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Static spherically symmetric solutions to a system of generalized Einstein-Maxwell field equations

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In this paper I investigate the static, spherically symmetric, pure electric, source-free solutions to the most general second-order vector-tensor theory of gravitation and electromagnetism which is such that its field equations are (i) derivable from a variational principle, (ii) consistent with the notion of conservation of charge, (iii) in agreement with Einstein's equations in the absence of electromagnetic fields, and (iv) compatible with Maxwell's equations in a flat space. These solutions (which are given in series form) bear a strong resemblance to the Reissner-Nordström solution (of the Einstein-Maxwell field equations) in the asymptotic domain; however, they differ quite radically from the Reissner-Nordström solution in the vicinity of the source. In addition it appears as though many of these solutions are only compatible with electric monopoles of finite extent.

In Einstein's theory of gravitation, the field equations governing the symmetric Lorentzian metric tensor g_{ab} and the electromagnetic vector potential ψ_a are[1]

$$G^{ij}=8\pi(T^{ij}+T^{ij}_M)$$

and $$F^{ij}{}_{;j}=4\pi J^i\,, \tag{1}$$

where

$$F_{ab}\equiv\psi_{b,a}-\psi_{a,b}\,,$$
$$T^{ij}\equiv(1/4\pi)(F^{ia}F^{j}{}_{a}-\tfrac{1}{4}g^{ij}F^{ab}F_{ab})\,, \tag{2}$$

and the tensors T^{ij}_M and J^i appearing in Eq. (1) represent the energy-momentum tensor and charge-current vector of the sources.

The results of Refs. 2 and 3 suggest that a possible alternative to (1) is

$$\lambda G^{ij}+\mu g^{ij}=8\pi(\chi T^{ij}+kA^{ij})+8\pi T^{ij}_M \tag{3}$$

and

$$\chi F^{ij}{}_{;j}+\tfrac{1}{2}kF_{bc;a}\,{}^*R^{*iabc}=4\pi J^i\,, \tag{4}$$

where λ, μ, χ, and k are constants[4] and

$$A^{ij}\equiv\frac{1}{8\pi}(F_{al}F_{b}{}^{l}\,{}^*R^{*iajb}+{}^*F^{ia;b}\,{}^*F^{j}{}_{b;a})\,. \tag{5}$$

The uniqueness of these generalized Einstein-Maxwell field equations is discussed in Ref. 2.

In order to guarantee that Eq. (3) reduces to Einstein's field equations in the absence of electromagnetic fields and that Eq. (4) reduces to Maxwell's equations in a flat space *it will henceforth be assumed that* $\lambda=1$, $\mu=0$, *and* $\chi=1$.

We now seek a static, spherically symmetric, pure electric, source-free solution to Eqs. (3) and (4) whose underlying manifold M is an open connected subset of $\mathrm{R}\times\mathrm{R}^3$. Let r, θ, and ϕ denote spherical polar coordinates on R^3 and let t denote the standard coordinate on R. We assume that the line element ds^2 and electromagnetic field tensor F have the following forms on M:

$$ds^2=-\omega^0\otimes\omega^0+\omega^1\otimes\omega^1+\omega^2\otimes\omega^2+\omega^3\otimes\omega^3 \tag{6}$$

and

$$F=f(\omega^0\otimes\omega^1-\omega^1\otimes\omega^0)=E(dt\otimes dr-dr\otimes dt)\,, \tag{7}$$

where

$$\omega^0\equiv e^a dt,\quad \omega^1\equiv e^b dr,\quad \omega^2\equiv r\,d\theta,\quad \omega^3\equiv r\sin\theta\,d\phi$$

and a, b, and f are functions only[5] of r. The linearly independent differential equations resulting from (3), (4), and (5) are[6]

$$e^{-2b}\left[\frac{2b'}{r}+\frac{(e^{2b}-1)}{r^2}\right]=f^2+\frac{kf^2}{r^2}(e^{-2b}-1)\,, \tag{8}$$

$$e^{-2b}\left[\frac{2a'}{r}+\frac{(1-e^{2b})}{r^2}\right]=-f^2-\frac{kf^2}{r^2}(3e^{-2b}-1)\,, \tag{9}$$

$$e^{-2b}\left[a''-a'b'+(a')^2+\frac{(a'-b')}{r}\right]=f^2+\frac{kfe^{-2b}}{r}[f(b'-a')-2f']\,, \tag{10}$$

and

$$f'+\frac{2f}{r}=\frac{k}{r^2}[f'(1-e^{-2b})+2fb'e^{-2b}]\,, \tag{11}$$

where a prime denotes a derivative with respect to r. This system of differential equations is not functionally independent since (10) is a consequence of the other three equations.

