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Classical algebraic chromodynamics

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I develop an extension of the usual equations of SU(n) chromodynamics which permits the consistent introduction of classical, noncommuting quark source charges. The extension involves adding a singlet gluon, giving a U(n)-based theory with outer product $P^a(u,v)=(1/2)(d^{abc}+if^{abc})(u^b v^c - v^b u^c)$ which obeys the Jacobi identity, inner product $S(u,v) = (1/2)(u^a v^a + v^a u^a)$, and with the n^2 gluon fields elevated to algebraic fields over the quark color charge C^* algebra. I show that provided the color charge algebra satisfies the condition $S(P(u,v),w) = S(u,P(v,w))$ for all elements u,v,w of the algebra, all the standard derivations of Lagrangian chromodynamics continue to hold in the algebraic chromodynamics case. I analyze in detail the color charge algebra in the two-particle (qq, $q\bar{q}$, $\bar{q}\bar{q}$) case and show that the above consistency condition is satisfied for the following unique (and, interestingly, asymmetric) choice of quark and antiquark charges: $Q_q^a = \xi^a$, $Q_{\bar{q}}^a = \bar{\xi}^a + \delta^{a0}(n/2)^{3/2}1$, with $\xi^a\xi^b = (1/2)(d^{abc}+if^{abc})\xi^c$, $\bar{\xi}^a\bar{\xi}^b = -(1/2)(d^{abc}-if^{abc})\bar{\xi}^c$. The algebraic structure of the two-particle U(n) force problem, when expressed on an appropriately diagonalized basis, leads for all n to a classical dynamics problem involving an ordinary SU(2) Yang-Mills field with uniquely specified classical source charges which are nonparallel in the color-singlet state. An explicit calculation shows that local algebraic U(n) gauge transformations lead only to a rigid global rotation of axes in the overlying classical SU(2) problem, which implies that the relative orientations of the classical source charges have physical significance. (For an application to the static $q\bar{q}$ force problem, see my later paper). I conclude with a series of conjectures about the extension of the algebraic results to the general N-particle case, and about the extension of the classical theory developed here to a full field theory.

I. GENERAL FORMALISM

I propose in this paper and a following paper a new approach to the problem of quark dynamics, based on the idea of following as closely as possible an analogy with familiar methods of classical electrodynamics.[1] In electrodynamics, a distribution of stationary charges with charge density

$$J^0 = \sum_n Q_{(n)}\delta^3(x-x_n)$$

produces a static electric field E^j obtained by solving $(\partial/\partial x^j)E^j = eJ^0$. From the electric field the forces on the charges can be calculated either directly from the Lorentz force law, or indirectly from a potential obtained by integrating the field energy density. As soon as one attempts to generalize these simple statements to the non-Abelian case, one encounters fundamental conceptual difficulties. Given a distribution of SU(n) color charges with density

$$J^{A0} = \sum_m Q^A_{(m)}\delta^3(x-x_m), \quad A=1,\ldots,n^2-1\,,$$

the color electric field is determined by the equation

$$(\partial/\partial x^j)E^{jA} + gf^{ABC}b^{jB}E^{jC} = gJ^{A0} = g\sum_n Q^A_{(n)}\delta^3(x-x_n)\,. \quad (1)$$

If the charges $Q^A_{(n)}$ are regarded as c numbers, in color-singlet states they must be parallel (or commuting) vectors, leading to a Coulomb force law.[2] Thus, for a classical charge formalism to be useful, it must incorporate the fact that the color charges $Q^A_{(n)}$ are q numbers, satisfying the color commutation relation

$$[Q^A_{(n)}, Q^B_{(m)}] = \delta_{nm}\, if^{ABC}Q^C_{(n)}\,.$$

However, since Eq. (1) is a constraint on independent Cauchy data E^{jA} and b^{jB} on any given time slice, when integrated with q-number source charges $Q^A_{(n)}$ it yields potential and field components which are noncommuting for spacelike separations. This noncommutativity cannot be attributed to conventional quantum dynamics of the gluon field, which produces noncommutativity only for causally related quantities.[3] Furthermore, if gluon field components do not commute for spacelike separations, then Eq. (1) is in fact not the correct gluon equation of motion, since its derivation from local gauge invariance[4] assumes commutativity of gluon field components.

In order to deal with these problems, I will construct a modified form of chromodynamics in which the gluon fields are regarded from the outset as algebraic fields over the quark color charge C^* algebra. The construction starts from the fundamental relation of gauge field theory, the statement that under the gauge transformation

$$\frac{\partial}{\partial x^{\mu}}+igB'_{\mu}=S\left(\frac{\partial}{\partial x^{\mu}}+igB_{\mu}\right)S^{-1},$$
$$B'_{\mu}=SB_{\mu}S^{-1}+\frac{i}{g}\frac{\partial S}{\partial x^{\mu}}S^{-1}, \tag{2}$$

the field strength

$$F_{\mu\nu}=\frac{\partial B_{\mu}}{\partial x^{\nu}}\frac{\partial B_{\nu}}{\partial x^{\mu}}-ig(B_{\mu}B_{\nu}-B_{\nu}B_{\mu}) \tag{3}$$

transforms as

$$F'_{\mu\nu}=\frac{\partial B'_{\mu}}{\partial x^{\nu}}-\frac{\partial B'_{\nu}}{\partial x^{\mu}}-ig(B'_{\mu}B'_{\nu}-B'_{\nu}B'_{\mu})$$
$$=SF_{\mu\nu}S^{-1}: \tag{4}$$

Let the n^2-1 matrices λ^A be the usual SU(n) matrices, normalized to $\mathrm{tr}(\lambda^A\lambda^B)=2\delta^{AB}$, and let us extend the set to n^2 matrices λ^a, $a=0,\ldots,n^2-1$ by adjoining $\lambda^0=(2/n)^{1/2}1$. These matrices satisfy the multiplication rule

$$\tfrac{1}{2}\lambda^a\,\tfrac{1}{2}\lambda^b=q^{abc}\,\tfrac{1}{2}\lambda^c,$$
$$q^{abc}=\tfrac{1}{2}(d^{abc}+if^{abc}) \tag{5a}$$

with d^{abc} (f^{abc}) totally symmetric (antisymmetric), and with the 0-index extension defined by $f^{0bc}=0$, $d^{0bc}=(2/n)^{1/2}\delta^{bc}$. Decomposing the fields into algebra-valued (hence noncommuting) amplitudes over the λ's,

$$B_{\mu}=b^{a}_{\mu}\tfrac{1}{2}\lambda^a,$$
$$F_{\mu\nu}=f^{a}_{\mu\nu}\tfrac{1}{2}\lambda^a, \tag{6a}$$

