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Maximal extension of a nonsingular solution in a generalized theory of gravitation*

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A transformation of coordinates is presented that removes the event-horizon singularities in an exact spherically symmetric solution of a generalized theory of gravitation. The resulting maximally extended solution corresponds to a gravitational and electromagnetic field that is nonsingular everywhere in physical space-time.

I. INTRODUCTION

An exact solution of a generalized theory of gravitation[1] which includes the electromagnetic field within the framework of a non-Riemannian geometry, with nonsymmetric fundamental tensor $g_{\mu\nu}$, has been shown to be a regular solution with the exception of event horizons.[2] The metric for the static spherically symmetric solution takes the form[1]

$$ds^2=\left(1-\frac{2m}{r}+\frac{4\pi Q^2}{r^2}\right)\left(1-\frac{\kappa^2Q^2}{r^4}\right)dt^2 - \left(1-\frac{2m}{r}+\frac{4\pi Q^2}{r^2}\right)^{-1}dr^2-r^2d\Omega^2, \tag{1.1}$$

where

$$d\Omega^2=d\theta^2+\sin^2\theta\, d\phi^2 \tag{1.2}$$

and $k=i\kappa$ is a purely imaginary constant. We shall choose the universal constant $\kappa=\hbar/e$ ($\kappa=\hbar G/c^3e$ in the units of Ref. 2). Thus the metric (1.1) is well approximated by the Reissner-Nordström metric down to distances $r\gtrsim\sqrt{\kappa Q}=a$.

The signature of the metric in the generalized theory changes from -2 to -4 for $r<a$. Thus we have a local Minkowski metric M^4 for $r>a$ and a four-dimensional Euclidean metric E^4 for $r<a$. We define *physical*[2] space-time as the region $r\geq a$, since no timelike or null world lines can penetrate a sphere S of radius $r=a$.

The paths in such a non-Riemannian geometry are described by the equation

$$\frac{d^2x^\mu}{ds^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{ds}\frac{dx^\beta}{ds}=0, \tag{1.3}$$

where $\Gamma^\mu_{\alpha\beta}$ is a Hermitian nonsymmetric connection, related to $g_{\mu\nu}$ by[1,2]

$$\partial_\alpha g_{\mu\nu}-g_{\sigma\nu}\Gamma^\sigma_{\mu\alpha}-g_{\mu\sigma}\Gamma^\sigma_{\alpha\nu}=0\,. \tag{1.4}$$

To examine the global properties of the manifold, it is necessary to find the maximal coordinates for the solution. The new coordinates will provide an analytic extension such that the metric is non-singular everywhere in physical space-time. It will then be seen that the generalized theory is time-like and null world line complete.

II. THE EVENT HORIZONS

As in the Reissner-Nordström solution coordinate singularities occur at

$$r_\pm=m\pm(m^2-4\pi Q^2)^{1/2} \tag{2.1}$$

when $m^2>4\pi Q^2$, while for $m^2=4\pi Q^2$ only one coordinate singularity occurs at $r=m$.

Let us first consider the case $4\pi Q^2<m^2$ and extend the (r,t) manifold across the coordinate singularities at r_+ and r_-. We define the advanced and retarded time coordinates v and w, respectively, by

$$v=t+\tilde{r}\,,\qquad w=t-\tilde{r}\,, \tag{2.2}$$

where

$$d\tilde{r}=-\frac{(r_++r_-)r-r_+r_-}{(r-r_+)(r-r_-)}\frac{dr}{(1-\kappa^2Q^2/r^4)^{1/2}}\,. \tag{2.3}$$

The function $\tilde{r}(r)$ can be integrated explicitly everywhere in the region $r\geq a$ (see Appendix A). The line element in terms of (v,r) is

$$ds^2=\left(1-\frac{2m}{r}+\frac{4\pi Q^2}{r^2}\right)\left(1-\frac{a^4}{r^4}\right)dv^2 -2\left(\frac{2m}{r}-\frac{4\pi Q^2}{r^2}\right)\left(1-\frac{a^4}{r^4}\right)^{1/2}dr\,dv -\left(1+\frac{2m}{r}-\frac{4\pi Q^2}{r^2}\right)dr^2-r^2d\Omega^2\,. \tag{2.4}$$

In terms of (w,r) coordinates we get

$$ds^2=\left(1-\frac{2m}{r}+\frac{4\pi Q^2}{r^2}\right)\left(1-\frac{a^4}{r^4}\right)dw^2 +2\left(\frac{2m}{r}-\frac{4\pi Q^2}{r^2}\right)\left(1-\frac{a^4}{r^4}\right)^{1/2}dr\,dw -\left(1+\frac{2m}{r}-\frac{4\pi Q^2}{r^2}\right)dr^2-r^2d\Omega^2. \tag{2.5}$$

We observe that the line elements (2.4) and (2.5) are regular everywhere for $r>a$, although we have sacrificed time-reversal invariance by including $drdv$ (or $drdw$) terms. Equation (2.5) is just the time-reversed solution of (2.4).

