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Lagrangian and Hamiltonian descriptions of Yang-Mills particles

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A new Lagrangian L is proposed for the description of a particle with a non-Abelian charge in interaction with a Yang-Mills field. The canonical quantization of L is discussed. At the quantum level L leads to both irreducible and reducible multiplets of the particle depending upon which of the parameters in L are regarded as dynamical. The case which leads to the irreducible multiplet is the minimal non-Abelian generalization of the usual Lagrangian for a charged point particle in an electromagnetic field. Some of the Lagrangians proposed before for such systems are either special cases of ours or can be obtained from ours by simple modifications. Our formulation bears some resemblance to Dirac's theory of magnetic monopoles in the following respects: (1) Quantization is possible only if the values of certain parameters in L are restricted to a certain discrete set, this is analogous to the Dirac quantization condition; (2) in certain cases, L depends on external (nondynamical) directions in the internal-symmetry space. This is analogous to the dependence of the magnetic-monopole Lagrangian on the direction of the Dirac string.

I. INTRODUCTION

The classical interactions of a particle carrying non-Abelian charge with the corresponding Yang-Mills field are of considerable physical interest. One need only consider the analogy with the Abelian system (a classical point charge interacting with the electromagnetic field) to appreciate the rich range of phenomena which needs to be understood. Some years ago, Wong[1] proposed a system of equations to describe the classical dynamics of such a system, equations which are the non-Abelian generalizations of the Lorentz force and Maxwell equations of electrodynamics. The non-Abelian particle is characterized by an "isovector" $\vec{I}(\tau)$ which transforms under the adjoint representation of the internal-symmetry group $\mathcal{G}$ and for which there is one more equation of motion with no Abelian counterpart. Wong also gave a Hamiltonian formulation of the system.

For many purposes, it is useful to define an action whose extrema give the classical equations of motion. The invariances of the associated Lagrangian lead straightforwardly to the symmetries and conservation laws of the theory. The classical action can also be employed, for example, in quantization and semiclassical calculations by path-integral techniques.

Lagrangians which lead to Wong's equations have been discussed in the recent literature.[2] These approaches in general introduce extra observable degrees of freedom and equations of motion not originally present in the Wong equations. Hence they do not represent a minimal non-Abelian extension of the Lorentz force and Maxwell equations. Moreover, when quantized they describe particles which in general belong to reducible representations of $\mathcal{G}$. In this paper we propose a classical Lagrangian whose observable degrees of freedom (that is, those with well-defined time evolution) are just those described by the Wong equations. Thus it gives a truly minimal non-Abelian generalization of the standard Lagrangian for a charged particle in an electromagnetic field. When quantized, this minimal Lagrangian describes a particle which belongs to an irreducible representation of $\mathcal{G}$. Actually however, we will show that our Lagrangian can describe *both* irreducible and reducible multiplets depending on which of the parameters in the Lagrangian are varied. Further, some of the previous Lagrangians are either special cases of ours or can be obtained from ours by simple modifications.

There are two interesting facts we find which are analogous to the situation in magnetic-monopole theory: (1) Quantization is possible only if the values of certain parameters in the Lagrangian [namely, $\hat{K}_a$ of (2.10)] are restricted to a certain discrete set; (2) in certain cases, the Lagrangian depends on external (nondynamical) directions in internal-symmetry space. The first resembles the Dirac quantization condition. It arises from the fact that the spectrum of the Casimir invariants of $\mathcal{G}$ is discrete. The second is similar to the dependence of the monopole Lagrangian on the direction of the Dirac string.

In Sec. II the Lagrangian is written down. In discussing the properties of the associated system, there are several cases to be considered. This classification is discussed in this section. The salient features of the corresponding quantum systems are also summarized. In the next six sections, each of these cases is treated in some detail. In the final section we indicate how some of the previous Lagrangians can be obtained from ours. The two appendices contain some technical calculations.

II. THE LAGRANGIAN

Before discussing the Lagrangian, we shall first write down Wong's equations. They are

$$m\frac{d}{d\tau}[\dot{z}^\mu/(-\dot{z}^2)^{1/2}]=-eF^{\mu\nu}_{\alpha}I_\alpha\dot{z}_\nu, \tag{2.1}$$

$$[D_\mu]_{\alpha\beta}F^{\mu\nu}_\beta(x)=e\int d\tau\,\delta^4(x-z(\tau))\dot{z}^\nu(\tau)I_\alpha(\tau). \tag{2.2}$$

Here $z^\mu(\tau)$ denotes the particle coordinate, $F^{\mu\nu}_\alpha(x)$ is the usual Yang-Mills field tensor, and D_μ denotes the covariant derivative. The range of the index α is equal to the dimension n of the internal-symmetry group $\mathcal{G}$. The vector $\vec{I}(\tau)$ transforms under the adjoint representation of $\mathcal{G}$. From (2.2) and the identity $[D_\mu, D_\nu]_{\alpha\beta}F^{\mu\nu}_\beta=0$, one finds the following consistency condition on $\vec{I}(\tau)$:

$$\frac{dI_\alpha}{d\tau}-e\dot{z}_\mu A^\mu_\rho(z)c_{\rho\alpha\beta}I_\beta=0. \tag{2.3}$$

Here $c_{\rho\alpha\beta}$ are the structure constants.

We shall now take up the Lagrangian. Let $\mathcal{G}$ be a compact connected Lie group with a simple Lie algebra $\mathcal{L}$. Let $\Gamma=\{g\}$ be a faithful unitary representation of $\mathcal{G}$. The associated Lie algebra γ has a basis $T(\rho)$ $(\rho=1,2,\ldots,n)$ with $T(\rho)^\dagger=T(\rho)$. (More precisely, this is a basis for $i\gamma$.) We normalize $T(\rho)$'s according to

$$\mathrm{Tr}\,T(\rho)T(\sigma)=R\delta_{\rho\sigma}, \tag{2.4}$$

where R is a constant. Further, they fulfill the commutation relations

$$[T(\rho),T(\sigma)]=ic_{\rho\sigma\lambda}T(\lambda). \tag{2.5}$$

The Lagrangian is given by

$$L=-m[-\dot{z}(\tau)^2]^{1/2}-(i/R)\,\mathrm{Tr}K_\rho T(\rho)g^{-1}(\tau)D_\tau g(\tau). \tag{2.6}$$

Here $g(\tau)\in\Gamma$ is an additional degree of freedom associated with the particle. The covariant derivative D_τ is defined by

$$D_\tau\equiv\frac{d}{d\tau}-ieA^\mu\dot{z}_\mu,\quad A^\mu\equiv A^\mu_\alpha(z(\tau))T(\alpha), \tag{2.7}$$

where A^μ_α are the Yang-Mills potentials. The quantities K_ρ are real valued. The Yang-Mills Lagrangian

$$-\tfrac{1}{4}\int d^3x F_{\mu\nu\alpha}F^{\mu\nu}_\alpha \tag{2.8}$$

can be added to (2.1). However, we will omit it, since the treatment of the Yang-Mills field is standard. Note also the following. In case 1 below, the K_ρ's are treated as constants. For this case, if the gauge group is U(1) ($g=e^{i\psi}$, where ψ is a real-valued function), the Lagrangian (2.6) differs from the usual Lagrangian for a charged particle in an electromagnetic field by a term proportional to $d\psi/d\tau$. Since this term is a total time derivative, in this case, for the Abelian gauge group, (2.6) is equivalent to the usual Lagrangian.

