved spacetime has raised the question of whether one can add terms to Einstein's equation containing fourth-order derivatives of the metric, and still maintain a reasonable theory. We investigate this question by considering the simple case of a conformally invariant field in a conformally flat spacetime, where one has a well-defined prediction for the vacuum stress energy of the field. We find that if a certain fourth-order term (associated with the $\Box R$ trace anomaly) enters with one sign, flat spacetime is unstable to conformally flat perturbations that grow exponentially on the Planck time scale. If this term enters with the other sign, conformally flat perturbations could cause spacetime to oscillate at the Planck frequency, resulting in high-energy radiation by charged test particles.

There has been a lot of interest recently in finding a well-defined, renormalized, stress-energy operator $T_{\mu\nu}$ for quantum fields in curved spacetime. This operator is of considerable importance for the theory of back reaction in the semiclassical version of general relativity, where one treats the gravitational field classically and the matter fields quantum mechanically. In analogy with the semiclassical theory of electromagnetic radiation, it is expected that (in certain regimes) this theory will be a valid approximation to a full quantum theory of gravity coupled to other fields.

Some progress has been made recently toward finding a renormalized $T_{\mu\nu}$. In particular, the stress-energy operator for a conformally invariant quantum field in a conformally flat spacetime has been derived via point separation,[1] dimensional regularization,[2] and axiomatic[3] approaches. There is full agreement that the form of $T_{\mu\nu}$ in this case is (using Misner-Thorne-Wheeler[4] sign conventions)

$$T_{\mu\nu} = :T_{\mu\nu}: + (K_1 H_{\mu\nu} + K_2 I_{\mu\nu})I\,, \tag{1}$$

where $:T_{\mu\nu}:$ denotes the normal-ordered operator, I denotes the identity operator,

$$H_{\mu\nu} = -R_\mu{}^\alpha R_{\alpha\nu} + \tfrac{2}{3}RR_{\mu\nu} + \tfrac{1}{2}R^{\alpha\beta}R_{\alpha\beta}g_{\mu\nu} - \tfrac{1}{4}R^2 g_{\mu\nu}\,, \tag{2}$$

$$I_{\mu\nu} = 2\nabla_\mu\nabla_\nu R - 2(\nabla^2 R)g_{\mu\nu} - 2RR_{\mu\nu} + \tfrac{1}{2}R^2 g_{\mu\nu}\,, \tag{3}$$

and K_1, K_2 are constants (which depend on the particular conformally invariant field being considered). There is also complete agreement on the numerical values of K_1. For example, in the case of a scalar field, $K_1 = (2880\pi^2)^{-1}$ (in units $\hbar = G = c = 1$). However, there is some disagreement over the numerical values of K_2. Although most renormalization methods yield nonzero values for K_2, it has been argued[5] that the presence of a term such as $I_{\mu\nu}$—which contains fourth-order erivatives of the metric—would drastically alter the dynamics of Einstein's equation. Some difficulties which occur when terms such as $I_{\mu\nu}$ are added to Einstein's equation have been discussed recently by Stelle.[6]

The purpose of this paper is to investigate further the question of whether a nonzero value of K_2 will result in a physically unacceptable semiclassical theory of back reaction. We show that if $I_{\mu\nu}$ is present, then depending on the sign of K_2, small initially well-behaved perturbations of flat spacetime will either grow exponentially or oscillate on the Planck time scale ($\sim 10^{-43}$ sec). In the former case, flat spacetime is violently unstable. In the latter case, a freely falling charged test particle would radiate very energetic photons ($h\nu \sim 10^{28}$ eV).

We begin with the semiclassical Einstein equation

$$G_{\mu\nu} = 8\pi\langle T_{\mu\nu}\rangle \tag{4}$$

for the case of a conformally invariant quantum field. Using our expression for $T_{\mu\nu}$ [Eq. (1)] we see that conformally flat spacetimes which satisfy

$$G_{\mu\nu} = 8\pi(K_1 H_{\mu\nu} + K_2 I_{\mu\nu}) \tag{5}$$

will be solutions to the semiclassical Einstein equation with the quantum field in the vacuum state. We will assume that the numerical values of K_1 and K_2 are roughly of order unity in Planck units ($\hbar = G = c = 1$). Our aim is to analyze the effect of a nonzero value of K_2 on the dynamics given by Eq. (5).

Equation (5) has an obvious solution: Minkowski spacetime, $G_{\mu\nu} = H_{\mu\nu} = I_{\mu\nu} = 0$. We now investigate whether this solution is stable, that is, do initially small perturbations remain small? In classical general relativity, Minkowski spacetime is, of course, stable and its stability is certainly a necessary requirement for a physically reasonable theory. We restrict ourselves to conformally flat perturbations because our expression for $T_{\mu\nu}$, Eq. (1), applies only to that case. In classical general relativity, there are no nontrivial, conformally flat, vacuum perturbations of Minkowski spacetime. However, as we shall now show, if $K_2 \neq 0$ there do exist nontrivial perturbations of this kind in the semiclassical theory.

