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Yang's R gauge for self-dual SU(3) gauge fields

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Yang's R-gauge formulation of self-dual SU(2) gauge fields is extended to the SU(N) gauge group. A simple and computationally useful expression for the Chern density is exhibited. For $N = 3$ the self-duality equations are given explicitly along with a Backlund-type transformation for generating "new" self-dual gauge fields from old ones. No attempt is made to resolve the severe singularity structure of these "new" gauge fields which preclude any physical interpretation.

Yang[1] has shown how, with a suitable choice of gauge (the R gauge), the self-dual SU(2) gauge field equations reduce to Laplace-type equations for one real variable (ϕ) and one complex variable (ρ). It has been realized[2] recently that the R gauge is especially suited for the construction of Backlund-type transformations (B) that generate "new" self-dual gauge fields from old ones. It can be further shown[3] that the Pontryagin density (*S) for these "new" self-dual gauge fields exhibits a striking "superposition" principle. Unfortunately, if one requires the gauge potentials to be real, *S for these "new" gauge fields is infested with severe singularities (not gauge artifacts) which preclude any physical interpretation. The purpose of this paper is to formally extend these observations to the SU(3) gauge group, which is generally believed to be the relevant gauge group for the description of strong interactions. As in Ref. 1, all considerations are local in character and do not refer to global properties. Arguments that hold generally for the SU(N) gauge group are notably emphasized. Explicit equations for the particular case of $N=3$, whenever cumbersome, are relegated to the appendixes.

Yang's formulation of self-dual SU(2) gauge fields is trivially generalized to the SU(N) gauge group by replacing the relevant 2×2 matrices with $N\times N$ matrices. The essential feature of the R gauge is that Yang's generating matrix D takes on a triangular form. It is a basic result of linear algebra[4] (Schmidt's orthogonalization process) that any nonsingular $N\times N$ matrix D can be factored in the form $D = TU$, where U is unitary and T is a nonsingular triangular matrix. Since U is unitary, one can always choose a gauge so that D is T, which we define to be the R gauge.

For $N>2$ we refer to the action density of self-dual SU(N) fields as the Chern density, and in the notation of Refs. 1 and 3 it is

$$S = {}^*S = -2\,\mathrm{Tr}(F_{y\bar z}F_{\bar y z} + F_{y\bar y}F_{z\bar z}). \tag{1}$$

We now simplify (1) by using the self-duality equation of Ref. 1 (i.e., $F_{yz} = F_{\bar y\bar z} = F_{y\bar y} + F_{z\bar z} = 0$). A slightly tedious amount of algebra gives one

$$\begin{aligned} S = {}^*S = -2\,\mathrm{Tr}\,[&\partial_y\partial_{\bar y}(A_z A_{\bar z}) + \partial_z\partial_{\bar z}(A_y A_{\bar y}) \\ &- \partial_y\partial_{\bar z}(A_{\bar y}A_z) - \partial_z\partial_{\bar y}(A_{\bar z}A_y) \\ &+ (\partial_y A_{\bar y})(\partial_z A_{\bar z}) + (\partial_{\bar y}A_y)(\partial_{\bar z}A_z) \\ &- (\partial_y A_{\bar z})(\partial_z A_{\bar y}) - (\partial_{\bar z}A_y)(\partial_{\bar y}A_z)], \end{aligned} \tag{2}$$

where $A_\mu = D^{-1}D_\mu$, $A_{\bar\mu} = \bar D^{-1}\bar D_{\bar\mu}$ ($\mu = y, z$) are the gauge potentials, and for real gauge fields $\bar D \doteq (D^+)^{-1}$.[5] Equation (2) simplifies considerably in the R gauge since then A_μ becomes triangular, giving (in the R gauge only)

$$\begin{aligned} S = {}^*S = -2\,\mathrm{Tr}[&\partial_y\partial_{\bar y}(A_z A_{\bar z} + d_z\bar d_{\bar z}) + \partial_z\partial_{\bar z}(A_y A_{\bar y} + d_y \bar d_{\bar y}) \\ &- \partial_y\partial_{\bar z}(A_{\bar y}A_z + \bar d_{\bar y} d_z) - \partial_z\partial_{\bar y}(A_{\bar z}A_y + \bar d_{\bar z} d_y)], \end{aligned} \tag{3}$$