There are four cases to be considered. In all the cases the z^μ's and g's are to be varied to obtain the Euler-Lagrangian equations. These already lead to (2.1) and (2.3) if we identify I_ρ by the equations

$$I=gKg^{-1},$$
$$I\equiv I_\rho T(\rho),\quad K\equiv K_\rho T(\rho) \tag{2.9}$$

and use the fact that the K_ρ's are either numerical constants or constants of motion in all the cases. The treatment of K_ρ is not the same in all the cases.

Case 1. Here the K_ρ's are treated as constants and are not varied in obtaining the Euler-Lagrange equations. We will show that the corresponding quantum system describes a particle which belongs to an irreducible representation (IRR) of $\mathcal{G}$. The specific IRR is determined by the values of K_ρ. The IRR occurs only once.

It is curious that the Lagrangian has an explicit dependence on a direction in the internal-symmetry space given by K. The choice of this direction, however, is arbitrary. The replacement of K by SKS^{-1}, where S is a τ-independent group element, does not change the system. A somewhat similar situation occurs in case 4 also.

Case 2. Here the K_ρ's are treated as dynamical variables. The quantum-mechanical Hilbert space carries the left regular representation of $\mathcal{G}$. The multiplicity of an IRR in this representation is equal to its dimension.

Case 3. Let $H(a)$ $(a=1,\ldots,k)$ be a basis for the Cartan subalgebra $\mathcal{C}$ of γ. It is well known[3] that we can write

$$K=h\hat{K}h^{-1},\quad \hat{K}\equiv\hat{K}_a H(a) \tag{2.10}$$

for a suitable $h\in\gamma$. In this case we vary h while the $\hat{K}_a$'s are held fixed. The values of $\hat{K}_a$ determine the particular IRR of $\mathcal{G}$ which occurs in quantum mechanics. This IRR occurs with a multiplicity equal to its own dimension.

Case 4. In this case, h is held fixed while the $\hat{K}_a$'s are varied. Here, we have not succeeded in fully investigating the quantum system except in some special cases. Some of the Lagrangians written down by previous authors[2] can be obtained from case 4.

III. CASE 1

We assume that the group elements are parametrized by an independent set of variables $\xi = (\xi_1, \xi_2, \ldots, \xi_n)$. Thus, in the Lagrangian, $\xi = \xi(\tau)$. Before discussing the variational problem and the Hamiltonian formulation, we note some simple results.

Since $\exp[iT(\sigma)\epsilon_\sigma] \in \Gamma$ we can write

$$e^{iT(\sigma)\epsilon_\sigma} g(\xi) = g[\xi(\epsilon)], \quad \xi(0) \equiv \xi. \tag{3.1}$$

Differentiating on ϵ_σ and setting $\epsilon = 0$ we find

$$iT(\sigma)g(\xi) = \frac{\partial g(\xi)}{\partial \xi_\rho} N_{\rho\sigma},$$
$$N_{\rho\sigma} \equiv \left.\frac{\partial \xi_\rho(\epsilon)}{\partial \epsilon_\sigma}\right|_{\epsilon=0}. \tag{3.2}$$

Here

$$\det N \neq 0, \tag{3.3}$$

for if not, there exist x_σ, not all zero, such that $N_{\rho\sigma} x_\sigma = 0$. By (3.2), $x_\sigma T(\sigma) g(\xi) = 0$, and hence $x_\sigma T(\sigma) = 0$. But this contradicts the linear independence of the $T(\rho)$'s.

A. The Euler-Lagrange equations

Recall that the K_ρ's are not dynamical variables for case 1.

Variation of ξ_ρ

The Lagrangian (2.6) gives

$$\frac{d}{d\tau}\left(\frac{\partial L}{\partial \dot\xi_\rho}\right) = -\frac{i}{R} \mathrm{Tr}\left[-Kg^{-1}\dot g g^{-1}\frac{\partial g}{\partial \xi_\rho} + Kg^{-1}\left(\frac{\partial}{\partial \xi_\lambda}\frac{\partial}{\partial \xi_\rho} g\right)\dot\xi_\lambda\right], \tag{3.4}$$

$$\frac{\partial L}{\partial \xi_\rho} = -\frac{i}{R}\mathrm{Tr}\left[-Kg^{-1}\frac{\partial g}{\partial \xi_\rho} g^{-1} D_\tau g + Kg^{-1}\left(\frac{\partial}{\partial \xi_\lambda}\frac{\partial}{\partial \xi_\rho} g\right)\dot\xi_\lambda - ieKg^{-1}A\frac{\partial g}{\partial \xi_\rho}\right], \tag{3.5}$$

where

$$A \equiv \dot z_\mu A^\mu_\alpha(z(\tau))T(\alpha), \quad \dot g = \frac{dg(\tau)}{d\tau}. \tag{3.6}$$

Equating (3.4) and (3.5), multiplying by $N_{\rho\sigma}$, and using (3.2) we find

$$\mathrm{Tr}\, T(\alpha)[\dot g g^{-1} - ieA, I] = 0, \quad I \equiv gKg^{-1}. \tag{3.7}$$

This manipulation is legitimate due to (3.3). Since the commutator is in γ, and $\{T(\alpha)\}$ is a basis for γ, it follows that

$$[\dot g g^{-1} - ieA, I] = 0. \tag{3.8}$$

But the definition of I gives

$$\dot I = [\dot g g^{-1}, I]. \tag{3.9}$$

Thus

$$\dot I - ie[A, I] = 0 \tag{3.10}$$

which is identical to (2.3).

Variation of z_μ

We find,

$$\frac{d}{d\tau}\frac{\partial L}{\partial \dot z_\mu} = m\frac{d}{d\tau}[\dot z^\mu/(-\dot z^2)^{1/2}] - \frac{e}{R}\frac{d}{d\tau}\mathrm{Tr}(IA^\mu),$$
$$\frac{\partial L}{\partial z_\mu} = -\frac{e}{R}\mathrm{Tr}(I\partial^\mu A^\nu)\dot z_\nu. \tag{3.11}$$

Thus, using (3.10) we get

$$m\frac{d}{d\tau}[\dot z^\mu/(-\dot z^2)^{1/2}] = -\frac{e}{R}\mathrm{Tr}(IF^{\mu\nu})\dot z_\nu,$$
$$F^{\mu\nu} \equiv F^{\mu\nu}_\alpha T(\alpha) \tag{3.12}$$

which is the same as (2.1).

The variation of the vector fields A^μ_α gives (2.2) in an obvious way.

B. The Hamiltonian formalism

As mentioned previously, the canonical formalism for the vector field A^μ_α is standard and will not be treated here. Thus we deal with the Lagrangian (2.6) which omits the term (2.8). Further, for notational simplicity we replace the first term in (2.6) by its nonrelativistic form

$$\tfrac{1}{2}m\dot z^2. \tag{3.13}$$

The canonical momenta which are conjugate to z_i and ξ_ρ will be denoted by p^i and π_ρ, respectively. The latter are given by

$$\pi^\rho = -\frac{i}{R}\mathrm{Tr}\left(Kg^{-1}\frac{\partial g}{\partial \xi_\rho}\right). \tag{3.14}$$

Since $N_{\rho\sigma}$ [defined in (3.2)] is nonsingular, we can replace π_ρ by the equivalent set of variables $\mathcal{I}_\sigma$ where

$$\mathcal{I}_\sigma = \pi_\rho N_{\rho\sigma}. \tag{3.15}$$

By (3.14), (3.2), and (2.9), we thus find the primary constraints[4]

$$\phi_\sigma \equiv \mathcal{I}_\sigma - I_\sigma \tag{3.16}$$

which vanish weakly.