From (2.3) we see that $\partial\tilde{r}/\partial r$ has a branch point at $r=a$. The (w,r) manifold is just the extension of the (v,r) manifold onto the second Riemann sheet of $\partial\tilde{r}/\partial r$, where $d\tilde{r}$ in (2.3) goes into $-d\tilde{r}$. We shall choose (v,r) as the coordinates corresponding to the "physical" sheet. Radial null world lines in this manifold are given by the equation

$$(1-\Lambda)\left(1-\frac{a^4}{r^4}\right)dv^2-2\Lambda\left(1-\frac{a^4}{r^4}\right)^{1/2}drdv-(1+\Lambda)dr^2=0\ , \qquad (2.6)$$

where

$$\Lambda=\frac{2m}{r}-\frac{4\pi Q^2}{r^2}\,. \qquad (2.7)$$

The solutions to Eq. (2.6) are given by

$$\frac{dv}{dr}=\left(\frac{\Lambda\pm1}{1-\Lambda}\right)\left(1-\frac{a^4}{r^4}\right)^{-1/2}=\begin{cases}\dfrac{1+2m/r-4\pi Q^2/r^2}{1-2m/r+4\pi Q^2/r^2}\,\dfrac{1}{(1-a^4/r^4)^{1/2}}\\[2ex] -\dfrac{1}{(1-a^4/r^4)^{1/2}}\,.\end{cases} \qquad (2.8)$$

The light cones determined by (2.8) are shown in Fig. 1(a), while those in the (w,r) manifold are described in Fig. 1(b). Provided that $r_+ > r_- \gg a$ (which is certainly reasonable for $\kappa\sim\hbar/e$), light rays near the event horizon behave qualitatively the same as in the Eddington-Finkelstein coordinate transformation of the Reissner-Nordström solution.[3] Ingoing null world lines can cross both r_+ and r_- in the (v,r) manifold, while outgoing null lines approach r_- asymptotically from $r=a$.

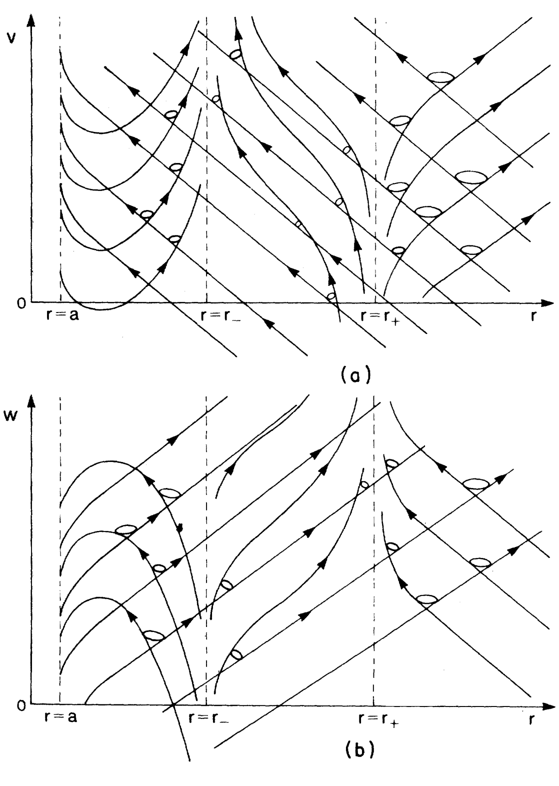

FIG. 1. (a), (b) Diagrams for the Eddington-type coordinates (v,r) and (w,r) respectively, showing null paths and light cones. All paths reach the boundary $r=a$ with zero velocity and positive acceleration.

We observe that in our Eddington-Finkelstein-type coordinates, described above, the outgoing [ingoing] null paths in the (v,r) [(w,r)] manifold below r_- turn over as they move away from $r=a$. This would result in a causality violation if v [w] was a pure timelike coordinate. In these coordinates this behavior only occurs if a is less than the turning point of these light rays. For astronomical bodies this will in general be the case, as can be shown for typical astronomical values of m and Q.

III. WORLD LINES

Let us now investigate the behavior of timelike world lines in the physical (v,r) manifold. In what follows we restrict ourselves to the radial case, although the conclusions can easily be shown to apply to nonradial world lines as well. We have

$$1=\hat{g}_{\mu\nu}\frac{d\hat{x}^\mu}{ds}\frac{d\hat{x}^\nu}{ds}\,, \qquad (3.1)$$

which in the (v,r) coordinate system yields

$$\left[\left(1-\frac{2m}{r}+\frac{4\pi Q^2}{r^2}\right)\left(1-\frac{a^4}{r^4}\right)-2\left(\frac{2m}{r}-\frac{4\pi Q^2}{r^2}\right)\left(1-\frac{a^4}{r^4}\right)^{1/2}\frac{dr}{dv}-\left(1+\frac{2m}{r}-\frac{4\pi Q^2}{r^2}\right)\left(\frac{dr}{dv}\right)^2\right]\left(\frac{dv}{ds}\right)^2=1\,. \qquad (3.2)$$

From (1.3) we obtain a constant of the motion

$$\left(1-\frac{2m}{r}+\frac{4\pi Q^2}{r^2}\right)\frac{dt}{ds}=\text{constant}\equiv U_0\,, \qquad (3.3)$$

where U_0 is related to the energy of the particle. For a massless particle we have $ds=0$, whereby $U_0=\infty$.

After some algebraic manipulations of Eqs. (3.2) and (3.3), with the use of (2.2), we get[4]

$$\left(\frac{dr}{dv}+\frac{\hat{g}_{(01)}Y}{Z}\right)^2=\frac{(\hat{g}_{(01)}Y)^2-\hat{g}_{00}YZ}{Z^2}, \tag{3.4}$$

where

$$\begin{aligned}\hat{g}_{(01)}&=-\left(\frac{2m}{r}-\frac{4\pi Q^2}{r^2}\right)\left(1-\frac{a^4}{r^4}\right)^{1/2},\\ \hat{g}_{00}&=\left(1-\frac{2m}{r}+\frac{4\pi Q^2}{r^2}\right)\left(1-\frac{a^4}{r^4}\right),\\ Y&=U_0^2\left(1-\frac{a^4}{r^4}\right)^2-\hat{g}_{00},\\ Z&=\left(-1-\frac{2m}{r}+\frac{4\pi Q^2}{r^2}\right)U_0^2\left(1-\frac{a^4}{r^4}\right)-\hat{g}_{(01)}{}^2.\end{aligned} \tag{3.5}$$

