Inelastic eikonal phenomenology in a stationary-phase approximation. II

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The interplay of unitarity with solutions of certain nonlinear Euler equations, obtained in a semiclassical approximation of *all* the eikonal pionization graphs (tower and nonplanar), is shown to generate reasonable features of hadron collisions at ultrahigh energies: total and inclusive cross sections rising as (rapidity)^2, flat inclusive rapidity plateaus, and an absence of correlations between emitted particles.

I. INTRODUCTION

We would like to describe a calculation in which a "semiclassical", or "averaged" estimate of the relevant elastic and inelastic eikonal functions is used to provide a realistic model of hadron reactions at ultrahigh energies. The model is ambitious in the sense that an estimate is attempted for *all* the connected, nonplanar (checkerboard) eikonal graphs which contribute to pionization effects; the model is realistic in the sense that a detailed formulation of s-channel unitarity is satisfied by certain solutions of the nonlinear, semiclassical "Euler" equations. Even though interacting Pomerons (and fragmentation effects in general) are not included, the results obtained resemble those recently found[1] in supercritical triple-Pomeron theories—total cross sections rising as $(\ln s)^2$, descriptions of inclusive reactions, and essentially no correlations between emitted particles. The latter feature is to be expected in any model wherein quantum fluctuations are averaged over by a semiclassical approximation; but what is most agreeable here is the role played by unitarity in selecting between possible solutions of the Euler equations. One finds a legitimate dynamical procedure for treating high-density, large-rapidity collisions in terms of effectively "free" modes, constructed out of the strongly interacting quanta of a typical Lagrangian field theory.

The present calculation proceeds as follows: Corresponding to a fundamental Lagrangian interaction

$$\mathcal{L}' = ig\bar{\psi}\gamma_\mu A_\mu \psi - \frac{\lambda}{2}\pi A_\mu{}^2\,, \tag{1}$$

one first considers the eikonal approximation to the scattering of a pair of nucleons ψ by the exchange of all possible neutral vector mesons (NVM) A_μ; and between different NVM's, one exchanges pions π in all possible ways. From this array of possible terms, one then extracts only those pionization contributions corresponding to the leading rapidity dependence of all ladder graphs containing pions with ordered rapidities. This "hybrid" model has strong resemblences to ϕ^3 theory and to QED, and it has proved extremely convenient in reproducing phenomenological versions of many cumbersome calculations, including a representation of effective triple-Pomeron physics.[2]

As previously discussed in several other contexts,[3] this set of interactions and extractions may be given an eikonal representation in a functional way by the expressions

$$e^{i\chi(s,b)} = \exp\left(-\frac{i}{2}\int \frac{\delta}{\delta\pi} D_c \frac{\delta}{\delta\pi}\right) \times \exp\left(ig^2 \int \mathfrak{F}_1 \bar{\Delta}_c[\pi]\mathfrak{F}_2\right)\bigg|_{\pi\to 0}\,, \tag{2}$$

or

$$i\chi = \left[\exp\left(-\frac{i}{2}\int \frac{\delta}{\delta\pi} D_c \frac{\delta}{\delta\pi}\right) \times \exp\left(ig^2 \int \mathfrak{F}_1 \bar{\Delta}_c[\pi]\mathfrak{F}_2\right)\right]_{\substack{\pi\to 0\\ \text{conn}}} -1\,, \tag{3}$$

where

$$\mathfrak{F}^\mu_{1,2}(u) = p^\mu_{1,2}\int_{-\infty}^{+\infty} d\xi\, \delta(u - z_{1,2} + \xi p_{1,2})$$

denote the classical currents of the interacting nucleons, with momenta $p_{1,2}$ and spatial coordinates $z_{1,2}$; the impact paramter is $\vec{b} = (\vec{z}_1 - \vec{z}_2)_\perp$; D_c and $\delta_{\mu\nu}\Delta_c$ denote pion and NVM propagators of masses μ and m, respectively; and $\bar{\Delta}_c(\pi) = \Delta_c(1 + \lambda\pi\Delta_c)^{-1}$ represents the NVM propagator in the presence of a ficticious, external, c-number field $\pi(x)$. A special choice of this propagator,