The Hamiltonian is

$$H = \frac{1}{2m}[p^i - eA^i_\alpha(z)\mathcal{I}_\alpha]^2 + eA^0_\alpha(z)\mathcal{I}_\alpha + v_\rho\phi_\rho, \tag{3.17}$$

where we have used the constraints in rearranging terms. The v_ρ's are Lagrange multipliers. We prove the following Poisson bracket (PB) identities in Appendix A:

$$\{\mathcal{J}_\rho, \phi_\sigma\} = -c_{\rho\sigma\lambda}\phi_\lambda, \tag{3.18}$$

$$\{\phi_\rho, \phi_\sigma\} = c_{\rho\sigma\lambda}(\mathcal{J}_\lambda - 2\phi_\lambda), \tag{3.19}$$

$$\{\mathcal{J}_\rho, \mathcal{J}_\sigma\} = -c_{\rho\sigma\lambda}\mathcal{J}_\lambda, \tag{3.20}$$

$$\{\phi_\rho, g\} = \{\mathcal{J}_\rho, g\} = -iT(\rho)g. \tag{3.21}$$

Thus $\{\phi_\sigma, H\} = 0$ gives

$$[v_\rho \,\mathrm{ad}T(\rho)]_{\sigma\lambda}\mathcal{J}_\lambda = 0, \tag{3.22}$$

where $\{\mathrm{ad}T(\rho)\}$ spans the adjoint representation of γ:

$$[\mathrm{ad}T(\rho)]_{\sigma\lambda} = -ic_{\rho\sigma\lambda}.$$

[Owing to (2.4) $c_{\rho\sigma\lambda}$ is totally antisymmetric (and, of course, it is real).] It follows that there are no secondary constraints and v_ρ is restricted by (3.22). The latter admits any v_ρ such that $v_\rho \,\mathrm{ad}T(\rho)$ is in the little group of $\vec{g}$. Note also that $v_\rho\phi_\rho$ are the only first-class constraints due to (3.19) and (3.22).

It is evident from (3.17) that only those variables which have (weakly) zero PB's with $v_\rho\phi_\rho$ will have a well-defined time evolution. Only such variables are of physical interest. From (3.18), we see that $\mathcal{J}_\rho$ are one such set of variables. Further, from (3.21),

$$\{v_\rho\phi_\rho, g\} = -iv_\rho T(\rho)g. \tag{3.24}$$

Thus only those functions of g which are invariant under the action of the little group of $\vec{\mathcal{J}}$ are of interest. But on the constrained surface where $\vec{\mathcal{J}} = \vec{\mathrm{I}}$, these are exactly functions of $\vec{\mathcal{J}}$. Thus the independent variables of interest are exhausted by $\mathcal{J}_\alpha$ and, of course, z_i and p^i.

It remains to compute Dirac brackets (DB's) of $\mathcal{J}_\alpha$. Since they are first-class variables by (3.18), their DB's are equal to their PB's:

$$\{\mathcal{J}_\rho, \mathcal{J}_\sigma\}^* = -c_{\rho\sigma\lambda}\mathcal{J}_\lambda. \tag{3.25}$$

There is one more important fact to be noticed. From (2.9), (2.10), and the constraints (3.16), we find

$$\mathcal{J} = gh\hat{K}(gh)^{-1},\ \mathcal{J} \equiv \mathcal{J}_\sigma T(\sigma). \tag{3.26}$$

Since $\hat{K}$ is a fixed matrix in case 1, the allowed values of $\mathcal{J}_\sigma$ are restricted by the requirement that it lies on the orbit of $\hat{K}$ under the adjoint action. The orbits of $\hat{K}$ are completely labeled by $\hat{K}$. Any function of $\mathcal{J}_\sigma$'s which is constant on the orbits can therefore be written as a function of the orbit labels $\hat{K}_a$. In particular, the Casimir invariants can be expressed in terms of $\hat{K}_a$. These Casimir invariants are thus fixed in case 1.

C. Quantum mechanics

The commutation relations (CR's) of $\mathcal{J}_\rho$'s can be obtained from (3.25) in the usual fashion. Thus the $\mathcal{J}_\rho$'s form a basis for a representation of the Lie algebra of $\mathcal{G}$. Since the Casimir invariants are uniquely given in terms of $\hat{K}_a$, there occurs only one IRR of $\mathcal{G}$ in quantum mechanics. Note also that in general the spectrum of the Casimir invariants is not arbitrary, but consists of a certain set of discrete values. Thus, quantization is possible only if the $\hat{K}_a$'s are restricted to a certain discrete set. This is similar to the Dirac quantization condition in the magnetic-monopole theory which occurs when we insist that the rotation group be implementable in quantum mechanics.

IV. CASE 2

A. The Euler-Lagrange equations

Here the K_ρ's are treated as dynamical variables. *Variation K_ρ.* We find from (2.6) that

$$\mathrm{Tr}\,T(\rho)g^{-1}D_\tau g = 0. \tag{4.1}$$

Since $g^{-1}D_\tau g \in \gamma$,

$$g^{-1}D_\tau g = 0 \ \text{ or } \ D_\tau g = 0. \tag{4.2}$$

Variation of ξ_ρ. Since K_ρ is dynamical, (3.8) is replaced by

$$g\dot{K}_\rho T(\rho)g^{-1} + [\dot{g}g^{-1} - ieA, I] = 0. \tag{4.3}$$

Owing to (4.2), the second term is zero and thus

$$\dot{K}_\rho = 0. \tag{4.4}$$

Equation (4.3) reduces to (3.10).

Variation of z_μ. In the passage from (3.11) to (3.12), we used (3.10). Since (3.10) is still valid, (3.12) continues to be true.

B. The Hamiltonian formalism

Let χ_ρ be the canonical momenta conjugate to K_ρ. In addition to the constraints ϕ_σ [Eq. (3.16)], we find the additional primary constraints χ_ρ, which also vanish weakly. The Hamiltonian is

$$H = \frac{1}{2m}[p^i - eA^i_\alpha(z)\mathcal{J}_\alpha]^2 + eA^0_\alpha(z)\mathcal{J}_\alpha + v_\rho\phi_\rho + w_\rho\chi_\rho\,, \tag{4.5}$$

where the w_ρ are additional Lagrange multipliers.

Besides (3.18)–(3.21) we have also the PB's

$$\{\mathcal{J}_\sigma, \chi_\rho\} = 0, \tag{4.6}$$

$$\{\phi_\sigma, \chi_\rho\} = -(\mathrm{ad}g)_{\sigma\rho}, \tag{4.7}$$

$$\{\chi_\sigma, \chi_\rho\} = 0. \tag{4.8}$$

Here $\{\mathrm{ad}g\}$ is the adjoint representation of $\mathcal{G}$. It is defined by

$$gT(\rho)g^{-1}=(\mathrm{ad}g)_{\sigma\rho}T(\sigma). \tag{4.9}$$

Owing to (2.4), ad$g$ is orthogonal (and, of course, it is real).