Equation (3.4) implies that

$$Y(\hat{g}_{(01)}{}^2Y-\hat{g}_{00}Z)>0, \tag{3.6}$$

or

$$g(r)\equiv\frac{1-2m/r+4\pi Q^2/r^2}{1-a^4/r^4}<U_0^2. \tag{3.7}$$

In Fig. 2 we show $g(r)$ as a function of r. We see that for $U_0^2<1$, Eq. (3.7) implies that the particle oscillates between it minimum and maximum radii $r_{\rm MIN}>a$ and $r_{\rm MAX}$, respectively, while for $U_0^2>1$, there is just the minimum radius $R_{\rm MIN}>a$. Obviously as $U_0^2\to\infty$ (null path), the minimum radius tends to a. The world lines in the (w,r) manifold exhibit the same qualitative behavior, for in this

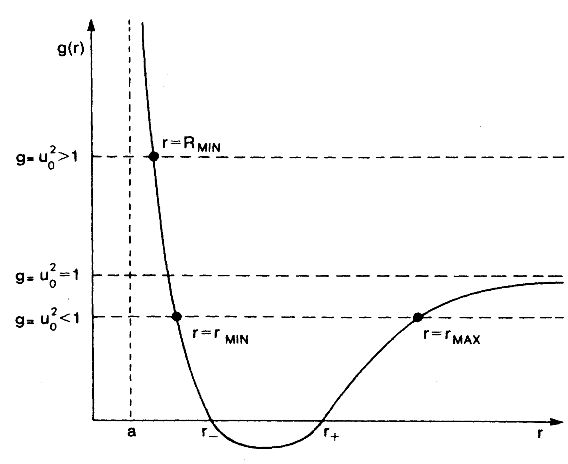

FIG. 2. Graph of $g(r)\equiv(1-2m/r+4\pi(Q^2/r^2)/(1-a^4/r^4)$ vs r. For $U_0^2<1$, Eq. (3.7) requires $r_{\rm min}<r<r_{\rm max}$. For $U_0^2\geq 1$, Eq. (3.7) requires $r>R_{\rm min}$; as $U_0^2\to\infty$, $R_{\rm min}\to a$.

coordinate system we simply replace $\hat{g}_{(01)}$ by $-\hat{g}_{(01)}$, so that Eq. (3.7) still holds.

As in the Reissner-Nordström solution, timelike world lines are prevented from reaching $r=0$. However, in the generalized theory, null (radial and nonradial) world lines are also prevented from hitting $r=0$. In the limit $a=\sqrt{kQ}\to 0$, the radial null world lines do reach $r=0$ in finite proper time, and the intrinsic singularity at $r=0$ *reappears in the physical manifold*. For nonzero a the radial light rays reach a in finite proper time with $dr/dv=0$ and with positive acceleration $d^2r/dv^2>0$. In fact, the ingoing solution in the (v,r) plane can be smoothly matched (C^∞) to the outgoing solution in the (w,r) plane. This suggests that a radial world line that reaches the branch point at $r=a$ passes smoothly onto the second Riemann sheet of $\partial\bar{r}/\partial r$, represented by the (w,r) coordinate system.

IV. THE MAXIMAL EXTENSION

The behavior of both timelike and null world lines is more clearly described in the maximal analytic extension of our original metric (1.1), in which we can simultaneously embed both the (v,r) and (w,r) planes.

Following the methods of Kruskal[5] and of Graves and Brill,[6] we seek a coordinate system in which light rays everywhere have the slope $dr'/dt'=\pm 1$, and the line element has the form

$$ds^2=f^2(r',t')(dr'^2-dt'^2)-r^2(r',t')d\Omega^2. \tag{4.1}$$

By using the transformation law

$$g_{\mu\nu}=\frac{\partial\hat{x}^\alpha}{\partial x^\mu}\frac{\partial\hat{x}^\beta}{\partial x^\nu}\hat{g}_{\alpha\beta} \tag{4.2}$$

We find that

$$\left[\left(\frac{\partial t'}{\partial t}\right)^2-\left(\frac{\partial r'}{\partial t}\right)^2\right]f^2(r',t')=\left(1-\frac{2m}{r}+\frac{4\pi Q^2}{r^2}\right)\left(1-\frac{a^4}{r^4}\right), \tag{4.3}$$

$$\left[\left(\frac{\partial t'}{\partial r}\right)^2-\left(\frac{\partial r'}{\partial r}\right)^2\right]f^2(r',t')=-\left(1-\frac{2m}{r}+\frac{4\pi Q^2}{r^2}\right)^{-1}, \tag{4.4}$$

$$\left(\frac{\partial t'}{\partial t}\frac{\partial t'}{\partial r}-\frac{\partial r'}{\partial t}\frac{\partial r'}{\partial r}\right)f^2(r',t')=0, \tag{4.5}$$

$$\left(\frac{\partial t'}{\partial t}\frac{\partial r'}{\partial r}-\frac{\partial r'}{\partial t}\frac{\partial t'}{\partial r}\right)\hat{g}_{[01]}=\frac{kQ}{r^2}. \tag{4.6}$$

If we combine (4.3), (4.4), and (4.5), we get

$$\frac{\partial r'}{\partial t}=\pm\left(1-\frac{2m}{r}+\frac{4\pi Q^2}{r^2}\right)\left(1-\frac{a^4}{r^4}\right)^{1/2}\frac{\partial t'}{\partial r}, \tag{4.7}$$

$$\frac{\partial t'}{\partial t}=\pm\left(1-\frac{2m}{r}+\frac{4\pi Q^2}{r^2}\right)\left(1-\frac{a^4}{r^4}\right)^{1/2}\frac{\partial r'}{\partial r}. \tag{4.8}$$