and specializing to the case of an infinitesimal gauge transformation S,

$$S=1+iu^a\tfrac{1}{2}\lambda^a, \tag{6b}$$

Eqs. (2)–(4) become

$$b'^{a}_{\mu}=b^{a}_{\mu}+iP^a(u,b_{\mu})-\frac{1}{g}\frac{\partial}{\partial x^{\mu}}u^a, \tag{2'}$$

$$f^{a}_{\mu\nu}=\frac{\partial b^{a}_{\mu}}{\partial x^{\nu}}-\frac{\partial b^{a}_{\nu}}{\partial x^{\mu}}-igP^a(b_{\mu},b_{\nu}), \tag{3'}$$

$$f'^{a}_{\mu\nu}=\frac{\partial b'^{a}_{\mu}}{\partial x^{\nu}}-\frac{\partial b'^{a}_{\nu}}{\partial x^{\mu}}-igP^a(b'_{\mu},b'_{\nu})$$
$$=f^{a}_{\mu\nu}+iP^a(u,f_{\mu\nu}), \tag{4'}$$

with the (anti-Hermitian) outer product P^a defined by

$$P^a(u,v)=-P^a(v,u)=q^{abc}(u^bv^c-v^bu^c). \tag{5b}$$

At this point the reason for adjoining 0 components becomes apparent: With 0 components included, the trace identity

$$\tfrac{1}{2}\mathrm{tr}(\lambda^c\lambda^a\lambda^e\lambda^f)=\tfrac{1}{2}\mathrm{tr}(\lambda^a\lambda^e\lambda^f\lambda^c) \tag{7a}$$

implies the identity[5]

$$q^{cam}q^{efm}=q^{aem}q^{fcm}, \tag{7b}$$

which in turn implies that the outer product P defined in Eq. (5) satisfies the Jacobi identity

$$P^a(u,P(v,w))+P^a(w,P(u,v))+P^a(v,P(w,u))=0 \tag{8}$$

for arbitrary (noncommuting) u,v,w. The Jacobi identity would not hold if 0 components were omitted from the construction. It is convenient to introduce a covariant derivative D_{μ},

$$D_{\mu}w^a\equiv\frac{\partial}{\partial x^{\mu}}w^a+igP^a(b_{\mu},w), \tag{9}$$

which by virtue of Eq. (8) satisfies

$$D_{\mu}P^a(u,v)=P^a(D_{\mu}u,v)+P^a(u,D_{\mu}v),$$
$$(D_{\mu}D_{\nu}-D_{\nu}D_{\mu})w^a=-igP^a(f_{\mu\nu},w). \tag{10}$$

Since for arbitrary small variations in the fields one has

$$\delta f^{a}_{\mu\nu}=D_{\nu}\delta b^{a}_{\mu}-D_{\mu}\delta b^{a}_{\nu}, \tag{11}$$

and since from Eq. (2′)

$$\delta_{\rm gauge}b^{a}_{\mu}=-g^{-1}D_{\mu}u^a, \tag{12}$$

we can immediately derive Eq. (4′) as an application of Eqs. (10) and (12),

$$\delta_{\rm gauge}f^{a}_{\mu\nu}=D_{\nu}\,\delta_{\rm gauge}b^{a}_{\mu}-D_{\mu}\delta_{\rm gauge}b^{a}_{\nu}$$
$$=g^{-1}(D_{\mu}D_{\nu}-D_{\nu}D_{\mu})u^a$$
$$=iP^a(u,f_{\mu\nu}). \tag{13}$$

It is also easy to show that Eq. (3′) implies that

$$D_{\lambda}f^{a}_{\mu\nu}+D_{\nu}f^{a}_{\lambda\mu}+D_{\mu}f^{a}_{\nu\lambda}=0. \tag{14}$$

As a further application of the formalism of Eqs. (5)–(10), I examine next the properties of an assumed gluon equation of motion

$$D_{\nu}f^{a\mu\nu}=gJ^{a\mu}, \tag{15}$$

with $J^{a\mu}$ the quark source current. This equation will be gauge covariant provided that

$$\delta_{\rm gauge}J^{a\mu}=g^{-1}\delta_{\rm gauge}(D_{\nu}f^{a\mu\nu})$$
$$=g^{-1}(\delta_{\rm gauge}D_{\nu})f^{a\mu\nu}+g^{-1}D_{\nu}(\delta_{\rm gauge}f^{a\mu\nu})$$
$$=g^{-1}igP^a(\delta_{\rm gauge}b_{\nu},f^{\mu\nu})+g^{-1}D_{\nu}iP^a(u,f^{\mu\nu})$$
$$=iP^a(u,J^{\mu}). \tag{16}$$

Hence if we express J^{μ} in terms of quark (antiquark) source charges, positions, and four-velocities by writing

$$J^{a\mu}=\int d\tau\sum_n Q^a_{(n)}(\tau)u^{\mu}_n(\tau)\delta^4(x-x_n(\tau)), \tag{17}$$

Eq. (16) specifies the gauge transformation properties of the quark source charges to be

$$\delta_{\text{gauge}} Q^a_{(n)}(\tau) = iP^a(u(x_n(\tau)), Q_{(n)}(\tau)) . \quad (18)$$

Applying a covariant divergence D_μ to Eq. (15) and using Eqs. (10) and (14), we get

$$D_\mu J^{a\mu} = 0 , \quad (19)$$

which implies that the proper-time evolution of the quark charges obeys

$$\frac{dQ^a_{(n)}(\tau)}{d\tau} = -igP^a(b_\nu(x_n(\tau)), Q_{(n)}(\tau))\, u_n^\nu(\tau) . \quad (20)$$

Up to this point my development parallels the c-number classical Yang-Mills formalism given by Wong,[6] except that in the above formulas all charges and fields can be arbitrary algebraic variables.

To proceed further, we need a Lagrangian density and a stress-energy tensor. To construct these, it is necessary to introduce an identity on algebra-valued variables which is not valid for arbitrary algebras, but which, I conjecture, does hold for particular finite algebras (the "color-charge algebras" defined below). I introduce an inner product S defined by

$$S(u, v) = \tfrac{1}{2}(u^a v^a + v^a u^a) = S(v, u) \quad (21)$$

and restrict the discussion from now on to algebras satisfying

$$S(u, P(v,w)) = S(P(u, v), w) \quad (22)$$

for all elements u, v, w of the algebra, a condition which guarantees that the covariant derivative of Eq. (9) is distributive over the inner product,

$$\frac{\partial}{\partial x^\mu} S(u, v) \equiv D_\mu S(u, v) = S(D_\mu u, v) + S(u, D_\mu v) . \quad (23)$$