Let us next discuss secondary constraints. The requirement $\{\phi_\sigma, H\}=0$ gives

$$i[v_\rho\,\mathrm{ad}T(\rho)]_{\sigma\lambda}\mathcal{G}_\lambda+(\mathrm{ad}g)_{\sigma\rho}w_\rho=0, \tag{4.10}$$

while $\{\chi_\sigma, H\}=0$ gives

$$v_\rho(\mathrm{ad}g)_{\rho\sigma}=0. \tag{4.11}$$

By (4.11), $v_\rho=0$ and hence by (4.10) $w_\rho=0$. Thus there are no secondary constraints.

We have incidentally shown that all constraints are second class.

It remains to compute the DB's of the independent variables. We can choose the latter to be $\mathcal{G}_\rho$ and g. The variables K_ρ are given in terms of these via the constraint $\mathcal{G}_\rho=I_\rho$ and (2.9). Since the $\mathcal{G}_\rho$'s are first-class variables, any DB which involves them is the same as the corresponding PB. Thus from (3.20) and (3.21),

$$\begin{aligned}\{\mathcal{G}_\rho,\mathcal{G}_\sigma\}^*&=-c_{\rho\sigma\lambda}\mathcal{G}_\lambda,\\ \{\mathcal{G}_\rho,g\}^*&=-iT(\rho)g.\end{aligned} \tag{4.12}$$

It remains to find $\{g_{\alpha\beta},g_{\rho\sigma}\}^*$. If c_μ denotes any constraint, let us define $\Delta_{c_\mu c_\nu}=\{c_\mu, c_\nu\}$. Thus Δ is a matrix whose elements are labeled by the constraints. Since $\{g_{\alpha\beta},\chi_\sigma\}=0$, we have

$$\{g_{\alpha\beta},g_{\rho\sigma}\}^*=-\{g_{\alpha\beta},\phi_\lambda\}\Delta^{-1}{}_{\phi_\lambda\phi_\nu}\{\phi_\nu,g_{\rho\sigma}\}. \tag{4.13}$$

To find $\Delta^{-1}{}_{\phi_\lambda\phi_\nu}$ we examine the ϕ,χ matrix elements of $\Delta^{-1}\Delta=1$:

$$\Delta^{-1}{}_{\phi_\lambda\phi_\nu}\Delta_{\phi_\nu\chi_\rho}+\Delta^{-1}{}_{\phi_\lambda\chi_\nu}\Delta_{\chi_\nu\chi_\rho}=0. \tag{4.14}$$

From (4.7) and (4.8), it follows that $\Delta^{-1}{}_{\phi_\lambda\phi_\nu}=0$ and that

$$\{g_{\alpha\beta},g_{\rho\sigma}\}^*=0. \tag{4.15}$$

C. Quantum mechanics

Since all the components $g_{\alpha\beta}$ of g can be simultaneously diagonalized, the wave functions ψ can be taken to be functions of z and g, $\psi=\psi(z,g)$. The $\mathcal{G}_\rho$'s are simply the differential operators which represent the elements $T(\rho)$ in the left regular representation of the group. In particular,

$$\{e^{i\mathcal{G}_\rho\theta_\rho}\psi\}(z,g)=\psi(z,e^{-iT(\rho)\theta_\rho}g). \tag{4.16}$$

The scalar product with respect to which the $\mathcal{G}_\rho$'s are Hermitian is given by

$$(\psi',\psi)=\int d\mu(g)d^3z\,\psi'^*(z,g)\psi(z,g), \tag{4.17}$$

where $d\mu(g)$ is the invariant measure on the group. The left regular representation is highly reducible. Every IRR occurs with a multiplicity equal to its own dimension.

V. CASE 3

A. The Euler-Lagrange equations

Here, in the decomposition (2.10), we regard h as a dynamical variable, while $\hat{K}_a$ is held fixed. It is convenient to take the variables in the Lagrangian (2.6) to be h and

$$G\equiv gh. \tag{5.1}$$

Then (2.6) becomes

$$\begin{aligned}L=&-m[-\dot{z}(\tau)^2]^{1/2}-(i/R)\,\mathrm{Tr}\hat{K}G^{-1}D_\tau G\\ &+(i/R)\,\mathrm{Tr}\hat{K}h^{-1}\dot{h}.\end{aligned} \tag{5.2}$$

We parametrize the group elements G and h by ξ and η, respectively.

Variation of ξ_ρ *and* η_ρ. As in Sec. III, we find that the ξ variation gives equation (3.10):

$$\dot{I}-ie[A,I]=0,\quad I\equiv G\hat{K}G^{-1}. \tag{5.3}$$

Similiarly the η variation gives

$$\dot{J}=0,\quad J\equiv h\hat{K}h^{-1}. \tag{5.4}$$

Variation of z_μ. As before, (3.12) follows.

B. The Hamiltonian formalism

Let

$$\pi_\rho^I=\frac{\partial L}{\partial\dot{\xi}_\rho},\quad \pi_\rho^J=\frac{\partial L}{\partial\dot{\eta}_\rho}. \tag{5.5}$$

Then the primary constraints which vanish weakly are the analogs of (3.16):

$$\phi_\rho^I\equiv\mathcal{G}_\rho-I_\rho, \tag{5.6}$$

$$\phi_\rho^J\equiv\mathcal{J}_\rho-J_\rho. \tag{5.7}$$

Here, $\mathcal{G}_\rho$ and $\mathcal{J}_\rho$ are defined in analogy to (3.15).

The Hamiltonian is

$$\begin{aligned}H=&\frac{1}{2m}[p^i-eA_\alpha^i(z)\mathcal{G}_\alpha]^2+eA_\alpha^0(z)\mathcal{G}_\alpha\\ &+v_\rho\phi_\rho^I+w_\rho\phi_\rho^J.\end{aligned} \tag{5.8}$$

The following PB's are zero:

$$\{\mathcal{G}_\rho,\mathcal{J}_\sigma\}=\{\mathcal{G}_\rho,\phi_\sigma^J\}=\{\mathcal{J}_\rho,\phi_\sigma^I\}=\{\phi_\rho^I,\phi_\sigma^J\}=0. \tag{5.9}$$

The remaining PB's of interest can be obtained from (3.18)–(3.21) by either of the following replacements: (a) $\mathcal{G}_\rho\to\mathcal{G}_\rho$, $\phi_\sigma\to\phi_\sigma^I$, $g\to G$; (b) $\mathcal{G}_\rho\to\mathcal{J}_\rho$, $\phi_\sigma\to\phi_\sigma^J$, $g\to h$.

The requirement $\{\phi_\sigma^I,H\}=\{\phi_\sigma^J,H\}=0$ leads as in (3.22) to

$$[v_\rho\,\mathrm{ad}T(\rho)]_{\sigma\lambda}\mathcal{G}_\lambda=[w_\rho\,\mathrm{ad}T(\rho)]_{\sigma\lambda}\mathcal{J}_\lambda=0. \tag{5.10}$$

There are no secondary constraints.