$$\bar{\Delta}_c(z\,|\,\pi)=\int \frac{d^4q}{(2\pi)^4}\,\frac{e^{iqz}}{q^2+m^2}$$
$$\times \exp\left(\lambda \int d^4\kappa\,\frac{\bar{\pi}(\kappa)}{(q-\kappa)_\perp^2+m^2}\right), \quad (4)$$

together with the subsequent replacement

$$\int \frac{d^4\kappa}{\kappa^2+\mu^2} \to i\pi Y \int d^2\kappa_\perp,\quad Y\sim \ln(s/\mu^2), \quad (5)$$

will again be used[4] to generate the leading-rapidity dependence of the pion ladder graphs exchanged between any pair of NVM's. The subscript "conn" specifies the retention of connected graphs only, with at least one pion linkage between any NVM and the rest of that graph. The elastic-scattering amplitude of a pair of nucleons is then given in the familiar way:

$$T(s,t)=i\frac{s}{2}\int d^2b\, e^{i\vec{q}\cdot\vec{b}}(1-e^{i\chi(b,s)}),$$
$$\vec{q}^{\,2}=-t,\quad |t|/s\ll 1. \quad (6)$$

with the normalization chosen such that $(d\sigma_{el}/dt) = |T|^2/\pi s^2$. Unitarity requires that $\sigma_{tot}=\sigma_{el}+\sigma_{in}$, with

$$\sigma_{tot}=\frac{4}{s}\,\mathrm{Im}\,T(s,0)=2\int d^2b\,\mathrm{Re}(1-e^{i\chi}), \quad (7a)$$

$$\sigma_{el}=\int d^2b\,|1-e^{i\chi}|^2, \quad (7b)$$

$$\sigma_{in}=\int d^2b\,(1-e^{-2\,\mathrm{Im}\chi}). \quad (7c)$$

Expansion of the g^2-dependent exponential of (3) provides a representation of the conventional eikonal approximations. Omitting all self-linkages (all pion radiative corrections to the same NVM line) to order g^2, one obtains just the old Lévy-Sucher result[5] $i\chi_1(s,b)\simeq -(ig^2/2\pi)\gamma(s)K_0(mb)$, where $\gamma(s)\to 1$ as $s/m^2\to\infty$, in this NVM theory (were the NVM replaced by scalar mesons A, then $\gamma(s)\to -m^2/s$ as $s/m^2\to\infty$). The g^4 term of this expansion contains all the ladder and crossed pion linkages between a pair of NVM lines. Upon extracting the leading rapidity dependence from the set of ladder graphs [the simplest way is via (4) and (5)], one reproduces an absorptive eikonal $i\chi_2\sim -a_1 s^{a_2}/\ln s\,\exp[-a_3b^2/\ln s]$ equivalent in physical content to the original Cheng-Wu, Chang-Yan calculations[6]; here, the a_i are constants and $a_2\sim\lambda^2/m^2$. (In a scalar-A-meson theory, $a_2+2\sim\lambda^2/m^2$ and the phenomenological choice $a_2=0$ produces an effective, unitarized Pomeron, complete with the correct cuts.) Estimates of the remaining $i\chi_n, n\geq 3$ are exceedingly difficult to obtain, and when obtained are difficult to believe.[4]

In a recent paper,[7] an attempt was made to incorporate the effects of all the χ_n, in an approximate way, by performing a semiclassical calculation for an averaged, "self-consistent" pion field, whose properties would reflect the probable importance of the higher eikonal terms in the limit of ultra-high energies and in the region of small impact parameters. One intuitively expects this to be an important limit and region because there exists the possibility of rapidly rising, and oscillating s dependence coming from all the remaining χ_n, which can tend to overpower the dampening effects of the impact-parameter dependence of such higher t-channel thresholds; what is at issue here is the interplay of s and b dependence, as both variables increase. In Ref. 7, a formalism was set up to perform such an "averaged" estimate by first converting the functional differential operations of (2) and (3) to an equivalent functional integral—along with the corresponding functional representations of all the pion inelastic cross sections—and then approximating these functional integrals by a stationary phase method, with the condition for stationary phase providing the "semiclassical" Euler equation for the "averaged" field. At very high energies, one may expect that every pion exchanged between a pair of NVM's will, in a leading-rapidity, ladder-graph approximation, generate a factor of Y essentially corresponding to the available phase space for the inelastic production of a pion with limited transverse momentum, and hence one is faced with an effective strong-coupling problem, in which it is not acceptable to neglect any particular sets of eikonal graphs. Mathematically, one evaluates an integral in such a strong-coupling limit by a method such as stationary phase; physically, one is shifting the description from one of overly many quanta to one of a single, averaged field.