In terms of the inner product, a Lagrangian density can be defined by

$$\mathfrak{L} = -\tfrac{1}{4} S(f_{\mu\nu}, f^{\mu\nu}) + \mathfrak{L}_{(\text{interaction} + \text{quark kinetic})} , \quad (24)$$

with $\mathfrak{L}_{(\text{interaction} + \text{quark kinetic})}$ obeying[7] in the isolated quark limit

$$\delta_{(\text{Euler-Lagrange})} \mathfrak{L}_{(\text{interaction} + \text{quark kinetic})} = -gS(\delta b_\mu, J^\mu) + \sum_n m_{(n)} u_n^\alpha(\tau) \delta_{n\alpha}(\tau) , \quad (25a)$$

$$\delta_{\text{gauge}} \mathfrak{L}_{(\text{interaction} + \text{quark kinetic})} = 0 \text{ (or total derivative)} , \quad (25b)$$

while a stress-energy tensor can be defined by

$$T^{\alpha\beta} = T^{\alpha\beta}{}_{\text{gluon}} + T^{\alpha\beta}{}_{\text{quark}} ,$$
$$T^{\alpha\beta}{}_{\text{gluon}} = -S(f^\alpha{}_\gamma, f^{\beta\gamma}) + \tfrac{1}{4}\eta^{\alpha\beta} S(f_{\gamma\delta}, f^{\gamma\delta}) ,$$
$$T^{\alpha\beta}{}_{\text{quark}} = \int d\tau \sum_n m_{(n)} u_n^\alpha(\tau) u_n^\beta(\tau) \times \delta^4(x - x_n(\tau)) . \quad (26)$$

It is now easy to check, by use of Eq. (22) and Eqs. (8)–(14), that all the standard formulas of Lagrangian chromodynamics follow from Eqs. (23)–(26). For example, varying b_μ we find

$$\begin{aligned} 0 &= \delta \int d^4x\, \mathfrak{L} \\ &= \int d^4x [-\tfrac{1}{2} S(\delta f_{\mu\nu}, f^{\mu\nu}) - gS(\delta b_\mu, J^\mu)] \\ &= \int d^4x [-S(D_\nu \delta b_\mu, f^{\mu\nu}) - gS(\delta b_\mu, J^\mu)] \\ &= \int d^4x\, S(\delta b_\mu, D_\nu f^{\mu\nu} - gJ^\mu) , \end{aligned} \quad (27)$$

which yields the equation of motion of Eq. (15). Similar calculations give for the quark equation of motion (Lorentz force law)

$$m_{(n)} \frac{du_{n\alpha}(\tau)}{d\tau} = gS(f_{\mu\alpha}(x_n(\tau)), Q_{(n)}(\tau) u_n^\mu(\tau)) , \quad (28)$$

which implies

$$\frac{\partial}{\partial x^\beta} T^{\alpha\beta}{}_{\text{quark}} = -gS(f^\alpha{}_\gamma, J^\gamma) , \quad (29)$$

and which together with

$$\frac{\partial}{\partial x^\beta} T^{\alpha\beta}{}_{\text{gluon}} = gS(f^\alpha{}_\gamma, J^\gamma) \quad (30)$$

implies overall stress-energy–tensor conservation. It is also easily checked, again using Eq. (22), that $\mathfrak{L}$ and $T^{\alpha\beta}$ are gauge invariant.

II. THE N=2 COLOR-CHARGE ALGEBRA

I turn next to the crucial issue of defining a suitable quark color-charge algebra, and examining whether it satisfies the condition of Eq. (22). The elements of the color-charge algebra will be operators acting on the product Hilbert space constructed by taking the direct product of the color Hilbert spaces for the various quarks and antiquarks which are present. Since classical chromodynamics must be the isolated quark limit of a quantum color field theory, I assume that (up to possible shifts in 0 components) the color charges behave algebraically as do the one-quark projections of the charges $\psi^\dagger \lambda^a \psi$, $\psi^\dagger(-\lambda^*)^a \psi$, with $\psi^\dagger = (a_1^\dagger, \ldots, a_n^\dagger)$ an SU(n) color spin operator. (The $a_i^\dagger$ are color state creation operators for a given quark or antiquark, with $\{a_i, a_j\} = \{a_i^\dagger, a_j^\dagger\} = 0$, $\{a_i, a_j^\dagger\} = \delta_{ij}$.) Specifically, I define a one-quark-projected color spinor

$$\psi_P = \begin{bmatrix} a_1 \prod_{j\neq 1}(1-n_j) \\ a_2 \prod_{j\neq 2}(1-n_j) \\ \cdot \\ \cdot \\ \cdot \\ a_n \prod_{j\neq n}(1-n_j) \end{bmatrix}, \tag{31}$$

with $n_j = a_j^\dagger a_j$ the number operator for color type j, and introduce n^2-plet charges $\xi^a, \bar{\xi}^a$ defined by

$$\xi^a = \psi_P^\dagger \tfrac{1}{2}\lambda^a \psi_P, \quad \bar{\xi}^a = \psi_P^\dagger(-\tfrac{1}{2}\lambda^{*a})\psi_P. \tag{32}$$

Since

$$\psi_{P\alpha}\psi_{P\beta}^\dagger = \delta_{\alpha\beta}\prod_{j=1}^{n}(1-n_j),$$

and since

$$\prod_{j=1}^{n}(1-n_j)\psi_P = \psi_P,$$

the charges defined in Eq. (32) satisfy the simple algebras

$$\begin{aligned} \xi^a\xi^b &= \psi_P^\dagger \tfrac{1}{2}\lambda^a \tfrac{1}{2}\lambda^b \psi_P = q^{abc}\xi^c, \\ \bar{\xi}^a\bar{\xi}^b &= \psi_P^\dagger(-\tfrac{1}{2}\lambda^{*a})(-\tfrac{1}{2}\lambda^{*b})\psi_P = -q^{bac}\bar{\xi}^c. \end{aligned} \tag{33}$$

For the quark and antiquark color charges, I take

$$\begin{aligned} Q_q^a &= \xi^a + \delta^{a0}K_q 1, \\ Q_{\bar{q}}^a &= \bar{\xi}^a + \delta^{a0}K_{\bar{q}} 1, \end{aligned} \tag{34}$$

where I have allowed for possible unit-operator shifts in the 0 components, which will be specified shortly.

Consider now a system of N ($=N_q + N_{\bar{q}}$) quarks and antiquarks, with color charges $Q_{(1)}^a, \ldots, Q_{(N)}^a$. I assume that the color charges for different particles commute,

$$[Q_{(i)}^a, Q_{(j)}^b] = 0, \quad i \neq j; \tag{35}$$

hence the inner algebraic properties of the charges are completely specified by Eqs. (33)–(35). Putting things in less abstract terms, the Q_q^a and the $Q_{\bar{q}}^a$ can be regarded simply as the matrices $\frac{1}{2}\lambda^a$ and $-\frac{1}{2}\lambda^{*a}$, with shifted 0 components, and the color-charge algebra I construct is an algebra built from direct products of the λ matrices for the various quarks and antiquarks which are present.