The analysis of Sec. III can now be repeated. The independent variables on the constrained surface are now z_i, p^i, $\mathcal{G}_\rho$, and $\mathcal{J}_\rho$. Further, we have the DB's

$$\{\mathcal{G}_\rho, \mathcal{G}_\sigma\}^* = -c_{\rho\sigma\lambda}\mathcal{G}_\lambda, \tag{5.11}$$

$$\{\mathcal{J}_\rho, \mathcal{J}_\sigma\}^* = -c_{\rho\sigma\lambda}\mathcal{J}_\lambda, \tag{5.12}$$

$$\{\mathcal{G}_\rho, \mathcal{J}_\sigma\}^* = 0. \tag{5.13}$$

On the constrained surface, from (5.3), (5.4), and the constraints, we have

$$\mathcal{G} = G\hat{K}G^{-1}, \quad \mathcal{G} \equiv \mathcal{G}_\sigma T(\sigma), \tag{5.14}$$

$$\mathcal{J} = h\hat{K}h^{-1}, \quad \mathcal{J} = \mathcal{J}_\sigma T(\sigma). \tag{5.15}$$

Thus both $\mathcal{G}$ and $\mathcal{J}$ lie on the same orbit characterized by $\hat{K}$. It follows that the Casimir invariants constructed out of $\mathcal{G}$'s and $\mathcal{J}$'s are identical.

C. Quantum mechanics

We have now two sets of commuting generators $\{\mathcal{G}_\sigma\}$ and $\{\mathcal{J}_\sigma\}$ for the group $\mathcal{G}$. The quantum-mechanical Hilbert space $\mathcal{H}$ thus carries a representation of $\mathcal{G}\otimes\mathcal{G}$. Since the Casimir invariants for the two sets are fixed and equal, $\mathcal{H}$ carries the same IRR's of either $\mathcal{G}$. It follows that the multiplicity of either IRR is equal to its own dimension.

The $\mathcal{J}$ spin is absolutely conserved [cf. (5.4)]. The Yang-Mills field carries no $\mathcal{J}$ spin. Thus, external Yang-Mills fields can produce these particles only in $\mathcal{J}$ singlet states.

VI. LAGRANGIAN FORMALISM FOR CASE 4

The Euler-Lagrange equations

Here, in the decomposition (2.10), we regard the $\hat{K}_a$ as dynamical variables while h is held fixed. In the notation (5.1), the Lagrangian becomes

$$L = -m[-\dot{z}(\tau)^2]^{1/2} - (i/R)\,\mathrm{Tr}\hat{K}G^{-1}D_\tau G. \tag{6.1}$$

Variation of ξ_ρ. In analogy to (4.3), we obtain

$$G\dot{\hat{K}}G^{-1} + [\dot{G}G - ieA, I] = 0, \quad I \equiv G\hat{K}G^{-1}. \tag{6.2}$$

Taking the trace of (6.2) with $GH(a)G^{-1}$ and using the identities

$$\begin{aligned} &\mathrm{Tr}A[B,C] = \mathrm{Tr}B[C,A], \\ &[I, GH(a)G^{-1}] = 0, \end{aligned} \tag{6.3}$$

we find

$$\dot{\hat{K}}_a = 0, \tag{6.4}$$

Thus, (6.2) implies

$$[\dot{G}G^{-1} - ieA, I] = 0 \tag{6.5}$$

which is the same as (3.10).

Variation of $\hat{K}_a$. Here we find

$$\mathrm{Tr}H(a)G^{-1}D_\tau G = 0. \tag{6.6}$$

Variation of z_μ. As usual, (3.12) is valid.

Implications of (6.5) and (6.6). Equation (6.5) says that

$$x \equiv G^{-1}D_\tau G \in \gamma_{\hat{K}}, \tag{6.7}$$

where $\gamma_{\hat{K}}$ is the Lie algebra of the little group of $\hat{K}$. Equation (6.6) says that x has no nonzero component in the Cartan subalgebra $\mathcal{C}$.

There are two cases to be considered:

4a. Generic case

Here $\gamma_{\hat{K}}$ is $\mathcal{C}$ itself. This is the usual case. For example, when $\mathcal{G} = \mathrm{SU}(3)$ and $\mathcal{C}$ is spanned by I_3 and Y, most elements of $\mathcal{C}$ have their little groups generated by I_3 and Y.

In the generic case,

$$x = 0 \quad \text{and} \quad D_\tau G = 0. \tag{6.8}$$

Thus there is no arbitrariness in the time evolution of G.

4b. Nongeneric case

Here $\gamma_{\hat{K}}$ is larger than $\mathcal{C}$. When $\mathcal{G} = \mathrm{SU}(3)$, one example of such a $\hat{K}$ is Y. The little group of Y is U(2). The generators of this U(2) are $\vec{I}$ and Y.

In the nongeneric case, we can choose x to be any arbitrary time-dependent function subject to the preceding restrictions. Thus the time evolution of G is not completely determined:

$$D_\tau G = Gx. \tag{6.9}$$

Let Ω be the group generated by all the allowed x's. Then only those functions $f(g)$ which fulfill

$$f(G) = f(G\omega) \tag{6.10}$$

for any $\omega \in \Omega$ have a unique time evolution. Only such functions are of physical interest. In the example above, $\Omega = \mathrm{SU}(2)$.

VII. THE HAMILTONIAN FORMALISM AND QUANTUM MECHANICS FOR CASE 4a

Let $\hat{\chi}_a$ denote the momenta conjugate to $\hat{K}_a$. The primary constraints are ϕ_ρ and $\hat{\chi}_a$. They vanish weakly. The Hamiltonian is

$$H = \frac{1}{2m}(p^i - eA^i_\alpha \mathcal{G}_\alpha)^2 + eA^0_\alpha \mathcal{G}_\alpha + v_\rho \phi_\rho + w_a \hat{\chi}_a. \tag{7.1}$$

In addition to the PB's (3.18)–(3.21), we also have

$$\{\mathcal{G}_\sigma, \hat{\chi}_a\} = 0, \quad \{\phi_\sigma, \hat{\chi}_a\} = -(\mathrm{ad}G)_{\sigma a}, \quad \{\hat{\chi}_a, \hat{\chi}_b\} = 0. \tag{7.2}$$

The requirements $\{\phi_\sigma, H\} = \{\hat{\chi}_a, H\} = 0$ lead to

$$GWG^{-1} = -i[V, I], \tag{7.3}$$

$$\mathrm{Tr}H(a)G^{-1}VG = 0, \tag{7.4}$$

where

$$V=v_\rho T(\rho), \quad W=w_a H(a). \tag{7.5}$$

Here, we have used the constraint $\mathcal{G}_\rho = I_\rho$

We first simplify (7.3). Since

$$\mathrm{Tr}\,GH(a)G^{-1}[V,I]=\mathrm{Tr}\,V[I,GH(a)G^{-1}]=0\,,$$

(7.3) gives $\mathrm{Tr}[H(a)W]=0$. Using (7.5), it follows that

$$W=0, \quad [V,I]=0. \tag{7.6}$$

Thus $G^{-1}VG \in \gamma_{\hat{K}}$. In the generic case, $\gamma_{\hat{K}}$ is spanned by $\{H(a)\}$. Thus from (7.4),

$$V=0. \tag{7.7}$$

There are no secondary constraints.

The preceding analysis also shows that all constraints in the generic case are second class.

The absence of Lagrange-multiplier terms in the Hamiltonian means that the time evolution of all the variables is unambiguous. They are all thus physically meaningful. A complete set of variables describing the constrained phase space (fulfilling $\phi_\rho=\hat{\chi}_a=0$) can be chosen to be $G_{\lambda\nu}$ and $\hat{K}_a$.