This type of calculation has one distinct advantage over other more properly classical computations, wherein one merely hopes that quantum fluctuations are small, and will not disturb the classical effects obtained. Because one is here dealing with an eikonal representation of quantum effects (which are then to be approximated by a semiclassical technique), one has the added feature of s-channel unitarity, relating inelastic pion emission at every impact parameter to the imaginary part of the elastic eikonal. After the transition to an averaged field has been made, this requirement of unitarity translates into a restriction on the possible solutions of the Euler equations. Hence an averaged field that satisfies this unitarity condition provides an approximate representation of all the complicated, nonlinear, internal dynamics specified by the original Lagrangian interaction, and one which is also consistent with the quantum-mechanical restrictions of probability for scattering and all production processes. Such a solution has every right to be considered as providing an "average" description of the physics in the regions in which it happens to be valid. Because such averaged solutions are intrinsically noninteracting, one has a framework in which the dynamics may be thought of as essentially that of uncoupled degrees of freedom, with the averaged solutions providing an effective representation of noninteracting "parton" modes.

One may add that it is precisely because the Euler equations are nonlinear that the method works at all. That is, solutions to a particular nonlinear Euler equation have no reason to satisfy *another* nonlinear relation, that of unitarity; one cannot expect miracles. But because the semiclassical equations are nonlinear, they involve the magnitude, phase and *branch* of their solutions—three parameters, or functions, rather than just two. In essence, unitarity determines the third quantity, for example, the branch; and in this sense, the nonlinearity of the problem is essential.

II. CALCULATIONS

In this section, solutions to the Euler equations are studied, approximated, and subjected to the restrictions of unitarity.

A. General forms

We first reproduce the basic equations of Ref. 7, without further discussion of their derivation. Evaluation of the elastic amplitude and inelastic cross sections in this eikonal model, by the method of a stationary-phase approximation to the relevant functional integrals, was there shown to lead to the nonlinear Euler equation

$$K_x\, \varphi_0(x) = -i\, \frac{\delta \mathfrak{F}}{\delta\varphi(x)}\bigg|_{\varphi=\varphi_0}, \tag{8}$$

with

$$\mathfrak{F}[\varphi] = i g^2 \int \mathfrak{F}_1 \bar{\Delta}_c[\varphi]\mathfrak{F}_2\,, \tag{9}$$

and

$$K = \mu^2 - \partial^2 = D_c^{-1}$$

Equation (8) is "self-consistent" in the sense that each "quasipion" φ_0 emitted by a NVM is defined by the behavior of many (an infinite number of such) quasipions along that line. Inelastic emission then turns out to be described by the simplest of all unitary mechanisms, as if no more than a single particle is emitted by each multiperipheral chain. The pionization form of (4), along with the replacement (5), is here adopted for the $\bar{\Delta}_c[\varphi]$ of (9); physically, this corresponds to the modeling of leading-rapidity, ladder-graph pions between *every* pair of NVM's, with the latter themselves exchanged in all possible ways between the scattering nucleons.

In terms of possible solutions φ_0 to (8), the eikonal is given by

$$\chi = -i\mathfrak{F}[\varphi_0] - \tfrac{1}{2}\int \varphi_0 K \varphi_0 + \frac{i}{2}\,\mathrm{Tr}\ln\left(1 + iD_c\, \frac{\delta^2\mathfrak{F}}{\delta\varphi\delta\varphi}\bigg|_{\varphi=\varphi_0}\right). \tag{10}$$

We do take seriously the trace-logarithm contribution of (10), which might be thought of as a normalization correction to the eikonal, and physically corresponds to a sum of quasipion linkages between the NVM's such that no more than two virtual particles are emitted or absorbed by any NVM line. However, due to the nonlinearity imposed by unitarity, inelastic emissions still have the form of just one particle ejected per chain.