where

$$d_\mu \equiv \text{diagonal part of } A_\mu,$$

$$\bar d_{\bar\mu} \equiv \text{diagonal part of } A_{\bar\mu},$$

and $\bar d_{\bar\mu} \doteq -(d_\mu)^*$ for real gauge fields. For the case of $N=2$, Eq. (3) becomes that given in Ref. 3. So far our considerations have been completely general and we now focus attention on the $N=3$ case.

Yang's R gauge for self-dual SU(3) gauge fields is best parametrized as

$$R=\begin{bmatrix}\dfrac{1}{\sqrt{\phi_1}} & 0 & 0\\ \dfrac{\rho_1}{\sqrt{\phi_1}} & \left(\dfrac{\phi_1}{\phi_2}\right)^{1/2} & 0\\ \dfrac{\rho_3}{\sqrt{\phi_1}} & \rho_2\left(\dfrac{\phi_1}{\phi_2}\right)^{1/2} & \sqrt{\phi_2}\end{bmatrix},\quad \overline{R}=\begin{bmatrix}\sqrt{\phi_1} & -\overline{\rho}_1\left(\dfrac{\phi_2}{\phi_1}\right)^{1/2} & \dfrac{\overline{\rho}_1\overline{\rho}_2-\overline{\rho}_3}{\sqrt{\phi_2}}\\ 0 & \left(\dfrac{\phi_2}{\phi_1}\right)^{1/2} & -\dfrac{\overline{\rho}_2}{\sqrt{\phi_2}}\\ 0 & 0 & \dfrac{1}{\sqrt{\phi_2}}\end{bmatrix}, \tag{4}$$

where

$$\phi_1 \doteq \text{real},\quad \phi_2 \doteq \text{real},\qquad \overline{\rho}_1 \doteq \rho_1^*,\quad \overline{\rho}_2 \doteq \rho_2^*,\quad \overline{\rho}_3 \doteq \rho_3^* \tag{5}$$

in order for the gauge field to be real. The ensuing gauge potentials $A_\mu = R^{-1}R_\mu$ ($A_{\overline{\mu}} = \overline{R}^{-1}\overline{R}_{\overline{\mu}}$), their diagonal parts d_μ ($\overline{d}_{\overline{\mu}}$), and the self-duality equations $F_{y\overline{y}} + F_{z\overline{z}} = 0$, which are eight in number, can be found explicitly in Appendix A.

Equations (A4) to (A12) are covariant under a number of algebraic transformations, of which the most important and nonobvious is a discrete inversion I that is explicitly constructed in Appendix B. As Appendix B suggests, all such transformations can be shown to be gauge transformations and, therefore, by themselves are of no use in generating new self-dual gauge fields from old ones.

A far more important covariance of the self-duality equations is a Backlund-type transformation B, which states that if (ϕ_1, ϕ_2, ρ_1, $\rho_2, \ldots, \overline{\rho}_3$) satisfy Eqs. (A5) to (A12), then so do (ϕ_1^B, ϕ_2^B, $\ldots, \overline{\rho}_3^B$), where