Next we shall compute the DB's of this complete set of variables. For this purpose, it is convenient to choose the basis $\{T(\rho)\}$ in the following way. Until now we have labeled the basis $\{T(\rho)\}$ purely by integers. However, it will be more convenient for us here to use a labeling composed partially of integers and partially of roots. Let $H(a)$ ($a=1,\ldots,k$) and $E(\alpha)$ denote the basis in the Cartan canonical form. Here α is a root:

$$[H(a),E(\alpha)]=\alpha_a E(\alpha). \tag{7.8}$$

We choose $E(\alpha)$'s and $H(\alpha)$'s so that they fulfill

$$E(\alpha)^\dagger = E(-\alpha), \tag{7.9}$$

$$\mathrm{Tr}\{E(\alpha)E(\beta)\}=2R\delta_{\alpha,-\beta}, \tag{7.10}$$

$$\mathrm{Tr}\{H(a)H(b)\}=R\delta_{ab}. \tag{7.11}$$

Of course,

$$\mathrm{Tr}\{H(a)E(\alpha)\}=0. \tag{7.12}$$

In terms of $H(a)$ and $E(\alpha)$, all the $T(\rho)$'s are given by

$$T(a)=H(a), \quad a=1,\ldots,k$$

$$T(\alpha)=\frac{E(\alpha)+E(-\alpha)}{2}, \tag{7.13}$$

$$T(-\alpha)=\frac{E(\alpha)-E(-\alpha)}{2i}. \tag{7.14}$$

In (7.13) and (7.14), α is a positive root, that is, its first nonvanishing component is positive. Note that $\mathrm{Tr}[T(\rho)T(\sigma)]=R\delta_{\rho\sigma}$ and that

$$[H(a),T(\alpha)]=i\alpha_a T(-\alpha) \tag{7.15}$$

for any root α.

In the notation of Sec. IV, the following can be proved:

$$\Delta^{-1}{}_{\phi_\rho\phi_\sigma}=-\sum_\alpha (\mathrm{ad}G)_{\rho\alpha}\frac{1}{\hat{K}\cdot\alpha}(\mathrm{ad}G)_{\sigma,-\alpha}\,, \tag{7.16}$$

$$\Delta^{-1}{}_{\phi_\rho\hat{\chi}_a}=-\Delta^{-1}{}_{\hat{\chi}_a\phi_\rho}=\mathrm{ad}G_{\rho a}, \tag{7.17}$$

$$\Delta^{-1}{}_{\hat{\chi}_a\hat{\chi}_b}=0. \tag{7.18}$$

Here, $\hat{K}\cdot\alpha$ is $\hat{K}_a\alpha_a$ and the summation in (7.16) is over all the nonzero roots α. In Appendix B, we prove (7.16). The proofs of (7.17) and (7.18) proceed in a similar fashion.

Using (7.16)–(7.18), we find

$$\{G_{\lambda\nu},G_{\lambda'\nu'}\}^*=-\{G_{\lambda\nu},\phi_\rho\}\Delta^{-1}{}_{\phi_\rho\phi_\sigma}\{\phi_\sigma,G_{\lambda'\nu'}\}$$
$$=\sum_\alpha [GT(\alpha)]_{\lambda\nu}\frac{1}{\hat{K}\cdot\alpha}[GT(-\alpha)]_{\lambda'\nu'}\,, \tag{7.19}$$

$$\{\hat{K}_a,G_{\lambda\nu}\}^*=-\{\hat{K}_a,\hat{\chi}_b\}\Delta^{-1}{}_{\hat{\chi}_b\phi_\rho}\{\phi_\rho,G_{\lambda\nu}\}$$
$$=-i[GT(a)]_{\lambda\nu}, \tag{7.20}$$

$$\{\hat{K}_a,\hat{K}_b\}^*=-\{\hat{K}_a,\hat{\chi}_c\}\Delta^{-1}{}_{\hat{\chi}_c\hat{\chi}_d}\{\hat{\chi}_d,\hat{K}_b\}$$
$$=0. \tag{7.21}$$

We shall now briefly examine these equations when the group is SU(2) and the representation Γ is the defining one. Then we can write

$$G=\begin{pmatrix}\hat{\theta}_1 & -\hat{\theta}_2^* \\ \hat{\theta}_2 & \hat{\theta}_1^*\end{pmatrix}, \quad \hat{\theta}_a^*\hat{\theta}_a=1\,. \tag{7.22}$$

Then $\hat{K}_a$ has only one component; call it $\hat{K}_1$. We can choose

$$\theta_a=|\hat{K}_1|^{1/2}\hat{\theta}_a,\ \theta_a^* \tag{7.23}$$

as our independent variables. The variables θ_a are unconstrained.

We choose our basis as follows:

$$T(1)=H(1)=\tfrac{1}{2}\sigma_3,$$
$$T(\alpha)=\tfrac{1}{2}\sigma_1, \quad \text{for } \alpha=(+1), \tag{7.24}$$
$$T(\alpha)=\tfrac{1}{2}\sigma_2, \quad \text{for } \alpha=(-1),$$

where σ_i are the Pauli matrices and (+1) and (−1) are the roots.

It can be shown (by examining individual components, for instance) that

$$\{|\hat{K}_1|^{1/2}G_{\lambda\nu},|\hat{K}_1|^{1/2}G_{\lambda'\nu'}\}^*$$
$$=-\frac{\hat{K}_1}{|\hat{K}_1|}\tfrac{1}{2}(\sigma_2)_{\lambda\lambda'}(\sigma_1)_{\nu\nu'}\,. \tag{7.25}$$

From (7.22) and (7.23), it follows that the only nonvanishing DB's are $\{\theta_1, \theta_1^*\}^* = (\hat{K}_1/|\hat{K}_1|)\frac{1}{2}i$ and $\{\theta_2, \theta_2^*\}^* = (\hat{K}_1/|\hat{K}_1|)\frac{1}{2}i$. Since they are both proportional to constants they can be interpreted as coordinates of harmonic oscillators. This system and its quantization has been discussed elsewhere.[2] See also Sec. VIII.

The DB (7.19) is not in a form suitable for quantization. In particular, it suffers from factor ordering problems. Also the occurrence of $\hat{K}\cdot\alpha$ in the denominator may cause problems. Some simplification of (7.19) is possible when Γ is the defining representation of SU(N) (for any N). However, except for the SU(2) example, we have not succeeded in finding a choice of variables suitable for quantization.

VIII. HAMILTONIAN FORMALISM AND QUANTUM MECHANICS FOR CASE 4b

Here, $\hat{K}$ has a given little group $\Gamma_{\hat{K}}$. Its Lie algebra $\gamma_{\hat{K}}$ is larger than $\mathcal{C}$. All the equations from (7.1) to (7.6) are still valid. Thus $G^{-1}VG \in \gamma_{\hat{K}}$. By (7.4) the components of $G^{-1}VG$ in $\mathcal{C}$ are zero. There are no secondary constraints.

The Lagrange multiplier terms in H are not fully determined. Certain linear combinations $\alpha_{\gamma\sigma}\phi_\sigma$ of the constraints ϕ_σ appear with arbitrary coefficients in H. Thus physical variables with well defined time evolution should be invariant under the group Ω generated by $\alpha_{\gamma\sigma}\phi_\sigma$. [See also the discussion which follows Eq. (6.9).] The first-class constraints we find below form a basis for the Lie algebra $\gamma(\Omega)$ of Ω.