We write the conformally flat metric $g_{\mu\nu}$ as

$$g_{\mu\nu} = \Omega^2 \eta_{\mu\nu} , \tag{6}$$

where $\eta_{\mu\nu}$ is a flat metric. In terms of Ω and the flat derivative operator ∂_μ associated with $\eta_{\mu\nu}$, the Ricci tensor, $R_{\mu\nu}$, of $g_{\mu\nu}$ is

$$R_{\mu\nu} = 4\Omega^{-2}(\partial_\mu\Omega)(\partial_\nu\Omega) - 2\Omega^{-1}\partial_\mu\partial_\nu\Omega - \Omega^{-1}\eta_{\mu\nu}(\partial^\alpha\partial_\alpha\Omega) - \Omega^{-2}\eta_{\mu\nu}(\partial^\alpha\Omega\partial_\alpha\Omega) , \tag{7}$$

where here and throughout indices are raised with $\eta^{\alpha\beta}$ (so, e.g., $\partial^\alpha\partial_\alpha\Omega \equiv \eta^{\alpha\beta}\partial_\alpha\partial_\beta\Omega$). Using this expression for $R_{\mu\nu}$, we may write Eq. (5) as an equation for Ω in the flat metric $\eta_{\mu\nu}$. One solution is obviously $\Omega = 1$, corresponding to Minkowski spacetime. Writing $\Omega = 1 + \gamma$ and keeping only terms first order in γ, we obtain the linearization of Eq. (5) describing small conformally flat perturbations of flat spacetime:

$$-\partial_\mu\partial_\nu\gamma + (\Box\gamma)\eta_{\mu\nu} - 48\pi K_2[-\partial_\mu\partial_\nu(\Box\gamma) + \Box(\Box\gamma)\eta_{\mu\nu}] = 0 , \tag{8}$$

where $\Box \equiv \partial^\alpha\partial_\alpha$. Note that the term $K_1 H_{\mu\nu}$ does not enter the linearized equation since $H_{\mu\nu}$ is quadratic in the curvature.

Our perturbation equation can be rewritten as

$$(\partial_\mu\partial_\nu - \eta_{\mu\nu}\Box)f = 0 , \tag{9}$$

where $f = (\gamma - 48\pi K_2 \Box\gamma)$. The most general solution to Eq. (9) is

$$f = k^\alpha x_\alpha + k , \tag{10}$$

where k^α is a constant vector field in Minkowski space, x^α is the position vector field in Minkowski space (with respect to a certain choice of origin), and k is a constant.

We are now left with simply the following equation:

$$(-48\pi K_2)\Box\gamma + \gamma = k^\alpha x_\alpha + k . \tag{11}$$

The most general solution is

$$\gamma = (k^\alpha x_\alpha + k) + \varphi , \tag{12}$$

where φ satisfies the homogeneous equation

$$\Box\varphi - \frac{1}{48\pi K_2}\varphi = 0 , \tag{13}$$

which is, in fact, just the Klein-Gordon equation. The inhomogeneous solution $\gamma = k^\alpha x_\alpha + k$ is easily seen to be pure gauge. Let $\gamma_{\mu\nu}$ be the corresponding perturbation of the metric: $\gamma_{\mu\nu} = 2\gamma\eta_{\mu\nu} = 2(k^\alpha x_\alpha + k)\eta_{\mu\nu}$. Then one can verify that $\gamma_{\mu\nu} = \partial_{(\mu}\xi_{\nu)}$, where ξ_ν is the conformal Killing field,

$$\xi_\nu = -k_\nu(x^\alpha x_\alpha) + 2(x^\alpha k_\alpha)x_\nu + 2kx_\nu . \tag{14}$$

Hence, we will now concentrate on just the homogeneous solutions to Eq. (11). It is easy to check that these perturbations are not pure gauge.

Consider first the case $K_2 < 0$. (This is the sign of K_2 obtained for the scalar and neutrino fields using point separation,[7] dimensional regularization,[8] and ζ-function regularization,[9] and also for the electromagnetic field using dimensional regularization.[10]) Then Eq. (13) describing the nontrivial perturbation is just the Klein-Gordon equation with a negative value of (mass)^2. This equation admits solutions which are initially well behaved but rapidly blow up in time. As a specific example, consider the spatially homogeneous solution

$$\gamma = Ce^{t/\tau} , \tag{15}$$

where C is a constant and $\tau = (-48\pi K_2)^{1/2}$. Recalling that K_2 is roughly of order unity in Planck units, we see that the exponential growth time scale of this solution is the Planck time ($\sim 10^{-43}$ sec). Hence, if at $t = 0$ one were to perturb Minkowski spacetime slightly in the direction of this solution, the effects would grow at a catastrophic rate; if $K_2 < 0$, Minkowski spacetime is violently unstable.