$$\begin{gathered}
\phi_1^B = \frac{\phi_1}{\phi_2},\quad \phi_2^B = \frac{1}{\phi_2},\quad \rho_1^B = -\rho_1,\quad \overline{\rho}_1^B = -\overline{\rho}_1\,,\\
\rho_{2y}^B = +\frac{\overline{\rho}_{3\overline{z}} - \overline{\rho}_2\overline{\rho}_{1\overline{z}}}{\phi_1\phi_2},\quad \rho_{2z}^B = -\frac{\overline{\rho}_{3\overline{y}} - \overline{\rho}_2\overline{\rho}_{1\overline{y}}}{\phi_1\phi_2},\\
\overline{\rho}_{2\overline{y}}^B = -\frac{\rho_{3z} - \rho_2\rho_{1z}}{\phi_1\phi_2},\quad \overline{\rho}_{2\overline{z}}^B = +\frac{\rho_{3y} - \rho_2\rho_{1y}}{\phi_1\phi_2},\\
(\rho_{3y}^B - \rho_2^B\rho_{1y}^B) = +\frac{\phi_1}{\phi_2{}^2}\overline{\rho}_{2\overline{z}},\quad (\rho_{3z}^B - \rho_2^B\rho_{1z}^B) = -\frac{\phi_1}{\phi_2{}^2}\overline{\rho}_{2\overline{y}},\\
(\overline{\rho}_{3\overline{y}} - \overline{\rho}_2^B\overline{\rho}_{1\overline{y}}^B) = -\frac{\phi_1}{\phi_2{}^2}\rho_{2z},\quad (\overline{\rho}_{3\overline{z}}^B - \overline{\rho}_2^B\overline{\rho}_{1\overline{z}}^B) = +\frac{\phi_1}{\phi_2{}^2}\rho_{2y}\,.
\end{gathered} \tag{6}$$

Equation (6) follows immediately from observing the nature of Eqs. (A7) and (A8), which can be viewed as integrability conditions. Note that B, unlike I, is not an algebraic but an integrable covariance. It is now crucial to observe that in general B does not take a real gauge field (ϕ_1, $\phi_2, \ldots, \overline{\rho}_3$), as defined by Eq. (5), to a real gauge field, i.e. (ϕ_1^B, $\phi_2^B, \ldots, \overline{\rho}_3^B$) do not in general satisfy Eq. (5). It is obvious that in general B cannot be a gauge transformation, a fact which makes it the key to finding new self-dual gauge fields from old ones. Furthermore, since operating with B twice on a gauge field gives back the original field (i.e., $B^2 = 1$) one must, to generate nontrivial solutions, interpose an I (or any combination of algebraic transformations) between two B's.

Two simple solutions to Eqs. (A5) to (A12) representing real gauge fields are

$$\begin{gathered}
\mathcal{L}_1\phi_1 = \phi,\quad \rho_1 = \rho,\quad \phi_2 = 1,\quad \rho_2 = \rho_3 = 0,\\
\mathcal{L}_2\phi_1 = \phi_2 = \phi,\quad \rho_3 = \rho,\quad \rho_1 = \rho_2 = 0,
\end{gathered}$$

and

$$\begin{gathered}
\rho_y = \phi_{\overline{z}},\quad \rho_z = -\phi_{\overline{y}},\quad \overline{\rho}_{\overline{y}} = \phi_z,\\
\overline{\rho}_{\overline{z}} = -\phi_y,\quad (\partial_y\partial_{\overline{y}} + \partial_z\partial_{\overline{z}})\phi = 0,
\end{gathered} \tag{7}$$

which can be thought of as two independent imbeddings of self-dual SU(2) gauge fields into the SU(3) gauge group. The analog of the construction of Ref. 2 would be to operate on $\mathcal{L}_1$ and $\mathcal{L}_2$ with BI an integer n number of times, where for even n the resulting gauge field manifestly satisfies the reality conditions (5). Unfortunately, if one requires the resulting gauge potentials to be real (for odd or even n), then by using Eq. (3) for $*S$ one finds severe singularities which are not gauge artifacts. To calculate $*S$, one needs the following convenient result:

$$\mathrm{Tr}(A_\mu A_{\overline{\nu}} + d_\mu \overline{d}_{\overline{\nu}}) = -\frac{\phi_{1\mu}\phi_{1\overline{\nu}} + \rho_{1\mu}\overline{\rho}_{1\overline{\nu}}\phi_2}{\phi_1{}^2} - \frac{\phi_{2\mu}\phi_{2\overline{\nu}} + \rho_{2\mu}\overline{\rho}_{2\overline{\nu}}\phi_1}{\phi_2{}^2} + \frac{\phi_{1\mu}\phi_{2\overline{\nu}} + \phi_{2\mu}\phi_{1\overline{\nu}} - 2(\rho_{3\mu} - \rho_2\rho_{1\mu})(\overline{\rho}_{3\overline{\nu}} - \overline{\rho}_2\overline{\rho}_{1\overline{\nu}})}{2\phi_1\phi_2}, \tag{8}$$