We now examine solutions to the above equations:

Equations (8), (9), and (11) imply that

$$a' = -b' - \frac{kf^2}{r} \tag{12}$$

and

$$f = \frac{C}{r^2 + k(e^{-2b} - 1)}, \tag{13}$$

where C is a constant. Thus determining b specifies f and a (up to a quadrature). Equations (13) and (8) give

$$\frac{d}{dr}[r(1 - e^{-2b})] = \frac{C^2}{r^2 + k(e^{-2b} - 1)}. \tag{14}$$

When $k = 0$, (3) and (4) reduce to the Einstein-Maxwell field equations. Indeed, the general solution to Eq. (14) in this case is

$$e^{2b} = \left(1 - \frac{2M}{r} + \frac{C^2}{r^2}\right)^{-1}$$

(where M is a constant) which leads to the Reissner-Nordström solution of general relativity. Thus *we shall henceforth assume that* $k \neq 0$. Then Eq. (14) can be written as the Abel equation (of the second kind)

$$\tau' = 3r^2 + \frac{Br}{\tau}, \tag{15}$$

where

$$B \equiv -kC^2 \text{ and } \tau \equiv r[r^2 + k(e^{-2b} - 1)]. \tag{16}$$

In summary we have demonstrated that the problem of solving Eqs. (8)–(11) reduces to solving Eq. (15) for τ.

I have been unable to solve Eq. (15) in closed form and I believe that it is impossible to express the general solution to Eq. (15) in terms of elementary functions. Consequently, I shall now present series solutions for τ, e^{2a}, e^{2b}, and f which are valid in the asymptotic and near-field regimes.

The asymptotic form of the metric and electromagnetic field. We assume that as $r \to \infty$, e^{2a}, e^{2b}, and f have the form

$$\begin{aligned} e^{2a} &= 1 - 2M/r + O(r^{-2}), \\ e^{2b} &= 1 + O(r^{-1}), \\ &\text{and} \\ f &= C/r^2 + O(r^{-3}), \end{aligned} \tag{17}$$

where M and C represent the mass and charge of the source, respectively. In this case for large r,

$$\tau = r^3 + \gamma - \frac{B}{r} + \frac{B\gamma}{4r^4} - \frac{B^2}{5r^5} - \frac{B\gamma^2}{7r^7} + \frac{9B^2\gamma}{32r^8} - \frac{2B^3}{15r^9} + \frac{B\gamma^3}{10r^{10}} + O(r^{-11}),$$

where γ is an arbitrary real constant with units of (length)^3. This expression for τ, in conjunction with the boundary conditions presented in Eq. (17), gives the following for large r:

$$e^{2a} = 1 - \frac{2M}{r} + \frac{C^2}{r^2} + \frac{kC^2}{2r^4} - \frac{kMC^2}{2r^5} + O(r^{-6}), \tag{18}$$

$$e^{2b} = 1 + \frac{2M}{r} + \frac{(4M^2 - C^2)}{r^2} + \frac{4M(2M^2 - C^2)}{r^3} + \frac{(C^4 - 12M^2C^2 + 16M^4)}{r^4} + \frac{M(12C^4 - kC^2 - 64M^2C^2 + 64M^4)}{2r^5} + O(r^{-6}), \tag{19}$$

and

$$E = \frac{C}{r^2}\left[1 + \frac{2kM}{r^3} - \frac{3kC^2}{4r^4} + \frac{4k^2M^2}{r^6} - \frac{24k^2MC^2}{7r^7} + \frac{117k^2C^4}{160r^8} + \frac{8k^3M^3}{r^9} + O(r^{-10})\right]. \tag{20}$$

Thus we see that Eqs. (3) and (4) predict that in the asymptotic regime the electric field will *not* go exactly as C/r^2 far from the source, but will have a "correction term" of order r^{-5} involving the mass of the source. Perhaps this can be used to set bounds on the magnitude or sign of k. In passing one should note the similarity between the above solution and the Reissner-Nordström solution.