Definition. The rank-N, type $(N_q, N_{\bar{q}})$ color-charge algebra is the minimal algebra containing N_q quark charges and $N_{\bar{q}}$ antiquark charges which is closed under composition using the outer product P of Eq. (5).

I have computed in detail the rank-2, types (2,0), (1,1), and (0,2) color-charge algebras, and find that they are all finite (containing 4 or 5 elements each), all have essentially the same inner algebraic structure, and all satisfy the condition of Eq. (22) provided the following choices of 0-component displacements are made:

$$\begin{aligned} K_q &= 0, \\ K_{\bar{q}} &= (n/2)^{3/2}. \end{aligned} \tag{36}$$

That is, charges transforming in the same sense as the outer product P (having q^{abc} in the charge multiplication law) have an undisplaced 0 component, while charges transforming in the opposite sense to the outer product P (having $-q^{bac} = q^{*abc}$ in the charge multiplication law) must have a displaced 0 component in order for Eq. (22) to be satisfied.[8]

The computations which lead to these statements are lengthy, but are greatly facilitated by the fact that the usual SU(n) identities[9] for the structure constants f^{ABC}, d^{ABC} (where indices range from 1 to n^2-1) take on a simpler form when reexpressed as identities for q^{abc} (which has indices ranging from 0 to n^2-1). All the identities which I have used are summarized in Table I. The building-up process leading to the rank-2, type (2,0) algebra is summarized in Table II. Starting from the quark charges $w_1^a = \xi_1^a$, $w_2^a = \xi_2^a$ as the initial elements of the algebra, one applies the outer product P^a, giving three more elements

TABLE I. U(n) structure-constant identities.

$q^{abc} = \frac{1}{2}(d^{abc} + if^{abc})$
$q^{abc} = q^{cab} = q^{bca}$
$q^{0bc} = \frac{1}{2}\left(\frac{2}{n}\right)^{1/2}\delta^{bc}$
$q^{mmc} = \delta^{c0} n \left(\frac{n}{2}\right)^{1/2}$
$q^{cam}q^{efm} = q^{aem}q^{fcm}$
$q^{map}q^{pbm} = \frac{n}{2}\delta^{ab}$
$q^{map}q^{mbp} = \frac{n}{2}\delta^{a0}\delta^{b0}$
$q^{map}q^{pbr}q^{rcm} = \frac{n}{2}q^{abc}$
$q^{map}q^{pbr}q^{mcr} = \frac{1}{2}\left(\frac{n}{2}\right)^{1/2}\delta^{c0}\delta^{ab} = \frac{n}{2}\delta^{c0}q^{ab0}$

TABLE II. Building-up process in the rank-2, type (2,0) case. Initial basis: $w_1^a=\xi_1^a$, $w_2^a=\xi_2^a$, $w_3^a=q^{abc}(\xi_1^b\xi_2^c-\xi_2^b\xi_1^c)$, $w_4^a=(\frac{1}{2}n)^{1/2}(\xi_2^a\xi_1^0-\delta^{a0}\xi_1^e\xi_2^e)$, $w_5^a=(\frac{1}{2}n)^{1/2}(\xi_1^a\xi_2^0-\delta^{a0}\xi_1^e\xi_2^e)$.

Outer-product table: Table gives $P(w_{\text{row}}, w_{\text{column}})$

	w_1	w_2	w_3	w_4	w_5
w_1	0	w_3	w_4	w_3	$\frac{1}{2}w_3$
w_2	$-w_3$	0	$-w_5$	$-\frac{1}{2}w_3$	$-w_3$
w_3	$-w_4$	w_5	0	$w_5-\frac{1}{2}w_4$	$-w_4+\frac{1}{2}w_5$
w_4	$-w_3$	$\frac{1}{2}w_3$	$-w_5+\frac{1}{2}w_4$	0	$-\frac{3}{4}w_3$
w_5	$-\frac{1}{2}w_3$	w_3	$w_4-\frac{1}{2}w_5$	$\frac{3}{4}w_3$	0

Inner-product table: Table gives $S(w_{\text{row}}, w_{\text{column}})$

	w_1	w_2	w_3	w_4	w_5
w_1	α	δ	0	0	γ
w_2	δ	β	0	γ	0
w_3	0	0	$-\gamma$	0	0
w_4	0	γ	0	γ	$\frac{1}{2}\gamma$
w_5	γ	0	0	$\frac{1}{2}\gamma$	γ

$\alpha=n(n/2)^{1/2}\xi_1^0$, $\beta=n(n/2)^{1/2}\xi_2^0$, $\gamma=\frac{1}{2}n^2\xi_1^0\xi_2^0-\frac{1}{2}\xi_1^e\xi_2^e$, $\delta=\xi_1^e\xi_2^e$.

$w_{3,4,5}^a$ as indicated in the table. From then on the algebra closes, yielding the outer- and inner-product tables shown. [Although there is one linear relation among the $w_1, \ldots, w_5$ in the (2,0) and (0,2) cases, I do not explicitly reduce to a four-element basis. As a result, the three rank-2 algebras are nominally five-element algebras and have the same diagonalized form. After diagonalization, the element z_2 is zero in the (2,0) and (0,2) cases, showing that these algebras really have four elements in minimal form.] It is straightforward at this point, but tedious, to verify directly from Table II that the identity of Eq. (22) is indeed satisfied. A much better procedure, which turns out to yield useful mathematical and physical insights, is to observe[10] that $w_{3,4,5}$ form an ideal of the P product table, which implies that the algebra can be diagonalized. As already noted, the diagonalized forms of all three rank-2 algebras [types (1,1) and (0,2) as well as (2,0)] are the same, and so I have summarized the relevant structural formulas for all three cases, in a uniform notation, in Tables III–VI. Before turning to a discussion of these tables, let me make two brief technical comments on the computations.