One point concerning self-linkages along every NVM line should be made, for the formalism of (2)–(5) will automatically include such dependence, even though the propagator (4) has really been designed to reproduce those pionization effects delivered by cross linkages between two different NVM's. But one may expect that this difficulty is relatively unimportant. For example, following the procedure of Ref. 4, the s dependence arising from cross linkages between n NVM lines produces a factor $s^{a(1/2)n(n-1)}$ $(a\sim\lambda^2/m^2)$, while the effective self-linkages along each NVM contribute a factor $(s^{a/2})^n$, so that the net s dependence still grows as $s^{(a/2)n^2}$, and it is the n^2 dependence of this exponent which suggests the possibility of strong cancellations generated by the higher $\chi_n, n\geq 3$. For $n=2$, the Cheng-Wu, Chang-Yan formalism is essentially unchanged; while for $n=1$, the Lévy-Sucher eikonal is multiplied by a factor of $s^{a/2}$, generating (by itself) a model of "rigid disc" scattering, with $\sigma_{\text{tot}}\sim Y^2$. In the context of the stationary phase approximation, such unwanted self-linkages may be removed by subtracting from χ the first-order expansion of the trace-logarithm of (10),

$$\chi \to \chi' = \chi - \frac{i}{2}\,\mathrm{Tr}\left(iD_c\, \frac{\delta^2\mathfrak{F}}{\delta\varphi\delta\varphi}\bigg|_{\varphi=\varphi_0}\right), \tag{11}$$

and this form will be used in what follows. For small b this has essentially no effect; but in the region of larger impact parameters close to $mb_{\max}\sim Y$, it ensures the Froissart bound in its usual form, rather than $mb_{\max}\sim(Y\ln Y)^{1/2}$.

In momentum space, (8) reads as follows:

$$\tilde{\varphi}_0(k) = -\frac{g^2\lambda}{(2\pi)^2}\,\frac{1}{k^2+\mu^2}\int \frac{d^2q\, e^{i\vec{q}\cdot\vec{b}}}{q^2+m^2}\,\frac{e^{R(q/m)}}{(q-k_\perp)^2+m^2}, \tag{12}$$

with

$$R(q/m) \equiv \lambda \int d^4p \tilde{\varphi}_0(p) [m^2 + (q - p_\perp)^2]^{-1}, \quad (13)$$

where we reject, at the outset, any additional mass-shell contribution to $\tilde{\varphi}_0$. It is immediately clear that finding solutions to (12) is a difficult task. To simplify the analysis, we look for solutions under the assumption $mb \gtrsim 1$, which suggests that the only significant values of q, in the integrand of (12), will be limited by $q \lesssim m$. Hence the $R(q/m)$ factor may be approximated as

$$R(q/m) \simeq R_0 + i\vec{q}\cdot\vec{R}_1 - q_i q_j R_2^{ij}, \quad (14)$$

retaining only (the integrable) quadratic q dependence. If the $|R_2^{ij}|$ are of size m^{-2}, this manipulation is equivalent to a cutoff of $\int d^2q$ at $q_{max} \sim m$; but nonlinear effects can serve to significantly decrease q_{max}. For example, if $m^2|R_2^{ij}| \sim Y$, or larger (and here we are assuming ultrahigh energies, where $Y \gg 1$), then $q_{max}^2 \ll m^2$. It is solutions of this form, corresponding to sizable φ_0, that we have in mind.

The substitution of (14) into (13) and (12), together with the use of (5), and the assumption $q_{max}^2 \ll m^2$, generates

$$\vec{R}_1 = 0 \text{ (by symmetry)}, \quad (15)$$

$$R_2^{ij} = R_2\delta_{ij}, \quad R_2 = R_0/6m^2, \quad (16)$$

and

$$R_0^2 = -i\xi \exp[R_0 - \eta/R_0], \quad (17)$$

with

$$\xi = 3(g\lambda)^2\pi Y/2m^2, \quad \eta = \tfrac{3}{2}(mb)^2.$$

By the use of these approximations, which follow naturally for large R_2, or R_0, all the nonlinear complexity of the original Eq. (12) has been transformed into a single relation for a complex R_0, (17). If solutions to (17) can be found which have the property

$$\mathrm{Re}(R_0) > 0, \quad (18)$$

which is necessary for the convergence of the original q integral of (12), then φ_0 may be written in the form

$$\tilde{\varphi}_0(k) = -i\left(\frac{m^2R_0}{\lambda\pi^2Y}\right)\frac{1}{(k^2+\mu^2)(k_\perp^2+m^2)}. \quad (19)$$