We choose a basis $W(\gamma)$ for $\gamma(\Omega)$ with $W(s)$, $s = 1, \ldots, l < k$ being in $\mathcal{C}$. They are normalized in the usual way: $\mathrm{Tr}[W(\gamma)W(\gamma')] = R\delta_{\gamma\gamma'}$. For those $W(\gamma)$ which are not in $\mathcal{C}$, it is obvious that

$$\mathrm{Tr}[W(\gamma)\hat{K}] = 0, \quad W(\gamma) \notin \mathcal{C}. \tag{8.1}$$

On the other hand, $\gamma(\Omega)$ is generated by taking a sufficient number of commutators like $[W(\gamma), W(\gamma')]$, $[[W(\gamma), W(\gamma')], [W(\gamma''), W(\gamma''')]]$, etc., of such $W(\gamma)$'s and then taking their linear combinations. Thus, $W(s)$ $(s<k)$ is a superposition of terms of the form $[A,B]$. Here, A and B commute with $\hat{K}$ since the $W(\gamma)$'s not in $\mathcal{C}$ do so. Therefore,

$$\mathrm{Tr}([A,B]\hat{K}) = \mathrm{Tr}([B,\hat{K}]A) = 0.$$

Thus

$$\mathrm{Tr}\,W(s)\hat{K} = 0, \quad s = 1, 2, \ldots, l. \tag{8.2}$$

These are the new constraints we are imposing on the phase space in order to have a nongeneric $\hat{K}$ with a given little group $\Gamma_{\hat{K}}$.

It is straightforward to verify that the first-class constraints are

$$\psi_\gamma = (1/R)\,\mathrm{Tr}\,W(\gamma)(G^{-1}\Phi G + \hat{K}),$$
$$W(\gamma) \in \gamma(\Omega), \quad \Phi \equiv \phi_\rho T(\rho). \tag{8.3}$$

In the Hamiltonian, only those ψ_γ's with $W(\gamma) \notin \mathcal{C}$ appear. Such ψ's are linear combinations of the ϕ's alone due to (8.1).

We can choose the second-class constraints to be

$$\Sigma_s = (1/R)\,\mathrm{Tr}\,W(s)\hat{K}, \quad s = 1, 2, \ldots, l,$$
$$\Theta_\lambda = (1/R)\,\mathrm{Tr}\,W^\perp(\lambda)G^{-1}\Phi G,$$

and (8.4)

$$\hat{\chi}_a, \quad a = 1, 2, \ldots, k.$$

Here the $W^\perp(\lambda)$'s span the orthogonal complement of $\gamma(\Omega)$ in the full Lie algebra. They are normalized as usual. Thus,

$$\mathrm{Tr}\,W^\perp(\lambda)W(\gamma) = 0,$$
$$\mathrm{Tr}\,W^\perp(\lambda)W^\perp(\lambda') = R\delta_{\lambda\lambda'}. \tag{8.5}$$

The variables with well-defined time evolution must have weakly zero PB's with ψ_γ. Using all the constraints, we see that a complete set of such variables is given by (a) z_i and p^i, (b) the nonvanishing components $\hat{K}_a$ of $\hat{K}$, and (c) those functions $f(G)$ which fulfill (6.10). The latter are simply functions on the left cosets G/Ω. Except in some very degenerate situations to be discussed below, we have not found a convenient parametrization for G/Ω. Note that $I = G\hat{K}G^{-1}$ defines a partial parametrization of G/Ω. However, it is invariant under $G \to Gh$ where h is generated by *any* element of $\mathcal{C}$. Thus it does not fully parametrize G/Ω.

The DB's involving $G_{\lambda\nu}$ and $\hat{K}_a$ are simple modifications of those found in the generic case. The derivations are straightforward and we will only state the results. Let α denote the roots perpendicular to $\hat{K}$ on the constrained hypersurface:

$$\alpha_a \hat{K}_a = 0. \tag{8.6}$$

Then $T(\pm\alpha) \in \gamma(\Omega)$. Let β denote the remaining roots:

$$\beta_a \hat{K}_a \neq 0. \tag{8.7}$$

The DB's are given by[5]

$$\{G_{\lambda\nu}, G_{\lambda'\nu'}\}^* = \sum_\beta [GT(\beta)]_{\lambda\nu} \frac{1}{\hat{K}\cdot\beta} [GT(-\beta)]_{\lambda'\nu'}, \tag{8.8}$$

$$\{\hat{K}_a, G_{\lambda\nu}\}^* = -i[GT(a)]_{\lambda\nu}, \tag{8.9}$$

$$\{\hat{K}_a, \hat{K}_b\}^* = 0. \tag{8.10}$$

Any other relevant DB on the constrained surface can be extracted from these equations.

We see that (8.8) is not in a form suitable for quantization. We shall now briefly examine the content of these equations in some degenerate situations. Let $\{G\}$ be the defining representation of SU(N) and let $\hat{K}$ be an element with the maximal little group. Such a $\hat{K}$ can be chosen to be a diagonal matrix with the first $N-1$ entries equal. Since $\mathrm{Tr}\hat{K}$ is zero, there is only one independent component among the $\hat{K}_a$'s, call it $\hat{K}_1$. The little group of $\hat{K}$ is U($N-1$) and the group Ω is SU($N-1$) which acts nontrivially on the first $N-1$ entries of vectors. Under the action $G \to Gh$, $(G_{1N}, G_{2N}, \ldots, G_{NN})$ has the little group SU($N-1$). Thus the last column of G parametrizes G/Ω. [Note that $\sum_a G^*_{aN} G_{aN} = 1$. This is similar to (7.22).] Further, we can now take $\theta_a = |\hat{K}_1/R|^{1/2} G_{aN}$ and its complex conjugate as a complete set of physical variables. Their DB's are like those of N independent harmonic oscillators as in Sec. VII. This system is identical to one of the cases discussed in previous work[2] where quantization has also been carried out. There is a transparent way to show the relationship between our Lagrangian and that of previous work. We shall describe this in the next section.

IX. DERIVATION OF PREVIOUS LAGRANGIANS

In this section we consider the Lagrangian (6.1) when $\{G\}$ is the defining representation of SU(N) and $\hat{K}$ is most degenerate. Then $\hat{K}$ can be written as

$$\hat{K} = -\frac{\hat{K}_1}{N} \underline{1} + (0, 0, \ldots, \hat{K}_1)_{\text{diagonal}}, \tag{9.1}$$

where $\underline{1}$ is the unit matrix. Now $G^{-1} D_\tau G$ is in the Lie algebra and so its trace is zero. Further $G^{-1} = G^\dagger$. Thus (6.1) reduces to

$$L = -m[-\dot{z}(\tau)^2]^{1/2} - (i/R)\hat{K}_1 G^*_{aN}(D_\tau G)_{aN}. \tag{9.2}$$

Upon introducing $\theta_a = |\hat{K}_1/R|^{1/2} G_{aN}$ and discarding the total time derivative $(i/2R)(G^*_{aN} G_{aN})\partial_\tau \hat{K}_1 = (i/2R)\partial_\tau \hat{K}_1$ we find an interaction Lagrangian $-i\theta^*_a (D_\tau \theta)_a$ of previous work[2] with θ_a's transforming under the defining representation of SU(N). There, Lagrangians were also considered with anticommuting θ_a's. We can obtain these as well by first writing (6.1) in terms of G and G^* and then treating the components of G as anticommuting.

In Ref. 2, interactions of the form $-i\theta^*_a(D_\tau\theta)_a$, where θ transformed under an arbitrary representation of G, were also considered. It is not always true that these are also special cases of (6.1).