TABLE III. Rank-2 color-charge algebra tables: original and diagonalizing bases.[a]

Algebra	$(2,0)=qq$	$(1,1)=q\bar{q}$	$(0,2)=\overline{qq}$
Original basis			
w_1	ξ_1^a	ξ_1^a	$\bar{\xi}_1^a+\delta^{a0}(n/2)^{3/2}$
w_2	ξ_2^a	$\bar{\xi}_2^a+\delta^{a0}(n/2)^{3/2}$	$\bar{\xi}_2^a+\delta^{a0}(n/2)^{3/2}$
$w_3=P(w_1,w_2)$	$q^{abc}(\xi_1^b\xi_2^c-\xi_2^b\xi_1^c)$	$q^{abc}(\xi_1^b\bar{\xi}_2^c-\bar{\xi}_2^b\xi_1^c)$	$q^{abc}(\bar{\xi}_1^b\bar{\xi}_2^c-\bar{\xi}_2^b\bar{\xi}_1^c)$
$w_4=P(w_1,w_3)$	$(n/2)^{1/2}(\xi_2^a\xi_1^0-\delta^{a0}\xi_1^e\xi_2^e)$	$(n/2)^{1/2}(\xi_1^0\bar{\xi}_2^a-\delta^{a0}\xi_1^e\bar{\xi}_2^e)$	$(n/2)^{1/2}\bar{\xi}_2^0\bar{\xi}_1^a-(n/2)q^{abc}(\bar{\xi}_1^b\bar{\xi}_2^c+\bar{\xi}_2^b\bar{\xi}_1^c)$
$w_5=P(w_3,w_2)$	$(n/2)^{1/2}(\xi_1^a\xi_2^0-\delta^{a0}\xi_1^e\xi_2^e)$	$(n/2)^{1/2}\xi_1^0\bar{\xi}_2^a-(n/2)q^{abc}(\xi_1^b\bar{\xi}_2^c+\bar{\xi}_2^b\xi_1^c)$	$(n/2)^{1/2}\bar{\xi}_1^0\bar{\xi}_2^a-(n/2)q^{abc}(\bar{\xi}_1^b\bar{\xi}_2^c+\bar{\xi}_2^b\bar{\xi}_1^c)$
Diagonalizing basis			
z_1	$w_1+w_2-(\frac{2}{3})(w_4+w_5)$	$w_1+w_2-[2/(n^2-1)][w_5+(\frac{1}{2}n^2-1)w_4]$	$w_1+w_2-[2/(n^2-1)](w_5+w_4)$
z_2	$w_1-w_2+2(w_4-w_5)=0$	$w_1-w_2-[2/(n^2-1)][3w_5-(1+\frac{1}{2}n^2)w_4]$	$w_1-w_2+2(w_4-w_5)=0$
z_3	$(2i/\sqrt{3})w_3$	$[2i/(n^2-1)^{1/2}]w_3$	$[2i/(n^2-1)^{1/2}]w_3$
z_4	$(\frac{2}{3})(w_4+w_5)$	$[4/(n^2-1)^{1/2}(n^2+8)^{1/2}](w_4+w_5)$	$[2/(n^2-1)](w_4+w_5)$
z_5	$(2/\sqrt{3})(w_5-w_4)$	$[4/(n^2-1)(n^2+8)^{1/2}][3w_5-(1+\frac{1}{2}n^2)w_4]$	$[2/(n^2-1)^{1/2}](w_5-w_4)$
Inverse transformation			
w_1	$\frac{1}{2}(z_1+z_2+z_4+\sqrt{3}z_5)$	$\frac{1}{2}(z_1+z_2)+(n^2+8)^{-1/2}[3z_5+(n^2-1)^{1/2}z_4]$	$\frac{1}{2}[z_1+z_2+z_4+(n^2-1)^{1/2}z_5]$
w_2	$\frac{1}{2}(z_1-z_2+z_4-\sqrt{3}z_5)$	$\frac{1}{2}(z_1-z_2)+(n^2+8)^{-1/2}[-(1+\frac{1}{2}n^2)z_5+(n^2-1)^{1/2}z_4]$	$\frac{1}{2}[z_1-z_2+z_4-(n^2-1)^{1/2}z_5]$
w_3	$-(\sqrt{3}i/2)z_3$	$-[(n^2-1)^{1/2}i/2]z_3$	$-[(n^2-1)^{1/2}i/2]z_3$
w_4	$-(\sqrt{3}/4)(z_5-\sqrt{3}z_4)$	$-[(n^2-1)/2(n^2+8)^{1/2}][z_5-3(n^2-1)^{-1/2}z_4]$	$-[(n^2-1)^{1/2}/4][z_5-(n^2-1)^{1/2}z_4]$
w_5	$(\sqrt{3}/4)(z_5+\sqrt{3}z_4)$	$[(n^2-1)/2(n^2+8)^{1/2}][z_5+(1+\frac{1}{2}n^2)(n^2-1)^{-1/2}z_4]$	$[(n^2-1)^{1/2}/4][z_5+(n^2-1)^{1/2}z_4]$

[a]Repeated lower-case indices are summed from 0 to n^2-1.

TABLE IV. Rank-2 color-charge algebra tables: scalar products.[a]

Algebra	$(2,0)=qq$	$(1,1)=q\bar{q}$	$(0,2)=\bar{q}\bar{q}$
Independent scalar products in original basis			
$\alpha=S(w_1,w_1)$	$n(n/2)^{1/2}\xi_1^0$	$n(n/2)^{1/2}\xi_1^0$	$(n/2)^3$
$\beta=S(w_2,w_2)$	$n(n/2)^{1/2}\xi_2^0$	$(n/2)^3$	$(n/2)^3$
$\gamma=S(w_1,w_2)$	$\xi_1^e\xi_2^e$	$\xi_1^0[\bar{\xi}_2^0+(n/2)^{3/2}]+\xi_1^A\bar{\xi}_2^A$	$[\bar{\xi}_1^0+(n/2)^{3/2}][\bar{\xi}_2^0+(n/2)^{3/2}]+\bar{\xi}_1^A\bar{\xi}_2^A$
$\delta=S(w_1,w_5)$	$\frac{1}{2}n^2\xi_1^0\xi_2^0-\frac{1}{2}\xi_1^e\xi_2^e$	$(\frac{1}{2}-\frac{1}{2}n^2)\xi_1^0\bar{\xi}_2^0+(\frac{1}{2}-\frac{1}{4}n^2)\xi_1^A\bar{\xi}_2^A$	$(\frac{1}{2}n^2-\frac{1}{2})\bar{\xi}_1^0\bar{\xi}_2^0-\frac{1}{2}\bar{\xi}_1^A\bar{\xi}_2^A$
Independent scalar products in new basis			
$A=S(z_1,z_1)$	$\alpha+\beta+2\gamma-(4/3)\delta$	$\alpha+\beta+2\gamma-[n^2/(n^2-1)]\delta$	$\alpha+\beta+2\gamma-[4/(n^2-1)]\delta$
$B=S(z_2,z_2)$	$\alpha+\beta-2\gamma-4\delta=0$	$\alpha+\beta-2\gamma-[(n^2+8)/(n^2-1)]\delta$	$\alpha+\beta-2\gamma-4\delta=0$
$C=S(z_1,z_2)$	$\alpha-\beta=0$	$\alpha-\beta-[(4-n^2)/(n^2-1)]\delta$	$\alpha-\beta=0$
$D=S(z_3,z_3)$	$(4/3)\delta$	$[4/(n^2-1)]\delta$	$[4/(n^2-1)]\delta$

[a]Repeated lower-case indices are summed from 0 to n^2-1; repeated capital letter indices are summed from 1 to n^2-1.