The near-field form of the metric and electromagnetic field. If we seek a solution to Eq. (15) which is of the form

$$\tau = \sum_{n=0}^{\infty} a_n r^n + \sum_{n=1}^{p} b_n r^{-n}, \tag{21}$$

where p is a positive integer, then all of the b_n's must vanish. Consequently, we will look for a solution of the form $\sum_{n=0}^{\infty} a_n r^n$. There exist two cases to consider.

Case (i): a_0 is an arbitrary nonzero constant and $a_1 = 0$. This solution to Eq. (15) takes the form

$$\tau = a_0 + \frac{Br^2}{2a_0} + r^3 - \frac{B^2r^4}{8a_0^3} - \frac{Br^5}{5a_0^2} + \frac{B^3r^6}{16a_0^5} + \frac{6B^2r^7}{35a_0^4} + \left(\frac{B}{8a_0^3} - \frac{5B^4}{128a_0^7}\right)r^8 - \frac{16B^3r^9}{105a_0^6} + O(r^{10}).$$

For small r the metric and electromagnetic field are expressible as in Eqs. (6) and (7) with

$$e^{2a} = \frac{\alpha A}{r}\left[1 + \frac{r}{A} - \frac{3C^2r^2}{2kA^2} - \frac{C^2r^3}{kA^3} + \frac{3C^4r^4}{8k^2A^4} + \frac{C^2r^5}{k^2A^3} + O(r^6)\right], \tag{22}$$

$$e^{2b}=\frac{r}{A}\left[1-\frac{r}{A}+\left(1+\frac{C^2}{2k}\right)\frac{r^2}{A^2}\right.$$
$$\left.-\left(1+\frac{C^2}{k}\right)\frac{r^3}{A^3}+O(r^4)\right], \qquad (23)$$

and

$$E=\frac{\alpha^{1/2}Cr}{kA}\left[1-\frac{r^3}{kA}-\frac{3C^2r^5}{10k^2A^3}+\frac{r^6}{k^2A^2}+O(r^7)\right], \qquad (24)$$

where $A\equiv a_0k^{-1}$ and α is a positive unitless real constant that arises in the computation of e^{2a}.

Thus we see that Eqs. (3) and (4) predict that in the vicinity of a point source the electric field E need not go as C/r^2 (as it does in the Reissner-Nordström solution), but rather E can have the form $E=(\text{constant})r+O(r^4)$.

It is not clear to me whether any of the above near-field solutions can be joined to the asymptotically flat solutions (18)–(20) with or without intervening "solution patches." In addition I have no idea whether the above expressions for e^{2a} and e^{2b} are different from zero throughout their domain of definition—although they are nonzero on a deleted neighborhood of $r=0$.

Let us now consider the nature of the singularity at $r=0$: Equations (6) and (7) yield $F_{ab}\,{}^*F^{ab}=0$ and $F_{ab}F^{ab}=-2f^2$. Using Eqs. (22)–(24) we find that

$$F_{ab}F^{ab}=-\frac{2C^2r^2}{k^2A^2}\left[1+\frac{C^2r^2}{kA^2}-\frac{2r^3}{kA}+O(r^4)\right], \qquad (25)$$

and hence $F_{ab}F^{ab}$ is well behaved as $r\to 0$. However, the scalar curvature R is given by

$$R=\frac{6C^2}{kA}\left[\frac{1}{r}+\frac{1}{A}+\frac{C^2r}{2kA^2}+O(r^2)\right], \qquad (26)$$

which blows up as $r\to 0$. Thus the geometry of our spacetime experiences a "genuine singularity at $r=0$."

Case (ii): $a_0=0$ and $a_1{}^2=B$ in Eq. (21). Since $B=-kC^2$, k must be negative.

This case represents a "singular solution" to Eq. (15), viz., $\tau'=\Phi(r,\tau)$, where $\Phi(r,\tau)\equiv 3r^2+Br\tau^{-1}$ is analytic on a neighborhood of $(r,\tau)=(0,\tau_0)$ only so long as $\tau_0\neq 0$. However, case (ii) demands that $\tau=0$ at $r=0$.