It is then straightforward to substitute (19) into (11), and obtain

$$i\chi = +\left(\frac{m}{\pi\lambda}\right)^2\frac{R_0}{Y} - \left(\frac{m}{\pi\lambda}\right)^2\frac{R_0^2}{2Y} - \tfrac{1}{2}\ln(1-R_0) - \tfrac{1}{2}R_0. \quad (20)$$

Evaluation of the trace-logarithm terms is performed by an approximate evaluation of its nth iterate, using the same approximation methods followed in reaching (19), and then the sum over all iterates (each is finite) has been put into logarithmic form by a simple continuation argument.

With (20) one can now supply a specific input to the s-channel unitarity relation of Ref. 7, which demands, for any such eikonal, that

$$\mathrm{Im}\,\chi = \tfrac{1}{2}\int (K\varphi_0)^* D_{(+)}(k\varphi_0). \quad (21)$$

where $\tilde{D}_{(+)}$ represents the (positive-definite) pion's phase-space function,

$$\tilde{D}_{(+)}(k) = (2\pi)^{-3}\theta(k_0)\delta(k^2+\mu^2).$$

In terms of the explicit solution (19), this becomes

$$\mathrm{Im}\,\chi = \tfrac{1}{2}\left(\frac{m}{\pi\lambda}\right)^2\frac{|R_0|^2}{Y}, \quad (22)$$

where the left-hand side of (22) is given by the imaginary part of (20). We shall insist that any solution for R_0 to (17) and (18) must also satisfy (22).

B. Approximate solutions

To solve (17), set $R_0 = \rho\exp[i(\theta + 2\pi n)]$, where ρ, θ, and n denote magnitude, phase $(-\pi \leq \theta \leq \pi)$, and branch, respectively. Equating magnitude and phase on both sides of (17) leads to the pair of nonlinear equations

$$(\rho - \eta/\rho)\cos\theta = \ln(\rho^2/\xi), \quad (23)$$

$$(\rho + \eta/\rho)\sin\theta = 4\pi n + 2\theta - \pi/2. \quad (24)$$

Unitarity, as in (22), then becomes

$$\tfrac{1}{2}\rho^2 = -\rho\cos\theta + \tfrac{1}{2}\rho^2\cos(2\theta) + aY[\ln(1+\rho^2 - 2\rho\cos\theta) + 2\rho\cos\theta],$$

or

$$\rho^2 = \rho^2\cos^2\theta - \rho\cos\theta + aY[\ln(1+\rho^2-2\rho\cos\theta) + 2\rho\cos\theta], \quad (25)$$

with $a = (\pi\lambda/2m)^2$. It will also be convenient to write the ratio of (24) and (23),

$$\tan\theta = A(\theta + 2\pi n - \pi/4),$$

$$A = \frac{2}{\ln(\rho^2/\xi)}\left(\frac{1-\eta/\rho^2}{1+\eta/\rho^2}\right). \quad (26)$$

Since $\cos\theta \geq 0$, we further restrict θ to lie in the range $-(\pi/2) \leq \theta \leq \pi/2$. The value of θ specified by (26) then depends on the values of n and A. For most cases of interest here, n will be large and A will be small, such that the product An is either large and growing or large and finite as Y increases. Thus the inverse of (26) is approximately given by

$$\theta \sim \tan^{-1}(2\pi A n) \sim \frac{\pi}{2} - \frac{1}{2\pi n A} . \tag{27}$$

so that we have $\theta \to \pi/2$ in the first case and $\theta \to \theta_0 < \pi/2$ in the second.