(i) Although the computations are most readily done in n^2-plet notation using the identities of Table I, the role of the 0-component shifts is better understood if Eq. (5) for the outer product is rewritten with 0 components explicitly separated,

$$P^0(u,v)=\frac{1}{2}\left(\frac{2}{n}\right)^{1/2}[u^0,v^0]+\frac{1}{2}\left(\frac{2}{n}\right)^{1/2}[u^A,v^A],$$
$$P^A(u,v)=\frac{1}{2}\left(\frac{2}{n}\right)^{1/2}[u^0,v^A]+\frac{1}{2}\left(\frac{2}{n}\right)^{1/2}[u^A,v^0]+\tfrac{1}{2}d^{ABC}[u^B,v^C]+\tfrac{1}{2}if^{ABC}\{u^B,v^C\}. \tag{37}$$

Equation (37) shows that the 0 components appear only in commutators, never in the anticommutator term. Hence, since w_1 and w_2 do not reappear as entries in the outer-product tables (cf. Table II), unit operator shifts in the 0 components of $w_{1,2}$ have no effect on the structure of the outer-product algebra. The shifts, of course, do change

TABLE V. Rank-2 color-charge algebra tables: outer- and inner-product tables on the z basis.

Outer-product table: Table gives $P(z_{\text{row}},z_{\text{column}})$

	z_1	z_2	z_3	z_4	z_5
z_1	0	0	0	0	0
z_2	0	0	0	0	0
z_3	0	0	0	iz_5	$-iz_4$
z_4	0	0	$-iz_5$	0	iz_3
z_5	0	0	iz_4	$-iz_3$	0

Inner-product table: Table gives $S(z_{\text{row}},z_{\text{column}})$

	z_1	z_2	z_3	z_4	z_5
z_1	A	C	0	0	0
z_2	C	B	0	0	0
z_3	0	0	D	0	0
z_4	0	0	0	D	0
z_5	0	0	0	0	D

TABLE VI. Rank-2 color-charge algebra tables: matrix elements of the scalar products.

$(2,0)=qq$ algebra

Color type (n)	U(3)	U(3)	U(2)	U(2)
Matrix element	$\langle\,\rangle_{\bar{3}}$	$\langle\,\rangle_6$	$\langle\,\rangle_1$	$\langle\,\rangle_3$
A	$\frac{2}{3}$	$\frac{10}{3}$	0	$\frac{8}{3}$
D	$\frac{4}{3}$	$\frac{2}{3}$	1	$\frac{1}{3}$

$(1,1)=q\bar{q}$ algebra

Color type (n)	U(3)	U(3)	U(2)	U(2)	U(n)
Matrix element	$\langle\,\rangle_1$	$\langle\,\rangle_8$	$\langle\,\rangle_1$	$\langle\,\rangle_3$	$\langle\,\rangle_1$
A	0	$\frac{381}{64}$	0	$\frac{8}{3}$	0
B	0	$\frac{165}{64}$	0	0	0
C	0	$-\frac{105}{64}$	0	0	0
D	$\frac{3}{2}$	$\frac{3}{16}$	1	$\frac{1}{3}$	$n/2$

$(0,2)=\bar{q}\bar{q}$ algebra

Color type (n)	U(3)	U(3)	U(2)	U(2)
Matrix element	$\langle\,\rangle_3$	$\langle\,\rangle_{\bar{6}}$	$\langle\,\rangle_1$	$\langle\,\rangle_3$
A	9	$\frac{45}{4}$	0	$\frac{8}{3}$
D	$\frac{1}{2}$	$\frac{5}{12}$	1	$\frac{1}{3}$

the values of inner products of elements of the algebra.

(ii) The matrix-element computations summarized in Table VI are of the usual type familiar in the nonrelativistic quark model. As an illustrative example, let me compute the singlet expectations for general U(n) color in the $q\bar{q}$ case. According to Table IV, one needs $\langle\xi_1^0\rangle_1$, $\langle\xi_1^0\bar{\xi}_2^0\rangle_1$, and $\langle\xi_1^A\bar{\xi}_2^A\rangle_1$. [The notation $\langle\ \rangle_c$ used here and in Table VI indicates the expectation of the operator u (which acts on the two-particle product color Hilbert space) in the two-particle state transforming according to the representation c of the color group.] The first two are obtained from Eq. (33), which tells us that ξ^0 and $\bar{\xi}^0$ behave as multiples of the unit operator 1 in their respective algebras, giving in any color state

$$\begin{aligned}\langle\xi_1^0\rangle &= q^{000} = \frac{1}{2}\left(\frac{2}{n}\right)^{1/2},\\ \langle\bar{\xi}_2^0\rangle &= -q^{000} = -\frac{1}{2}\left(\frac{2}{n}\right)^{1/2},\\ \langle\xi_1^0\bar{\xi}_2^0\rangle &= \langle\xi_1^0\rangle\langle\bar{\xi}_2^0\rangle.\end{aligned} \tag{38}$$

To get $\langle\xi_1^A\bar{\xi}_2^A\rangle_1$ one uses the standard identity

$$\begin{aligned}\langle\xi_1^A\bar{\xi}_2^A\rangle_1 &= \tfrac{1}{2}[\langle(\xi_1^A+\bar{\xi}_2^A)^2\rangle_1 - \langle(\xi_1^A)^2\rangle_1 - \langle(\bar{\xi}_2^A)^2\rangle_1]\\ &= -\langle(\xi_1^A)^2\rangle_1\\ &= -(q^{aa0}-q^{000})\langle\xi_1^0\rangle\\ &= -(n^2-1)/2n.\end{aligned} \tag{39}$$

Equations (38) and (39), when substituted into Table IV, give the values $\langle A\rangle_1=\langle B\rangle_1=\langle C\rangle_1=0$, $\langle D\rangle_1=n/2$ listed in Table VI.

Let me now proceed to a discussion of the interesting structural features revealed in Tables III–VI.

(1) In the diagonalized form given in Table V, it is easy to check that the identity of Eq. (22) is satisfied. Since both the P and S product tables are block diagonal, the algebra decomposes into two independent subalgebras spanned by $z_{1,2}$ and $z_{3,4,5}$, respectively. Over the $z_{1,2}$ subalgebra, both sides of Eq. (22) vanish identically. The $z_{3,4,5}$ subalgebra is just an SU(2) Lie algebra, and so over this subalgebra we have

$$\left.\begin{aligned}P(z_i,z_j)&=i\epsilon^{ijk}z_k\\ S(z_i,z_j)&=D\delta_{ij}\end{aligned}\right\},\quad i,j,k=3,4,5 \tag{40}$$

ϵ = alternating tensor with $\epsilon^{345}=1$,

giving

$$\begin{aligned}S(P(z_i,z_j),z_k)-S(z_i,P(z_j,z_k)) &= D(\epsilon^{ijl}\delta_{lk}-\delta_{il}\epsilon^{jkl})\\ &= D(\epsilon^{ijk}-\epsilon^{jki})=0,\end{aligned} \tag{41}$$

so again Eq. (22) is satisfied.