The solutions for r' and t' are given by

$$r'(r,t)=r'(r^*,t)=h(r^*+t)+g(r^*-t)\,, \tag{4.9}$$

$$t'(r,t)=t'(r^*,t)=h(r^*+t)-g(r^*-t)\,, \tag{4.10}$$

where

$$\frac{\partial r^*}{\partial r}=\pm\left(1-\frac{2m}{r}+\frac{4\pi Q^2}{r^2}\right)\left(1-\frac{a^4}{r^4}\right)^{1/2}$$
$$=\pm\frac{r^4}{(r-r_+)(r-r_-)(r^4-a^4)^{1/2}}\,. \tag{4.11}$$

The double valuedness of $\partial r^*/\partial r$ is due to the branch point at $r=a$. We shall choose the positive sign in Eq. (4.11) to correspond to the physical space-time manifold. The other Riemann sheet, corresponding to a minus sign in (4.7) and (4.8), can then be interpreted as being associated with a second space-time manifold isometric with our own.

Because the physical space-time manifold consists only of the region $r\geq a$ we obtain (see Appendix A)

$$\begin{aligned}
r^*(r)=&\int_a^r \frac{r^4dr}{(r-r_+)(r-r_-)(r^4-a^4)^{1/2}}\\
=&\frac{(r^4-a^4)^{1/2}}{r}+\frac{r_++r_-}{2}\ln\left[\frac{r^2+(r^4-a^4)^{1/2}}{a^2}\right]\\
&+\frac{1}{2}\frac{r_+^2}{(r_+-r_-)(1-a^4/r_+^4)^{1/2}}\ln\left\{\frac{(r^2-r_+^2)a^2}{r_+^2r^2-a^4+[(r_+^4-a^4)(r^4-a^4)]^{1/2}}\right.\\
&\qquad\times\left.\frac{r[(r_+^2+a^2)(r^2-a^2)]^{1/2}-r_+[(r_+^2-a^2)(r^2+a^2)]^{1/2}}{r[(r_+^2+a^2)(r^2-a^2)]^{1/2}+r_+[(r_+^2-a^2)(r^2+a^2)]^{1/2}}\right\}\\
&-\frac{1}{2}\frac{r_-^2}{(r_+-r_-)(1-a^4/r_-^4)^{1/2}}\ln\left\{\frac{(r^2-r_-^2)a^2}{r_-^2r^2-a^4+[(r_-^4-a^4)(r^4-a^4)]^{1/2}}\right.\\
&\qquad\times\left.\frac{r[(r_-^2+a^2)(r^2-a^2)]^{1/2}-r_-[(r_-^2-a^2)(r^2+a^2)]^{1/2}}{r[(r_-^2+a^2)(r^2-a^2)]^{1/2}+r_-[(r_-^2-a^2)(r^2+a^2)]^{1/2}}\right\}\\
&+aH(r,r_+,r_-)\,.
\end{aligned} \tag{4.12}$$

Here $H(r,r_+,r_-)$ consists of terms proportional to elliptic integrals of the first, second, and third kinds, which vanish at $r=a$ and are finite as $a\to 0$. The constant of integration $r^*(a)$ has been set equal to zero.

For $r>r_+$ and $r<r_-$ we define

$$\begin{aligned}
h(r^*+t)&=Ae^{b(r^*+t)}\,,\\
g(r^*-t)&=Ae^{b(r^*-t)}\,,
\end{aligned} \tag{4.13}$$

where we have chosen the plus sign in Eq. (4.11) and b is a constant. Equations (4.3)–(4.5) and (4.9), (4.10) then give

$$\begin{aligned}
f^2(r,t)=&\frac{(r-r_+)(r-r_-)}{4A^2r^2b^2}\left(1-\frac{a^4}{r^4}\right)e^{-2br^*}\\
=&\frac{(r-r_+)(r-r_-)}{4A^2r^2b^2}\left(1-\frac{a^4}{r^4}\right)\left[\frac{r^2+(r^4-a^4)^{1/2}}{a^2}\right]^{-(r_++r_-)b}\\
&\times\exp\left\{-\left[\frac{2b}{r}(r^4-a^4)^{1/2}+2baH(r)\right]\right\}[\mathfrak{F}_+]^{-br_+^4/(r_+-r_-)(r_+^4-a^4)^{1/2}}[\mathfrak{F}_-]^{br_-^4/(r_+-r_-)(r_-^4-a^4)^{1/2}}\,,
\end{aligned} \tag{4.14}$$

where

$$\mathfrak{F}_\pm=\frac{(r^2-r_\pm^2)\{r[(r_\pm^2+a^2)(r^2-a^2)]^{1/2}-r_\pm[(r_\pm^2-a^2)(r^2+a^2)]^{1/2}\}a^2}{\{r_\pm^2r^2-a^4+[(r_\pm^4-a^4)(r^4-a^4)]^{1/2}\}\{r_\pm[(r_\pm^2-a^2)(r^2+a^2)]^{1/2}+r[(r_\pm^2+a^2)(r^2-a^2)]^{1/2}\}}. \tag{4.15}$$

For $r_- < r < r_+$ we choose

$$h(r^*+t) = Ae^{b(r^*+t)}\,, \qquad g(r^*-t) = -Ae^{b(r^*-t)} \tag{4.16}$$

so that we retain the positivity condition $f^2 > 0$.