On the other hand, if $K_2 > 0$ (the sign obtained for the electromagnetic field by point separation[7] and ζ-function regularization[9]) then the nontrivial perturbations are described by the ordinary Klein-Gordon equation [with positive (mass)^2]. The solutions now do not grow exponentially in time but rather oscillate at the Planck frequency. The spatially homogeneous solutions are

$$\gamma = C_1 \sin\omega t + C_2 \cos\omega t , \tag{16}$$

where $\omega = (48\pi K_2)^{-1/2}$. While the consequences of perturbations of this kind are not as drastic as those of the case $K_2 < 0$, we wish to argue that they could still lead to unacceptable observable consequences: Charged test particles in a universe oscillating via the above perturbation, Eq. (16), will emit photons at the Planck energy.[11]

To see this in more detail, we consider a freely falling (i.e., geodesic) charged test particle in the perturbed spacetime described by Eq. (16). If we write Maxwell's equations in the form

$$\nabla_{[\alpha}F_{\mu\nu]} = 0 , \tag{17a}$$

$$\nabla_{[\alpha} {}^*F_{\mu\nu]} = 4\pi\epsilon_{\alpha\mu\nu\beta}J^\beta , \tag{17b}$$

then the left-hand side, when expressed in terms of the flat metric $\eta_{\mu\nu}$ and the conformal factor, is in fact independent of the conformal factor. (This is an expression of the fact that the source-free Maxwell equations are conformally invariant.) Thus, the curved-space Maxwell equations are simply the flat-space equations with source given by the right-hand side of Eq. (17b). But this source corresponds to a particle following a geodesic in the curved metric $g_{\mu\nu}$ rather than the flat metric $\eta_{\mu\nu}$. It is easy to check that timelike geodesics in the curved metric oscillate about the geodesics of $\eta_{\mu\nu}$ at the frequency of the perturbation. (There is one exception: Geodesics of $g_{\mu\nu}$ moving in the preferred time direction defined by the perturbation coincide with the geodesics of $\eta_{\mu\nu}$.) Thus the electromagnetic radiation produced by a geodesic particle in the curved spacetime is identical to that of a particle in flat spacetime which is accelerated back and forth at the perturbation frequency. Such a particle, of course, emits radiation at that frequency. This means that if a perturbation of the type Eq. (16) were applied, all charged particles (except those moving exactly in the preferred time direction) would emit some radiation at the Planck frequency. This corresponds to a photon energy of $h\nu \sim 10^{28}$ eV. By comparison, the most energetic cosmic-ray event ever observed had an energy $\sim 10^{21}$ eV.

Thus, the oscillations introduced by the higher-order derivative term would lead to unacceptable consequences if they were to occur in the present universe. However, this does not necessarily mean that the appearance of $I_{\mu\nu}$ in Eq. (5) with $K_2 > 0$ must be ruled out. It is possible that any such oscillations which may have been present in the early universe would by now have been damped out by mechanisms such as the interactions with charged particles described above. Furthermore, it is possible that no such oscillations could be excited (with appreciable amplitude) by normal astrophysical processes in the present universe. Since we do not have the full theory of how these oscillations are sourced—Eq. (5) holds only for conformally flat spacetimes—these issues cannot be definitively settled at present.

In summary, we have found[12] that if $K_2 < 0$—the sign obtained for most fields by most renormalization methods—Eq. (5) leads to completely unphysical behavior: Flat spacetime is unstable with respect to perturbations which grow on the Planck time scale. The presence of the term $I_{\mu\nu}$ with $K_2 > 0$ may also be unacceptable, but the situation is not as clear. Since $I_{\mu\nu}$ is a conserved local curvature term (the one obtained from the Lagrangian R^2), an arbitrary multiple of it can be added to $T_{\mu\nu}$ without violating the required properties of $T_{\mu\nu}$ (see axioms 1–4 of Ref. 5). In view of this fact, we believe that the correct renormalized value of K_2 is zero. On the other hand, the value of K_1 cannot be arbitrarily changed in this manner and, as argued elsewhere,[13] must be nonzero. The effect of the term $K_1 H_{\mu\nu}$ on dynamics is discussed in Ref. 3.

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[12]One final point must be addressed before our conclusions can be safely drawn. We have studied the behavior of perturbations of flat spacetime, but is is not at all obvious *a priori* that these perturbations correspond to exact solutions. Indeed, if one seeks a one-parameter family of exact solutions which is conformly flat to all orders (not merely first order), it turns out that the higher-order equations put additional restrictions on the first-order perturbation, so some of the solutions of the perturbation equation may be spurious. However, we have verified that these restrictions are satisfied by spatially homogeneous perturbations having the instability behavior considered above. Furthermore, if the exact solutions are not required to be conformally flat in higher orders, then presumably no such restrictions would occur. Exact solutions (with Robertson-Walker symmetry) of Einstein's equation modified by the term $I_{\mu\nu}$ have been studied by T. V. Ruzmaĭkina and A. A. Ruzmaĭkin, Zh. Eksp. Teor. Fiz. 57, 680 (1969) [Sov. Phys.-JETP 30, 372 (1970)], M. Giesswein, R. Sexl, and E. Streeruwitz, Phys. Lett. 52B, 442 (1974) and M. Giesswein and E. Streeruwitz, Acta Phys. Austriaca 41, 41 (1975).

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