(2) Focusing now on the $q\bar{q}$ algebra, we see from Tables V and VI that in the color-singlet state, the scalar-product expectations $\langle A,B,C\rangle_1$ associated with the $z_{1,2}$ subalgebra all vanish. Hence despite the fact that the gluon structure is U(n) rather than SU(n), the color forces in the singlet state are entirely non-Abelian in character. This is just what is required for the color forces to be asymptotically free. Evidently, algebraic chromodynamics has two levels of Lie structure: an underlying U(n) algebraic gauge theory which always leads, in the two-particle case, to an overlying classical SU(2) Yang-Mills chromodynamics. In the color-singlet state, the only remnants in the overlying gauge theory of the original choice n of color group appear in two places: in the value of $\langle D\rangle_1$, the scalar product expectation in the non-Abelian sector, and in the effective charges for the classical SU(2) theory obtained by reexpressing the original charges w_1^a and w_2^a in the $z_{3,4,5}$ basis. Referring to Table III, we see that for the $q\bar{q}$ algebra the effective charges, in an obvious vectorial notation appropriate to SU(2), are

$$\begin{aligned}\vec{Q}_q^{\,\mathrm{eff}} &= \left(0,\left(\frac{n^2-1}{n^2+8}\right)^{1/2},\ \frac{3}{(n^2+8)^{1/2}}\right),\\ \vec{Q}_{\bar{q}}^{\,\mathrm{eff}} &= \left(0,\left(\frac{n^2-1}{n^2+8}\right)^{1/2},\ -\frac{(1+\frac{1}{2}n^2)}{(n^2+8)^{1/2}}\right),\end{aligned} \tag{42}$$

which satisfy

$$\begin{aligned}(\vec{Q}_q^{\,\mathrm{eff}})^2 &= 1,\\ (\vec{Q}_{\bar{q}}^{\,\mathrm{eff}})^2 &= n^2/4,\end{aligned} \tag{43}$$

$$\begin{aligned}\vec{Q}_q^{\,\mathrm{eff}}\cdot\vec{Q}_{\bar{q}}^{\,\mathrm{eff}} &= -\tfrac{1}{2},\\ \theta(\vec{Q}_q,\vec{Q}_{\bar{q}}) &= \cos^{-1}(-1/n).\end{aligned} \tag{44}$$

Hence the effective charges, for all $n\geq 2$, are nonparallel. Analogous effective charges can be defined for the $z_{3,4,5}$ sectors of the qq and $\bar{q}\bar{q}$ algebras. In color-nonsinglet states, the expectations $\langle A\rangle$, $\langle B\rangle$, $\langle C\rangle$ do not all vanish, and so the color forces contain Abelian as well as non-Abelian components.

(3) In order to study the physical significance of the effective charges, I have explicitly computed the effect of making a general infinitesimal local algebraic gauge transformation in the case of the qq algebra. According to Eq. (18), the gauge variation of each charge is determined by the value at the charge of the gauge parameter u; hence for an $N=2$ algebra the effect on the charges of a completely general gauge transformation is described by

$$
\begin{aligned}
&w_1' = w_1 + iP(u_1, w_1),\\
&w_2' = w_2 + iP(u_2, w_2),\\
&u_1 = \alpha_1 w_1 + \beta_1 w_2 + i\delta_1 w_3 + \gamma_1 w_4 + \epsilon_1 w_5,\\
&u_2 = \alpha_2 w_1 + \beta_2 w_2 + i\delta_2 w_3 + \gamma_2 w_4 + \epsilon_2 w_5.
\end{aligned} \tag{45}
$$

Working in the qq case ($w_1 = \xi_1$, $w_2 = \xi_2$; P table before diagonalization given in Table III) I have explicitly constructed the P and S product tables generated by starting from initial charges w_1' and w_2'. After transforming to new diagonalizing bases $z_1', \ldots, z_5'$, these tables become identical to Tables IV and V. The new diagonalizing bases are related to the original ones by

$$
\begin{aligned}
&z_1' = z_1,\\
&z_2' = z_2,\\
&(z_3', z_4', z_5') = (z_3, z_4, z_5)\\
&\qquad + \left(\frac{\sqrt{3}}{4}(\delta_1+\delta_2), -\tfrac{1}{2}(\theta_1-\theta_2), \frac{\sqrt{3}}{2}(\theta_1+\theta_2)\right)\\
&\qquad \times (z_3, z_4, z_5),\\
&\theta_1 = -(\beta_1 + \gamma_1 + \tfrac{1}{2}\epsilon_1),\\
&\theta_2 = \alpha_2 + \epsilon_2 + \tfrac{1}{2}\gamma_2,
\end{aligned} \tag{46}
$$

where I again use a vectorial notation for the 3, 4, 5 components and where $\times$ denotes the usual vector cross product. The new diagonalizing bases are thus obtained from the old ones simply by making an infinitesimal rotation. This calculation shows that the effect of making a local algebraic gauge transformation in the underlying algebraic chromodynamics is to induce a global rotation of axes in the overlying classical Yang-Mills theory. Hence the effective charges cannot be rotated to be parallel; their relative orientations have intrinsic physical significance. This result is the basis for the analysis of the static quark force problem given in the following paper.

(4) From Table VI it is apparent that within each algebra the expectation $\langle D\rangle_c$ decreases monotonically as the size of the color representation increases, and that the largest value of $\langle D\rangle$ occurs in the color-singlet state. Thus there is a sense in which algebraic chromodynamics gives color forces which are strongest in the singlet state.

(5) As is readily apparent from Eq. (42) and from the tables, the theory is not symmetric between quarks and antiquarks. I will have more to say about this below, when I speculate about the extension of classical algebraic chromodynamics into a full-fledged quantum field theory. It is easy to check from the tables that in the $n=2$ color case the qq, $q\bar{q}$, and $\bar{q}\bar{q}$ algebras become identical, reflecting the fact that the fundamental $\underline{2}$-representation of SU(2) is self-conjugate.