We now eliminate the zeros in $f^2(r,t)$ at r_+ and r_- by an appropriate choice of b. The zero at $r=a$ cannot be eliminated because a is not a coordinate singularity, but rather a *boundary* at which the topology changes. As in the Reissner-Nordström solution,[6] each coordinate singularity can only be eliminated in a neighborhood around it. Thus we define two coordinate patches for which b takes the values

$$b_\pm = \pm\left(\frac{2r_\pm^2}{r_+ - r_-}\right)^{-1}\left(1-\frac{a^4}{r_\pm^4}\right)^{1/2}, \tag{4.17}$$

where b_+ and b_- are the constants of integration associated with r_+ and r_-, respectively. By using L'Hospital's rule, it can then be shown that f^2 is nonzero at r_+ and r_- in their respective coordinate patches and, therefore, f^2 is nonvanishing, positive and analytic everywhere in the complete manifold, except of course at $r=a$ where it vanishes.

When $a\to 0$, the terms in (4.12) and (4.14) containing elliptic integrals disappear and we find that

$$r^* \to r + \frac{r_+^2}{r_+ - r_-}\ln\frac{r-r_+}{r_+} - \frac{r_-^2}{r_+ - r_-}\ln\frac{r-r_-}{r_-}. \tag{4.18}$$

Moreover, as $a\to 0$ we get

$$f^2(r,t)\to\left(\frac{1}{4b^2A^2}\right)\left[\frac{(r-r_+)(r-r_-)}{r^2}\right]e^{-2br} \times\left(\frac{r-r_+}{r_+}\right)^{-2br_+^2/(r_+-r_-)} \times\left(\frac{r-r_-}{r_-}\right)^{2br_-^2/(r_+-r_-)}, \tag{4.19}$$

$$b_\pm \to \pm\left(\frac{2r_\pm^2}{r_+ - r_-}\right)^{-1}. \tag{4.20}$$

These results give the Kruskal-type coordinates for the Reissner-Nordström solution[6] as desired.

The explicit transformations determined by (4.9), (4.10), and (4.13) are for $r\geq r_+$ and $r\leq r_-$:

$$r_\pm'(r,t) = 2Ae^{b_\pm r^*}\cosh b_\pm t\,, \tag{4.21}$$

$$t_\pm'(r,t) = 2Ae^{b_\pm r^*}\sinh b_\pm t\,. \tag{4.22}$$

In the region $r\geq r_+$, b_+ is positive and, since t is the time of an observer at infinity, a physical particle moves towards increasing t' as $t\to\infty$. Note that f^2 is everywhere positive, so that t' is timelike everywhere, as opposed to the coordinates used by Graves and Brill,[6] in which r' is the timelike variable in the second patch. The conditions of analyticity of the metric and the coordinates, in the appropriate patches, are retained.

In the region $r_- < r < r_+$, the explicit transformations are

$$r_\pm'(r,t) = 2Ae^{b_\pm r^*}\sinh b_\pm t\,, \tag{4.23}$$

$$t_\pm'(r,t) = 2Ae^{b_\pm r^*}\cosh b_\pm t. \tag{4.24}$$

The Kruskal-type diagrams for these extended coordinates are given in Figs. 3(a) and 3(b). Ra-

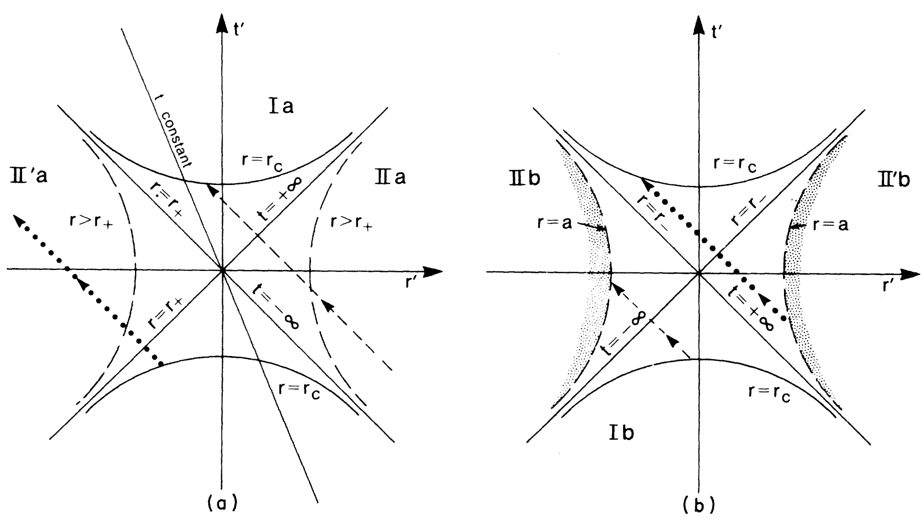

FIG. 3. (a), (b) Kruskal-type diagrams for extended coordinates (r', t'), corresponding to physical Riemann sheet of dr^*. Null paths have slopes ± 1.

dial null paths can cross the radius $r=r_+$ in both directions. Ingoing ones (dashed line) can come in from infinity in region IIa, cross r_+ and, at some r_c in region Ia, then pass into a second coordinate patch and cross r_-, reaching region IIb. Outgoing null world lines (dotted line) can start in region II$'b$, cross into the first patch at $r=r_c$ and subsequently cross r_+ and go to infinity in region II$'a$. The half plane, defined by regions I and II in both patches, is isometric to the (v,r) manifold, while the section defined by regions I and II$'$ is isometric to the (w,r) manifold. As shown previously,[2] ingoing timelike world lines "bounce" at some minimum radius and then pass through r_- again into a region isometric with the coordinate patch of Fig. 3(a); the particle escapes to infinity in some other asymptotically flat space-time. The particle cannot re-emerge from the event horizon in our own space-time, for then causality violations would occur. This situation is the same as in the maximally extended Reissner-Nordström solution,[7] although now it is true of both timelike and null radial world lines.