We begin with small b and work our way outward. For $\eta < \rho^2$ we suppose that $\rho^2 > \xi$, and that A is small although nA can be large. Thus $\theta \simeq \pi/2$ and (24) indicates that $\rho \sim 4\pi n$, with the detailed form of the Y dependence of ρ determined by the appropriate approximation to unitarity. From (25) one finds

$$\rho^2 \sim aY \ln \rho^2 , \tag{28}$$

or $\rho^2 \sim aY\ln Y + \dots$, for large Y. (For sufficiently large Y there are always two solutions to (28), one with $O(\rho^2) \sim 1$, and the other with large ρ^2, as used here.) Thus unitarity determines that $\rho \sim n \sim (Y\ln Y)^{1/2}$, which is consistent with $\rho^2 > \xi \sim Y$, large $A^{-1} \sim \ln(\ln Y)$ and the growth of $nA \sim (Y\ln Y)^{1/2}/\ln(\ln Y)$. In this region, the eikonal of (20) is given by

$$i\chi \sim -\tfrac{1}{8}\ln Y - i\left(\frac{a}{4}\, Y\ln Y\right)^{1/2} , \tag{29}$$

properly absorptive, and independent of b at these relatively small impact parameters, $\eta = \tfrac{3}{2}(mb)^2 < \rho^2 \sim Y\ln Y$.

As η increases towards ρ^2, ρ^2 must decrease towards ξ in order to satisfy (18); $\rho^2 \Rightarrow \eta$ when $\rho^2 \Rightarrow \xi$, and $\xi > \rho^2$ when $\eta > \rho^2$. It is clear, however, that since $\xi \sim Y$, no such solutions $\rho^2 \sim Y\ln Y$ of unitarity can co-exist for $\xi > \rho^2$, and it becomes necessary to examine closely the forms of possible solutions as ρ^2 approaches η and ξ from above.

We observe that, for somewhat larger impact parameters, it is possible to construct a solution with $\theta \to \theta_0 < \pi/2$, that is, with the product nA finite. Thus $\cos\theta_0 \neq 0$, and (25) suggests that the leading dependence of ρ is given by

$$\rho \sim \alpha Y + \beta \ln Y, \quad \alpha = \frac{2a\cos\theta_0}{\sin^2\theta_0} , \quad \beta^{-1} = 2\cos\theta_0 .$$

If this is appropriate as $\eta \lesssim \rho^2$, we find from (24) that

$$\rho \sim \frac{2\pi n}{\sin\theta_0} \sim n \sim Y + \gamma \ln Y ,$$

with $\gamma = (1/4a)\tan^2\theta_0$; hence we are now on a higher branch, $n \sim Y$. Then,

$$A \sim \left(\frac{\rho^2 - \eta}{2\rho^2}\right)\left[\ln\left(\frac{\rho^2}{\xi}\right)\right]^{-1} \sim \left[\frac{(Y+\gamma\ln Y)^2 - \eta/\alpha^2}{2(Y+\gamma\ln Y)^2}\right](\ln Y)^{-1} .$$

For the largest η in this region, $\eta_{\max} \equiv (\alpha Y)^2 \lesssim \rho^2$, $A \sim \gamma/Y$ and $nA \sim$ constant, so that $\theta \to \theta_0 < \pi/2$, as originally assumed. This maximum η corresponds to $mb_{\max} \sim Y$, that impact parameter at which one expects the Froissart bound to become operative. Here, $\mathrm{Re}(i\chi) \sim -Y$.

What happens as η is increased past $\eta_{\max}$? One sees that $A \to 0$, for any n, and therefore $\theta \to 0$. But then unitarity requires $\ln|1-\rho| = -\rho(1 - 1/2aY)$, to which there is always one solution with $\rho \sim O(1)$. Thus, when η increases past $\eta_{\max}$ and approaches ρ^2, ρ^2 falls sharply below $\xi \sim Y$; this drop must be discontinuous, so as to preserve $\cos\theta > 0$ with $\eta > \xi$.

For $\eta > \xi > \rho^2$, we expect the physically desirable solution of very small ρ, for which (25) requires that both ρ and $\cos\theta$ are small, with $\rho \sim Y^{-1}\cos\theta$. Hence $n \sim (\eta/\rho)\sin\theta$, $A^{-1} \sim \ln(Y^3/\cos^2\theta)$, and $nA \sim \eta Y \tan\theta / \ln(Y^3/\cos^2\theta)$ which we expect to be large. So again $\theta \to \pi/2$, with $\cos\theta \sim 1/nA$. But then $\tan\theta \sim (\cos\theta)^{-1}$ which implies $\ln(Y^3/\cos^2\theta) \sim \eta Y$, or $\cos\theta \sim Y^{3/2}\exp[-\alpha'\eta Y]$ with $\alpha' = a/4\pi$. Thus ρ falls off very rapidly indeed, as Yb^2 increases, and is effectively zero for $b > b_{\max} \sim (1/m)Y$. This behavior provides the Froissart bound for our inclusive, and total cross sections.