where $\mu, \nu = y, z$. Since I is a gauge transformation, it leaves $*S$ invariant. The transformation property of $*S$ under B is very simple, as one can easily check. For the particular solutions $\mathcal{L}_1$ and $\mathcal{L}_2$, the $*S$ derived from Eqs. (3)′ and (8) becomes that of Ref. 3. Until the singularity structure is resolved, it is not clear whether any of the "new" solutions are physically relevant.

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APPENDIX A

Using Eq. (4) we find for the gauge potential A_μ $(A_{\bar\mu})$ and their diagonal parts d_μ $(\bar d_{\bar\mu})$,

$$A_\mu = R^{-1}R_\mu = \begin{bmatrix} -\dfrac{\phi_{1\mu}}{2\phi_1} & 0 & 0 \\ \dfrac{\rho_{1\mu}\sqrt{\phi_2}}{\phi_1} & \dfrac{\phi_{1\mu}}{2\phi_1}-\dfrac{\phi_{2\mu}}{2\phi_2} & 0 \\ \dfrac{\rho_{3\mu}-\rho_2\rho_{1\mu}}{(\phi_1\phi_2)^{1/2}} & \dfrac{\sqrt{\phi_1}\,\rho_{2\mu}}{\phi_2} & \dfrac{\phi_{2\mu}}{2\phi_2} \end{bmatrix}, \tag{A1}$$

$$d_\mu = \begin{bmatrix} \dfrac{-\phi_{1\mu}}{2\phi_1} & 0 & 0 \\ 0 & \dfrac{\phi_{1\mu}}{2\phi_1}-\dfrac{\phi_{2\mu}}{2\phi_2} & 0 \\ 0 & 0 & \dfrac{\phi_{2\mu}}{2\phi_2} \end{bmatrix}, \tag{A2}$$

$$A_{\bar\mu} = \bar R^{-1}\bar R_{\bar\mu} = \begin{bmatrix} \dfrac{\phi_{1\bar\mu}}{2\phi_1} & -\dfrac{\bar\rho_{1\bar\mu}}{\phi_1}\sqrt{\phi_2} & \dfrac{\bar\rho_2\bar\rho_{1\bar\mu}-\bar\rho_{3\bar\mu}}{(\phi_1\phi_2)^{1/2}} \\ 0 & -\dfrac{\phi_{1\bar\mu}}{2\phi_1}+\dfrac{\phi_{2\bar\mu}}{2\phi_2} & -\dfrac{\sqrt{\phi_1}}{\phi_2}\bar\rho_{2\bar\mu} \\ 0 & 0 & -\dfrac{\phi_{2\bar\mu}}{2\phi_2} \end{bmatrix}, \tag{A3}$$

and

$$\bar d_{\bar\mu} = \begin{bmatrix} \dfrac{\phi_{1\bar\mu}}{2\phi_1} & 0 & 0 \\ 0 & \dfrac{-\phi_{1\bar\mu}}{2\phi_1}+\dfrac{\phi_{2\bar\mu}}{2\phi_2} & 0 \\ 0 & 0 & \dfrac{-\phi_{2\bar\mu}}{2\phi_2} \end{bmatrix}, \tag{A4}$$

where $\mu = y, z$. From (A1) to (A4) one can calculate the gauge field strength $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$, and the self-duality equations $F_{y\bar y} + F_{z\bar z} = 0$, eight in number, become two real equations:

$$(\partial_y\partial_{\bar y}+\partial_z\partial_{\bar z})\ln\phi_1 + \frac{\phi_2}{\phi_1^{\,2}}(\rho_{1y}\bar\rho_{1\bar y}+\rho_{1z}\bar\rho_{1\bar z}) + \frac{1}{\phi_1\phi_2}[(\rho_{3y}-\rho_2\rho_{1y})(\bar\rho_{3\bar y}-\bar\rho_2\bar\rho_{1\bar y}) + (\rho_{3z}-\rho_2\rho_{1z})(\bar\rho_{3\bar z}-\bar\rho_2\bar\rho_{1\bar z})] = 0, \tag{A5}$$

$$(\partial_y\partial_{\bar y}+\partial_z\partial_{\bar z})\ln\left(\frac{\phi_1}{\phi_2}\right) + \frac{\phi_2}{\phi_1^{\,2}}(\rho_{1y}\bar\rho_{1\bar y}+\rho_{1z}\bar\rho_{1\bar z}) - \frac{\phi_1}{\phi_2^{\,2}}(\rho_{2y}\bar\rho_{2\bar y}+\rho_{2z}\bar\rho_{2\bar z}) = 0, \tag{A6}$$

and six complex equations:

$$\left(\frac{\rho_{3y}-\rho_2\rho_{1y}}{\phi_1\phi_2}\right)_{\bar y} + \left(\frac{\rho_{3z}-\rho_2\rho_{1z}}{\phi_1\phi_2}\right)_{\bar z} = 0, \tag{A7}$$

$$\left(\frac{\bar\rho_{3\bar y}-\bar\rho_2\bar\rho_{1\bar y}}{\phi_1\phi_2}\right)_{y} + \left(\frac{\bar\rho_{3\bar z}-\bar\rho_2\bar\rho_{1\bar z}}{\phi_1\phi_2}\right)_{z} = 0, \tag{A8}$$

$$\left(\frac{\phi_2\rho_{1y}}{\phi_1^{\,2}}\right)_{\bar y} + \left(\frac{\phi_2\rho_{1z}}{\phi_1^{\,2}}\right)_{\bar z} - \frac{1}{\phi_1\phi_2}[\bar\rho_{2\bar y}(\rho_{3y}-\rho_2\rho_{1y}) + \bar\rho_{2\bar z}(\rho_{3z}-\rho_2\rho_{1z})] = 0, \tag{A9}$$

$$\left(\frac{\phi_2\bar\rho_{1\bar y}}{\phi_1^{\,2}}\right)_{y} + \left(\frac{\phi_2\bar\rho_{1\bar z}}{\phi_1^{\,2}}\right)_{z} - \frac{1}{\phi_1\phi_2}[\rho_{2y}(\bar\rho_{3\bar y}-\bar\rho_2\bar\rho_{1\bar y}) + \rho_{2z}(\bar\rho_{3\bar z}-\bar\rho_2\bar\rho_{1\bar z})] = 0, \tag{A10}$$

$$\left(\frac{\phi_1\rho_{2y}}{\phi_2^{\,2}}\right)_{\bar y} + \left(\frac{\phi_1\rho_{2z}}{\phi_2^{\,2}}\right)_{\bar z} + \frac{1}{\phi_1\phi_2}[\bar\rho_{1\bar y}(\rho_{3y}-\rho_2\rho_{1y}) + \bar\rho_{1\bar z}(\rho_{3z}-\rho_2\rho_{1z})] = 0, \tag{A11}$$

$$\left(\frac{\phi_1\bar\rho_{2\bar y}}{\phi_2^{\,2}}\right)_{y} + \left(\frac{\phi_1\bar\rho_{2\bar z}}{\phi_2^{\,2}}\right)_{z} + \frac{1}{\phi_1\phi_2}[\rho_{1y}(\bar\rho_{3\bar y}-\bar\rho_2\bar\rho_{1\bar y}) + \rho_{1z}(\bar\rho_{3\bar z}-\bar\rho_2\bar\rho_{1\bar z})] = 0, \tag{A12}$$