Null world lines that hit $r=a$ do not have well-defined velocities, for dr/dt is not properly determined at $r=a$ in view of the fact that $f^2=0$ at this point. However, we are free to extend the world line to $r=a$ by the requirement of continuity.[8] Indeed, the world line can be extended by allowing it to pass onto the second Riemann sheet of $\partial r^*/\partial r$ as defined by (4.11). Then $r^* \to -r^*$ and, to eliminate the zeros of f^2 at r_+ and r_-, we choose $b_\pm \to -b_\pm$. The line element remains unchanged, but the coordinates (r',t') for $r>r_+$, $r<r_-$, on the new sheet are

$$r'_\pm = 2Ae^{b_\pm r^*}\cosh(-b_\pm t) = 2Ae^{b_\pm r^*}\cosh(b_\pm t)\,, \tag{4.25}$$

$$t'_\pm = 2Ae^{b_\pm r^*}\sinh(-b_\pm t) = -2Ae^{b_\pm r^*}\sinh(b_\pm t)\,. \tag{4.26}$$

Thus the second sheet describes, in effect, the *time-reversed* manifold. As t increases in region IIa, t'_+ will now decrease; $(-t'_\pm)$ is now timelike and a null world line in region IIb moves away from $r=a$ in the $(r'_\pm, t'_\pm)$ coordinate system as shown in Figs. 4(a) and 4(b). The extension of this world line is again made unique by continuity requirements. Thus, the original (r',t') manifold and its time-reversed image, taken together and joined at the branch point $r=a$, constitute the complete analytic extension of the exact solution to the unified theory. As $k\to 0$, f^2, $b_\pm$, r', and t' all reduce to the corresponding quantities in the Kruskal coordinates of the Reissner-Nordström solution. When $Q\to 0$, the analytic extension reduces to that of the Schwarzschild solution.

If $4\pi Q^2 > m^2$, as in the case of elementary particles, no event horizons occur and the original (r,t) coordinates cover the complete manifold.

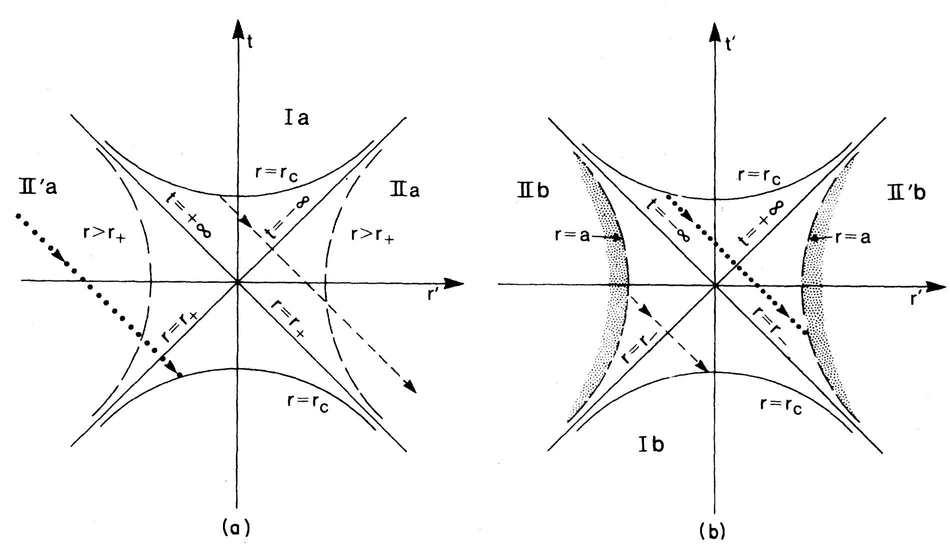

FIG. 4. (a), (b) Kruskal-type diagrams for extended coordinates (r',t'), corresponding to second Riemann sheet $(dr^* \to -dr^*)$. Now $(-t')$ is timelike and regions Ia and Ib contain only outgoing solutions. A null path reaching $r=a$ in Fig. 3(b) can emerge in region IIb of Fig. 4(b) (dashed line). The unprimed regions in the second Riemann sheet are equivalent to the primed regions on the physical sheet, as expected by the observation that the transformation $t\to -t$ takes us from one sheet to the other, as well as from one side of the Kruskal-type diagram to the other.

Timelike and null world lines still cannot penetrate the sphere described by the radius $r=a$, so that light rays bounce away to infinity in the same space-time region and the world lines are complete.[2] The solution is everywhere nonsingular in physical space-time.

V. PENROSE DIAGRAMS

By means of the usual transformation,

$$t'+r'=\tan\left(\frac{\psi+\xi}{2}\right),\qquad t'-r'=\tan\left(\frac{\psi-\xi}{2}\right), \tag{5.1}$$

we can construct the Penrose diagrams for the complete manifold. As seen in Fig. 5, the qualitative features in the neighborhoods of r_+, r_-, and $r=\infty$ are the same as those in the Penrose diagrams for the Reissner-Nordström solution.[9] The region $r<a$ is not included in our diagram, and therefore an intrinsic geometrical singularity does not occur in our solution. There is a timelike boundary at $r=a$.

A world line which reaches the boundary passes smoothly onto the second Riemann sheet of $\partial r^*/\partial r$, represented by the coordinates $(r', -t')$. In the language of Penrose diagrams, the transition to the second sheet corresponds to the transformation $\psi\to-\psi$. We can visualize two separate Penrose diagrams, related by the above transformation and connected on both sides ($\xi=\pm\pi/2$) at the boundaries $r=a$. The continuation of a null path hitting the boundary in one patch (path A in Fig. 5) can also be represented on the same diagram by the reflection of that path about the $\psi=0$ axis. For convenience this continuation, in Fig. 5 (path B), has been translated upwards by 2π, which is allowed by the symmetry properties of the diagram.