The series expansion for τ is

$$\tau=a_1r+\frac{3r^3}{4}+\frac{3r^5}{32a_1}-\frac{9r^7}{256a_1^2}+\frac{117r^9}{2^{11}5a_1^3}+O(r^{11}).$$

For small r the metric and electromagnetic field are given by Eqs. (6) and (7) with

$$e^{2a}=\alpha r^2\left[\left(1+\frac{a_1}{k}\right)-\left(\frac{3}{2a_1}+\frac{7}{4k}\right)r^2\right.$$
$$\left.+\left(\frac{75}{32ka_1}+\frac{15}{8a_1^2}\right)r^4+O(r^6)\right], \qquad (27)$$

$$e^{2b}=\frac{k}{k+a_1}\left[1+\frac{r^2}{4(k+a_1)}\right.$$
$$\left.-\frac{(3k+a_1)r^4}{32a_1(k+a_1)^2}+O(r^6)\right],$$

and

$$E=\frac{\alpha^{1/2}Cr}{a_1}\left[1-\frac{3r^2}{2a_1}+\frac{27r^4}{16a_1^2}-\frac{105r^6}{64a_1^3}\right.$$
$$\left.+\frac{2961r^8}{2^{11}a_1^4}+O(r^{10})\right],$$

where α is a positive real constant with units of $(\text{length})^{-2}$. Thus, as in the previous case, for small r the electric field E goes as $E=(\text{constant})r+O(r^3)$.

At present I doubt whether any of the above solutions can be joined to the asymptotically flat solutions presented in Eqs. (18)–(20), however, I have no proof of this claim.

As in the previous case as $r\to 0$ the scalar field $F_{ab}F^{ab}$ is well behaved, while, unless $2k+3a_1=0$, the scalar curvature R blows up since

$$F_{ab}F^{ab}=\frac{2}{k}\left[1-\frac{3r^2}{2a_1}+\frac{3r^4}{2a_1^2}+O(r^6)\right]$$

and

$$R=-2\left[\left(2+\frac{3a_1}{k}\right)\frac{1}{r^2}-\left(\frac{41}{4k}+\frac{15}{2a_1}\right)+O(r)\right].$$

These expressions for $F_{ab}F^{ab}$ and R show that none of the above solutions can be isometric to any of the solutions presented in case (i).

The two sets of solutions presented in cases (i) and (ii) above represent the *only* static, spherically symmetric, pure electric, source-free solutions to Eqs. (3) and (4) which arise from a solution τ to Eq. (15) of the form (21) and which are valid in a deleted neighborhood of $r=0$. Each of these two classes of solutions predicts behavior extremely different from that of the Reissner-Nordström solution.

In Ref. 3 it is argued that a possible alternative to the usual energy-momentum tensor T_{ij} [cf., Eq. (2)] of the electromagnetic field used in general relativity is provided by $\mathcal{T}_{ij}\equiv T_{ij}+kA_{ij}$, where A_{ij} is defined by Eq. (5). We shall now consider the behavior of T_{ij} and $\mathcal{T}_{ij}$ in the vicinity of $r=0$. To that end suppose that the constants A and a_1 appearing in Eqs. (22) and (27) have been chosen so that $A>0$ and $1+a_1/k>0$, so that the Killing vector field $\partial/\partial t$ is timelike in a deleted neighborhood V of $r=0$.

Let Θ be a Killing observer whose world line lies in V and has unit tangent vector $u = e^{-a}\partial/\partial t$. Equations (2), (6), (7), (8), and (13) give[7]

$$T(u,u)=\frac{f^2}{8\pi} \text{ and } \mathcal{T}(u,u)=\frac{Cf}{8\pi r^2}.$$

For case (i),

$$T(u,u)=\frac{C^2r^2}{8\pi k^2A^2}\left[1+\frac{C^2r^2}{kA^2}-\frac{2r^3}{kA}+O(r^4)\right],$$

$$\mathcal{T}(u,u)=\frac{C^2}{8\pi kAr}\left[1+\frac{C^2r^2}{2kA^2}-\frac{r^3}{kA}+O(r^4)\right],$$

and for case (ii),

$$T(u,u)=\frac{C^2}{8\pi a_1^2}\left[1-\frac{3r^2}{2a_1}+\frac{3r^4}{2a_1^2}+O(r^6)\right],$$

$$\mathcal{T}(u,u)=\frac{C^2}{8\pi a_1 r^2}\left[1-\frac{3r^2}{4a_1}+\frac{15r^4}{32a_1^2}+O(r^6)\right].$$

In either case $T(u,u)$ is extremely well behaved as Θ approaches the singularity while $\mathcal{T}(u,u)$ can go to either positive or negative infinity as $r\to 0$ depending upon our choice of k in case (i) and a_1 in case (ii).