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APPENDIX A.

Here we prove (3.18)–(3.21). Equation (3.21) is simply a consequence of (3.2) and canonical commutation relations. Equation (3.20) can be obtained by noting that

$$\begin{aligned}\{\{\mathcal{G}_\rho, \mathcal{G}_\sigma\}, g(\xi)\} &= -\{\{\mathcal{G}_\sigma, g(\xi)\}, \mathcal{G}_\rho\} \\ &\quad -\{\{g(\xi), \mathcal{G}_\rho\}, \mathcal{G}_\sigma\} \text{ by Jacobi identity} \\ &= -ic_{\sigma\rho\lambda} T(\lambda) g(\xi) \text{ by (3.21).}\end{aligned} \tag{A1}$$

Thus

$$\{\mathcal{G}_\rho, \mathcal{G}_\sigma\} = -c_{\rho\sigma\lambda}\mathcal{G}_\lambda + F, \tag{A2}$$

where $\{F, g(\xi)\} = 0$. Consequently F is independent of the π's. However, from the definition (3.15), both $\mathcal{G}_\lambda$ and $\{\mathcal{G}_\rho, \mathcal{G}_\sigma\}$ are seen to be linear and homogeneous in the π's. Substituting π's $=0$ in (A2), we find $F=0$. This proves (3.20).

To derive (3.18) and (3.19), first note that

$$\{\mathcal{G}_\rho, \mathrm{Tr}T(\sigma) gKg^{-1}\} = -c_{\rho\sigma\lambda} \mathrm{Tr}T(\lambda) gKg^{-1}. \tag{A3}$$

This follows from (3.21) and the trace identity in (6.3). Then, straightforward algebra gives (3.18) and (3.19).

APPENDIX B.

In the following, we prove (7.16), that is, (7.16) fulfills

$$\Delta^{-1}{}_{\phi_\rho \phi_\lambda} \Delta_{\phi_\lambda \phi_\sigma} + \Delta^{-1}{}_{\phi_\rho \hat{\chi}_a} \Delta_{\hat{\chi}_a \phi_\sigma} = \delta_{\rho\sigma} \tag{B1}$$

and

$$\Delta^{-1}{}_{\phi_\rho \phi_\lambda} \Delta_{\phi_\lambda \hat{\chi}_a} + \Delta^{-1}{}_{\phi_\rho \hat{\chi}_b} \Delta_{\hat{\chi}_b \hat{\chi}_a} = 0\,. \tag{B2}$$

First we summarize some elementary results:

(1) ad$G$ is real and orthogonal in the basis $T(\rho)$, while ad$\hat{K}$ is antisymmetric.

(2)

$$\begin{aligned}c_{\lambda\nu\sigma} I_\sigma &= i I_\sigma(\mathrm{ad}T(\sigma))_{\lambda\nu} \equiv i(\mathrm{ad}I)_{\lambda\nu} \\ &= i[\mathrm{ad}(G\hat{K}G^{-1})]_{\lambda\nu}.\end{aligned} \tag{B3}$$

(3) From $[\hat{K}, T(\alpha)] = i\hat{K}\cdot\alpha T(-\alpha) = (\mathrm{ad}\hat{K})_{\rho\alpha} T(\rho)$, we find

$$(\mathrm{ad}\hat{K})_{\rho\alpha} = i\hat{K}\cdot\alpha\,\delta_{\rho,-\alpha}. \tag{B4}$$

Using (3.19), (7.2), and the constraint (3.16), the left-hand side of (B1) can be written as

$$-\sum_\alpha (\mathrm{ad}G)_{\rho\alpha} \frac{1}{\hat{K}\cdot\alpha} (\mathrm{ad}G)_{\lambda,-\alpha} c_{\lambda\nu\sigma} I_\nu + \sum_{a=1}^{k} (\mathrm{ad}G)_{\rho a}(\mathrm{ad}G^{-1})_{a\sigma}. \tag{B5}$$

Now substitute (2) and use (1) and (3) to find

$$\sum_{\alpha} (\mathrm{ad}G)_{\rho\alpha}(\mathrm{ad}G^{-1})_{\alpha\sigma} + \sum_{a=1}^{k} (\mathrm{ad}G)_{\rho a}(\mathrm{ad}G^{-1})_{a\sigma} \qquad \text{(B6)}$$

which is, of course, $\delta_{\rho\sigma}$.

Similarly the left-hand side of (B2) is

$$-\sum_{\alpha} (\mathrm{ad}G)_{\rho\alpha} \frac{1}{\hat{K}\cdot\alpha} (\mathrm{ad}G)_{\lambda,-\alpha}(\mathrm{ad}G)_{\lambda a}, \qquad \text{(B7)}$$

where λ is summed over all values. So,

$$(\mathrm{ad}G)_{\lambda,-\alpha}(\mathrm{ad}G)_{\lambda a} = \delta_{-\alpha,a}. \qquad \text{(B8)}$$

Thus since α is summed over only the nonzero roots, (B7) vanishes. This proves (B2).

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[4]For exhaustive discussions of constrained Hamiltonian systems, see P. A. M. Dirac, *Lectures on Quantum Mechanics*, Belfer Graduate School of Science Monographs Series No. 2 (Yeshiva University, New York, 1964); E. C. G. Sudarshan and N. Mukunda, *Classical Dynamics: A Modern Perspective* (Wiley, New York, 1974); A. Hanson, T. Regge, and C. Teitelboim, *Constrained Hamiltonian Systems* (Accademia Nazionale dei Lincei, Rome, 1976).

[5]Choose $W^{\perp}(\lambda)$ such that $W^{\perp}(p)$ $(p = l+1, l+2, \ldots, k) \in \mathfrak{C}$. $\mathfrak{C}$ is then spanned by $W(s)$, $W^{\perp}(p)$. The remaining $W^{\perp}(\lambda)$ can be labeled by the roots β relative to this basis for $\mathfrak{C}$, as in (7.13) and (7.14) [see also (8.7)]. Further, the components of $\hat{K}$ in this basis fulfill $\hat{K}_a = 0$, $a \leq l$. Call the momenta conjugate to $\hat{K}$ in this basis $\hat{\chi}_a$ $(a \leq l)$, $\hat{\chi}_p$ $(p \geq l+1)$. Then, to compute (8.8)–(8.10), we need $\Delta^{-1}{}_{\hat{\chi}_p\hat{\chi}_{p'}}$, $\Delta^{-1}{}_{\theta_\lambda\hat{\chi}_p}$, and $\Delta^{-1}{}_{\theta_\lambda\theta_{\lambda'}}$ (for $p, p' \geq l+1$ and all allowed λ). It may be shown that $\Delta^{-1}{}_{\hat{\chi}_p\hat{\chi}_{p'}} = 0$, $\Delta^{-1}{}_{\theta_p\hat{\chi}_{p'}} = \delta_{pp'}$, $\Delta^{-1}{}_{\theta_\beta\hat{\chi}_p} = 0$, $\Delta^{-1}{}_{\theta_\lambda\theta_p} = 0$, $\Delta^{-1}{}_{\theta_\beta\theta_{\beta'}} = (1/\hat{K}\cdot\beta)\delta_{\beta,-\beta'}$. Here the θ's are evaluated in the basis above, $p, p' = l+1, l+2, \ldots, k$; β, β' are roots which fulfill (8.7) and λ can be either like a p or a β.