III. CONCLUSIONS

On the basis of the averaged solutions described above, we have obtained an absorptive eikonal which vanishes sharply when $b > b_{\max}$, thereby generating $\sigma_{\text{tot}} \sim Y^2$. Inclusive cross sections are obtained by calculating the functional derivatives of

$$\sigma_{\text{In}} = \int d^2b\, e^{-2\,\mathrm{Im}\chi}\left[\exp\left(\int (K\varphi_0)^* D_{(+)}(K\varphi_0)\right) - 1\right] \tag{30}$$

with respect to $\tilde{D}_{(+)}(k)$, and produce noncorrelated inclusive distributions. The one-particle inclusive cross section, for example, is given by

$$\omega\frac{d\sigma}{d^3k} = \pi \int d^2b\, |K\tilde{\varphi}_0(k)|^2 , \tag{31}$$

with an integrand that cuts off sharply at $b \gtrsim b_{\max}$. Using the larger b forms throughout, this gives approximately

$$\omega\frac{d\sigma}{d^3k} \sim b_{\max}{}^2 \frac{m^4}{(m^2 + k_\perp{}^2)^2} . \tag{32}$$

The limited k distribution of (32) is not significant, since it was effectively assumed at the beginning; but the growth of such an inclusive cross section with Y^2, and the flat plateau in the rapidity version of this result are specific predictions of the model. The other n-particle inclusive cross sections show a similar rise with Y^2, and contain no correlations between any such inclusive emissions.

Within the context of this averaged, or semiclassical calculation, the present computation thus provides an answer to the long-standing eikonal question: How important are *all* the $i\chi_n, n \geq 2$? We do not find the almost complete cancellation suggested in Ref. 4; rather there is an effective saturation of the Froissart bound, which would have been given by the pionization model of $i\chi_2$, except for a slight decrease in σ_{tot} and the inclusive cross sections coming from the region of small $b, mb < (Y \ln Y)^{1/2}$. One difference is that here the ratio of total one-particle inclusive cross section to σ_{tot} produces a multiplicity $\langle n \rangle \sim Y$, in contrast to the Cheng-Wu, Chang-Yan behavior $\langle n \rangle \sim s^{a/1+2a}$. Except for the complete absence of correlations, these results are not in overt disagreement with existing data, and may be looked upon as a prediction of future, higher-energy experiments.

It should be emphasized that a calculational procedure quite different from that of Ref. 4 has been performed, even though the same pionization input, $\overline{\Delta}_c[\pi]$ of (4), has been used. In Ref. 4, one in principle assumed the existence of an expansion of the eikonal in powers of g^2, $\chi = \sum_n g^{2n} \chi_n$, and then calculated at fixed n the large-Y limit of each $\chi_n(Y)$, subsequently summing over all n to obtain a σ_{tot} which vanished asymptotically. No such expansion in powers of g^2 is contemplated here, nor would one be possible based upon the results of the present semiclassical calculation, in which χ turns out to be a function of $g^2 Y$; an expansion in powers of g^2 is not sensible if $g^2 Y$ is large, as is evidenced by the form of the "g^2 expansion" one would build from our asymptotic rapidity forms, replacing Y there by $g^2 Y$. In addition, there is the very essential and nonlinear complication of impact-parameter dependence, accounting for at least one discontinuous change in the eikonal as the Froissart b_{max} is exceeded.

Finally, it should perhaps be remarked that the calculated behavior of σ_{tot} is not really the valuable point of the paper. Hadronic physics is far more complicated than sketched here, as is immediately clear from our neglect of diffractive disassociation effects (as in Ref. 1). What is important, and to our knowledge unique, is the use of unitarity as a restriction on the class of acceptable semiclassical solutions. It is hoped that this technique will find other applications in high-energy problems, and possibly in statistical mechanics, where semiclassical approximations are of high current interest.

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