We observe that timelike world lines never reach $r=a$ but continue oscillating through "mirror" universes if $U_0^2<1$, or bounce off to $r=\infty$ in a mirror universe if $U_0^2>1$.

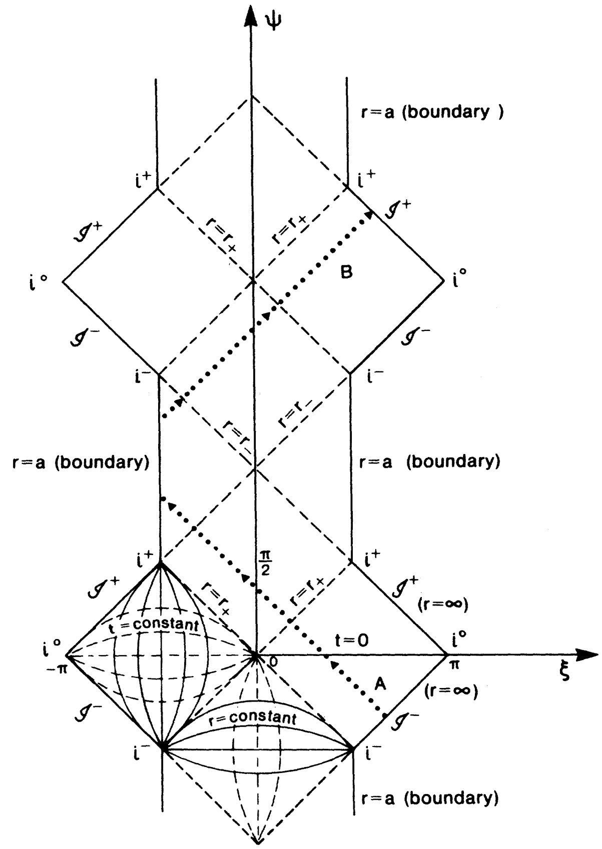

FIG. 5. Penrose-type diagram for extended coordinates, showing an infinite chain of asymptotically flat regions isometric with those in Figs. 3(a) and 3(b).

VI. CONCLUSIONS

We have presented the maximal analytic extension of a nonsingular solution to a generalized theory of gravitation and electromagnetism. The behavior of timelike world lines is qualitatively the same as in the Reissner-Nordström solution, in that a "wormhole" with a pulsating throat occurs in the solution. The manifold now has a boundary which hides the singularity at $r=0$ in a sphere of radius $r=\sqrt{\kappa Q}$ for all cases $4\pi Q^2>m^2$, $4\pi Q^2<m^2$, and $4\pi Q^2=m^2$. The physical universe is now free of infinite curvature and infinite energy densities. Radial null world lines bounce away from $r=a$ into a time-reversed manifold, and the solution possesses both timelike and null world line completeness.

APPENDIX A

We shall now provide explicit details of the integration of $\tilde{r}(r)$ and $r^*(r)$. The basic integral to be considered is

$$r^*(r)=\int_a^r\frac{r^2dr}{(r-r_+)(r-r_-)(1-a^4/r^4)^{1/2}}$$
$$=\int_a^r\left[1+\frac{r_+^2}{(r_+-r_-)(r-r_+)}-\frac{r_-^2}{(r_+-r_-)(r-r_-)}\right]\frac{r^2dr}{(r^4-a^4)^{1/2}}. \tag{A1}$$

We have immediately that

$$\int_a^r \frac{r^2dr}{(r^4-a^4)^{1/2}}=\frac{a}{\sqrt{2}}F(\epsilon,S)-\sqrt{2}\,aE(\epsilon,S)+\frac{(r^4-a^4)^{1/2}}{r}, \tag{A2}$$

where $\epsilon=\cos^{-1}(r/a)$, $S=1/\sqrt{2}$, and $F(\epsilon,S)$ and $E(\epsilon,S)$ are the elliptic integrals of the first and second kinds, respectively. Moreover,

$$\begin{aligned}\int_a^r \frac{r^2dr}{(r-r_\pm)(r^4-a^4)^{1/2}} &= \int_a^r \frac{r\,dr}{(r^4-a^4)^{1/2}}+r_\pm\int_a^r\frac{dr}{(r^4-a^4)^{1/2}}+r_\pm^2\int_a^r\frac{r\,dr}{(r^2-r_\pm^2)(r^4-a^4)^{1/2}}+r_\pm^3\int_a^r\frac{dr}{(r^2-r_\pm^2)(r^4-a^4)^{1/2}}\\ &=\tfrac{1}{2}\ln\frac{r^2+(r^4-a^4)^{1/2}}{a^2}+\frac{r_\pm}{\sqrt{2}a}F(\epsilon,S)\\ &\quad+r_\pm^2\frac{1}{2(r_\pm^4-a^4)^{1/2}}\ln\left\{\frac{-a^2(r^2-r_\pm^2)}{(r_\pm^2r^2-a^4)+[(r_\pm^4+a^4)(r^4-a^4)]^{1/2}}\right\}\\ &\quad+r_\pm^3\left[\frac{-a\Pi(\epsilon,r_\pm^2/(r_\pm^2-a^2),S)}{\sqrt{2}\,r_\pm^2(r_\pm^2-a^2)}-\frac{1}{\sqrt{2}\,ar_\pm^2}F(\epsilon,S)\right],\end{aligned} \tag{A3}$$

where $\Pi(\epsilon,r_\pm^2/(r_\pm^2-a^2),S)$ is the elliptic integral of the third kind.