Owing to the above work we see that if we were to regard $\mathcal{T}_{ij}$ as the energy-momentum tensor of the electromagnetic field in the theory of gravitation and electromagnetism based upon Eqs. (3) and (4), then it would predict regions of unbounded energy density for some solutions. Moreover, when $k<0$, $\mathcal{T}_{ij}$ admits regions of *unbounded negative* energy density for certain solutions to Eqs. (3) and (4).

Finally, note that the results presented here *do not* imply that for every choice of $k\neq 0$ there exists static, spherically symmetric, asymptotically flat, pure electric, source-free solutions to Eqs. (3) and (4) which are valid *for all* $r>0$. In fact it may be that for certain k's no asymptotically flat far-field solutions can be joined to solutions valid near $r=0$. My motivation for making this statement is based upon the following observations.

Suppose that we seek a source-free solution to Eq. (4) under the assumption that our metric is the Schwarzschild metric and that our electric field has the form presented in Eq. (7). In this case Eq. (4) reduces to Eq. (11) with $e^{-2b}=1-2Mr^{-1}$. As a result Eq. (13) gives

$$F=\frac{Cr}{r^3-2kM}(dt\otimes dr - dr\otimes dt),$$

where $2M<r<\infty$ for Schwarzschild coordinates. Transforming F to Eddington-Finkelstein coordinates (u,r,θ,ϕ) (for which $0<r<\infty$) involves only replacing dt by du. Consequently, under our present assumptions F is well behaved in the vicinity of $r=0$, as was the case for the near-field solutions discussed above. If $k>0$, then F is singular at $r=(2kM)^{1/3}$, and the scalar invariant $F_{ab}F^{ab}\to -\infty$ as $r\to(2kM)^{1/3}$, implying that the electric field experiences a singularity before we reach $r=0$. This *leads us to suspect* that a similar phenomenon may occur in some of the static spherically symmetric solutions to Eqs. (3) and (4) presented here. Our suspicions are heightened by the fact that when the function $\epsilon\equiv Cr(r^3-2kM)^{-1}$ is expanded in powers of r^{-1} we find that

$$\epsilon=\frac{C}{r^2}\left(1+\frac{2kM}{r^3}+\frac{4k^2M^2}{r^6}+\frac{8k^3M^3}{r^9}+\cdots+\frac{2^nk^nM^n}{r^{3n}}+\cdots\right).$$

Upon comparing this series with the expression for E presented in Eq. (20) we are naturally led to *conjecture* that E must contain ϵ in it and hence is probably singular when $k>0$.

Now if for a certain range of k the asymptotically flat far-field solutions presented in Eqs. (18)–(20) could not be joined to near-field solutions then we would conclude that the field Eqs. (3) and (4) do not admit asymptotically flat source-free solutions corresponding to an electric monopole *point source*. Thus for this range of k every electric monopole with an asymptotically flat external field would have to have a nonzero radius.

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[1]My notational conventions are the same as those employed in C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), with the exception that I denote tensor indices by lower-case Latin letters when the tensors components are determined with respect to a chart.

[2]G. W. Horndeski, J. Math. Phys. 17, 1980 (1976).

[3]G. W. Horndeski and J. Wainwright, Phys. Rev. D 16, 1691 (1977).

[4]The constants λ and χ are unitless while μ and k have units of (length)$^{-2}$ and (length)^2, respectively.

[5]If we were to allow a, b, and f to depend upon both r and t, then there would exist a time coordinate T on R such that in terms of the chart (T,r,θ,ϕ),

$$ds^2=-e^{2A}dT^2+e^{2B}dr^2+r^2(d\theta^2+\sin^2\theta\, d\phi^2)$$

and

$$F=\psi\,(dT\otimes dr-dr\otimes dT),$$

where A, B, and ψ are functions only of r. Consequently, the staticness of the spacetime is a consequency of the source-free version of Eqs. (3) and (4) and the assumption of spherical symmetry. For the proof of this fact see G. W. Horndeski, J. Math. Phys. (to be published).

[6]The details omitted from this paper can be found in G. W. Horndeski, University of Waterloo report (unpublished).

[7]Each side of Eq. (8) is equal to $8\pi\,\mathcal{T}(u,u)$.