APPENDIX B

The R gauge as defined in Eq. (4) is lower triangular. One can equally well take for the R gauge an upper triangular matrix such as

$$R^I = \begin{bmatrix} (\phi_2^I)^{1/2} & \rho_2^I \left(\dfrac{\phi_1^I}{\phi_2^I}\right)^{1/2} & \dfrac{\rho_3^I}{(\phi_1^I)^{1/2}} \\ 0 & \left(\dfrac{\phi_1^I}{\phi_2^I}\right)^{1/2} & \dfrac{\rho_1^I}{(\phi_1^I)^{1/2}} \\ 0 & 0 & \dfrac{1}{(\phi_1^I)^{1/2}} \end{bmatrix} . \tag{B1}$$

One can verify that the self-duality equations $F_{y\bar{y}} + F_{z\bar{z}} = 0$, resulting from the choice R^I, are identical in form to those in Appendix A, except that everything is labeled by I. This suggests that we make the identification

$$RR^\dagger = R^I R^{I\dagger} , \tag{B.2}$$

which implies $(R^{-1}R^I)$ is unitary, so that we can always go from the R gauge to the R^I gauge. The identification (B2) implies that if $(\phi_1, \phi_2, \ldots, \bar{\rho}_3)$ are solutions to Eqs. (A5) to (A12), then so are $(\phi_1^I, \phi_2^I, \ldots, \bar{\rho}_3^I)$, where

$$\begin{aligned} \phi_1^I &= \frac{\phi_1\phi_2}{\phi_1\phi_2{}^2 + \rho_2\bar{\rho}_2\phi_1{}^2 + \rho_3\bar{\rho}_3\phi_2} , \\ \rho_3^I &= \frac{\bar{\rho}_3\phi_2}{\phi_1\phi_2{}^2 + \rho_2\bar{\rho}_2\phi_1{}^2 + \rho_3\bar{\rho}_3\phi_2} , \\ \bar{\rho}_3^I &= \frac{\rho_3\phi_2}{\phi_1\phi_2{}^2 + \rho_2\bar{\rho}_2\phi_1{}^2 + \rho_3\bar{\rho}_3\phi_2} , \\ \rho_1^I &= \frac{\rho_1\bar{\rho}_3\phi_2 + \bar{\rho}_2\phi_1{}^2}{\phi_1\phi_2{}^2 + \rho_2\bar{\rho}_2\phi_1{}^2 + \rho_3\bar{\rho}_3\phi_2} , \\ \bar{\rho}_1^I &= \frac{\rho_3\bar{\rho}_1\phi_2 + \rho_2\phi_1{}^2}{\phi_1\phi_2{}^2 + \rho_2\bar{\rho}_2\phi_1{}^2 + \rho_3\bar{\rho}_3\phi_2} , \\ \phi_2^I &= \frac{\phi_1\phi_2}{\rho_1\bar{\rho}_1(\phi_2{}^2 + \phi_1\rho_2\bar{\rho}_2) + \phi_1(\phi_1\phi_2 + \rho_3\bar{\rho}_3 - \rho_1\rho_2\bar{\rho}_3 - \bar{\rho}_1\bar{\rho}_2\rho_3)} , \\ \rho_2^I &= \frac{\bar{\rho}_1(\phi_2{}^2 + \rho_2\bar{\rho}_2\phi_1) - \rho_2\bar{\rho}_3\phi_1}{\rho_1\bar{\rho}_1(\phi_2{}^2 + \phi_1\rho_2\bar{\rho}_2) + \phi_1(\phi_1\phi_2 + \rho_3\bar{\rho}_3 - \rho_1\rho_2\bar{\rho}_3 - \bar{\rho}_1\bar{\rho}_2\rho_3)} , \\ \bar{\rho}_2^I &= \frac{\rho_1(\phi_2{}^2 + \rho_2\bar{\rho}_2\phi_1) - \rho_3\bar{\rho}_2\phi_1}{\rho_1\bar{\rho}_1(\phi_2{}^2 + \phi_1\rho_2\bar{\rho}_2) + \phi_1(\phi_1\phi_2 + \rho_3\bar{\rho}_3 - \rho_1\rho_2\bar{\rho}_3 - \bar{\rho}_1\bar{\rho}_2\rho_3)} . \end{aligned} \tag{B3}$$

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