Since $r_\pm^2/(r_\pm^2-a^2)>1$, the elliptic integral of the third kind can be written as[10]

$$\begin{aligned}\Pi\left(\epsilon,\frac{r_\pm^2}{r_\pm^2-a^2},S\right) &= -\Pi\left(\epsilon,S^2\frac{r_\pm^2-a^2}{r^2},S\right)+F(\epsilon,S)\\ &\quad+\frac{1}{\sqrt{2}a}r_\pm\left(\frac{r_\pm^2-a^2}{r_\pm^2+a^2}\right)^{1/2}\ln\left\{\frac{r_\pm[(r_\pm^2-a^2)(r^2+a^2)]^{1/2}+r[(r_\pm^2+a^2)(r^2-a^2)]^{1/2}}{r_\pm[(r_\pm^2-a^2)(r^2+a^2)]^{1/2}-r[(r_\pm^2+a^2)(r^2-a^2)]^{1/2}}\right\}.\end{aligned} \tag{A4}$$

Combining (A1)–(A4) we finally obtain Eq. (4.12).

From Eq. (2.3) we have

$$\begin{aligned}\tilde{r}(r) &= -\int\frac{[r(r_+ + r_-)-r_+r_-]dr}{(r-r_+)(r-r_-)(1-a^4/r^4)^{1/2}}\\ &=\int\left[1-\frac{r^2}{(r-r_+)(r-r_-)}\right]\frac{dr}{(1-a^4/r^4)^{1/2}}\\ &=\int_a^r\frac{r^2dr}{(r^4-a^4)^{1/2}}-r^*(r).\end{aligned} \tag{A5}$$

The explicit result for $\tilde{r}(r)$ can then be obtained from (A2) and Eq. (4.12).

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[4]We use the notation $g_{(\mu\nu)}=\frac{1}{2}(g_{\mu\nu}+g_{\nu\mu})$ and $g_{[\mu\nu]}=\frac{1}{2}(g_{\mu\nu}-g_{\nu\mu})$.

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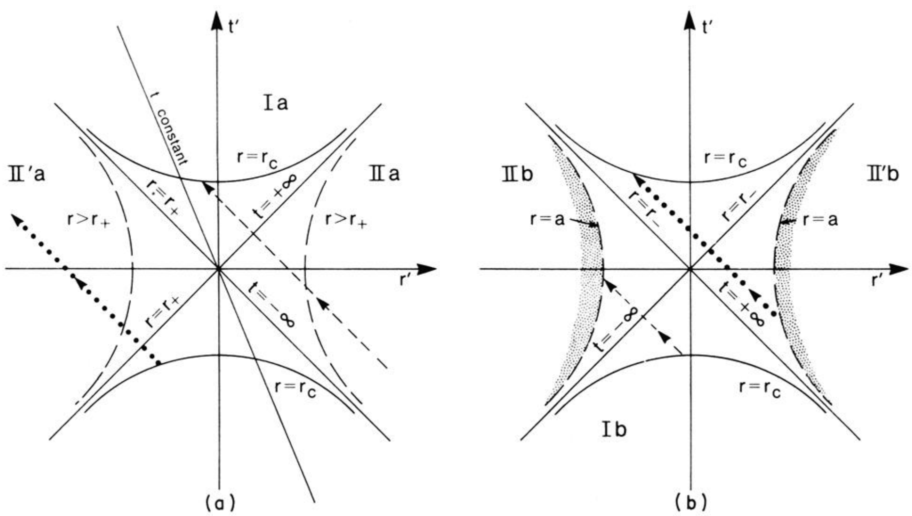

FIG. 3. (a), (b) Kruskal-type diagrams for extended coordinates (r', t'), corresponding to physical Riemann sheet of dr^*. Null paths have slopes ± 1.

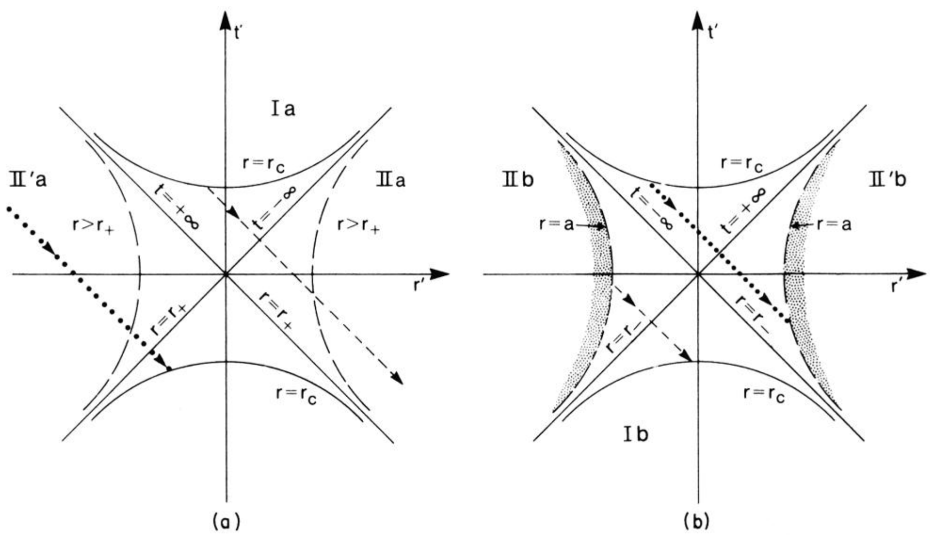

FIG. 4. (a), (b) Kruskal-type diagrams for extended coordinates (r',t'), corresponding to second Riemann sheet $(dr^* \rightarrow -dr^*)$. Now $(-t')$ is timelike and regions Ia and Ib contain only outgoing solutions. A null path reaching $r=a$ in Fig. 3(b) can emerge in region IIb of Fig. 4(b) (dashed line). The unprimed regions in the second Riemann sheet are equivalent to the primed regions on the physical sheet, as expected by the observation that the transformation $t \rightarrow -t$ takes us from one sheet to the other, as well as from one side of the Kruskal-type diagram to the other.