(6) After discovering the algebraic structures given in the tables, I searched to find known mathematical antecedents, and found a partial answer in papers by Albert[11] and Santilli[12] on nonassociative algebras. Albert, in his fundamental paper published in 1948, introduced the concept of a trace-admissible algebra. To explain this idea, let u, v, w be elements of an algebra $\mathfrak{A}$ with nonassociative product uv. The algebra $\mathfrak{A}$ is a Lie algebra if the product operation is anticommutative and satisfies the Jacobi identity

$$
\begin{aligned}
&uv = -vu,\\
&u(vw) + w(uv) + v(wu) = 0.
\end{aligned} \tag{47}
$$

(These conditions also imply that the associator $[u, v, w] = (uv)w - u(vw)$ satisfies the identities

$$
\begin{aligned}
&[u, v, w] = -[w, v, u],\\
&[u, v, w] + [v, w, u] + [w, u, v] = 0,
\end{aligned} \tag{48}
$$

which in Albert's terminology define a more general type of structure, called a flexible Lie-admissible algebra.) The algebra $\mathfrak{A}$ is trace-admissible if there is a symmetric bilinear function, or trace form, $\tau(u, v)$ with arguments u, v in $\mathfrak{A}$ and values over a field $\mathfrak{F}$, such that

$$
\tau(uv, w) = \tau(u, vw). \tag{49}
$$

[Albert's definition of trace includes other technical conditions, one of which, $\tau(u, v) = 0$ if $uv = 0$, is not satisfied by the color-charge algebras.] Referring now to the concepts I introduced above, if uv is identified with the exterior product $P(u, v)$, then the conditions of Eq. (47) [and of Eq. (48)] are satisfied. Similarly, when Eq. (22) is satisfied the inner product S is a trace form as defined above. [To see that Albert's extra condition mentioned above is not satisfied, note that $P(z_3, z_3) = 0$ but $S(z_3, z_3) \neq 0$.] Hence the $N=2$ color-charge algebras are trace-admissible Lie algebras.

III. DISCUSSION

I turn next to a discussion, based on the features of the $N=2$ algebras, of what might be expected in the general-N case.

(1) It will clearly be worth the effort to do an explicit calculation of the rank-3, types (3, 0), (2, 1), (1, 2), and (0, 3) algebra tables. These will give additional checks that the 0-component shifts given in Eq. (36) produce color-charge algebras satisfying Eq. (22), and assuming this test is passed, will provide the algebraic basis for the calculation of static baryon properties. A comparison of the (3, 0) and (0, 3) tables for $n=3$ color will also answer (or at least begin to answer) the following question: Are the diagonalized S and P

tables symmetric between quarks and antiquarks when specialized to color-singlet states? [This question cannot be usefully posed for the $N=2$ tables because in the color $n=2$ case, which is the only case for which qq and $\bar{q}\bar{q}$ can couple to form color singlets, the (2, 0), (1, 1), and (0, 2) algebras become trivially identical.] The answer to this question will determine whether a full color theory will require only one set of gluons coupling with outer product $P(u,v)$, or two sets of gluons, one set coupling with outer product $P(u,v)$ and with quark and antiquark charges as given above, and a second set coupling with outer product $*P^a(u,v) = -q^{*abc}(u^b v^c - v^b u^c)$ and with the role of quark and antiquark charges interchanged. In calculating the $N=3$ (and higher N) tables, generalizations of the identities of Table I to larger cycles of chain-linked q's will be needed, but it appears that the needed identities can be obtained by repeated application of the identities contained in Table I.

(2) Assuming that no problems are encountered in the $N=3$ case, I make the following conjectures (weaker conjectures are indicated with question marks):

(i) The rank-N color-charge algebras satisfy Eq. (22), that is, the inner product S is a trace operation as defined above.

(ii) When put in block-diagonal form, the P table has an Abelian sector containing projected initial charges (the generalization of $z_{1,2}$ above) and possible additional elements on which P vanishes, and non-Abelian sectors on which P takes the form of an SU(j) Lie algebra with $j \leq N$,

$$P(z_{(j)k}, z_{(j)l}) = if^{klm}_{(j)} z_{(j)m} . \tag{50}$$

On the non-Abelian sectors, S is a multiple of unity,

$$S(z_{(j)k}, z_{(j)l}) = D_{(j)}\delta_{kl} . \tag{51}$$

The S product vanishes for entries from different non-Abelian sectors or if one entry is from a non-Abelian sector and one is from the Abelian sector,

$$\begin{aligned} S(z_i, z_j) &= 0, \quad z_i, z_j \text{ in different non-Abelian sectors} \\ S(z_i, z_j) &= 0, \quad z_i \text{ in Abelian sector, } z_j \text{ in non-Abelian sector.} \end{aligned} \tag{52}$$

The expectation of S in color-singlet states, $\langle S \rangle_1$, vanishes identically on all Abelian sectors. The expectations $\langle D_{(j)} \rangle$, or at least some dominant subset of them, are largest in color-singlet states. A complete solution of the rank-N, type $(N_q, N_{\bar{q}})$ algebra, containing all information needed for physical calculations, would consist of explicit expressions or algorithms for the diagonalized P and S tables, and for the inverse transformations giving the initial charges $w_1, \ldots, w_N$ in terms of the z's.

(iii) Local algebraic gauge transformations in the underlying U(n) color theory induce rigid global axis rotations in the overlying SU(j) classical Yang-Mills structures. Hence the magnitudes and relative orientations of the effective charges in the SU(j) sectors have intrinsic physical significance.

(iv?) The scalar product expectations $\langle S \rangle_1$ and effective charges in color-singlet states are identical for the types (j,k) and (k,j) algebras, giving color forces in color-singlet states which are symmetric under quark-antiquark interchange.

(3) As an application of (ii), let me assume that when algebraic chromodynamics is quantized, the form of the color-singlet axial-vector current anomaly will turn out to be

$$\mathcal{A} = C\frac{g^2}{32\pi^2}\epsilon_{\mu\nu\sigma\tau} S(f^{\mu\nu}, f^{\sigma\tau}) . \tag{53}$$

Decomposing the algebraic fields $f^{\mu\nu}$ into classical Yang-Mills fields over the SU(j) sectors discussed in (ii),

$$f^{\mu\nu} = \sum_j f^{(j)k\mu\nu} z_{(j)k} , \tag{54}$$

Eq. (53) becomes

$$\mathcal{A} = C\frac{g^2}{32\pi^2}\sum_j D_{(j)}\mathcal{A}(j) , \tag{55}$$

with

$$\mathcal{A}(j) = \epsilon_{\mu\nu\sigma\tau} f^{(j)k\mu\nu} f^{(j)k\sigma\tau} , \tag{56}$$

a classical Yang-Mills anomaly. Since the $f^{(j)k\mu\nu}$ are c numbers, the standard argument that $\mathcal{A}(j)$ is a total divergence[13] applies to Eq. (56), and so the algebraic anomaly $\mathcal{A}$ will also be a total divergence.

(4) So far I have neglected quark flavor (and quark spin, which in the static limit behaves as simply another flavor variable). Assuming that each quark has $F=2n_f$ flavor-spin states, and that (at least in an approximate sense) color couplings are diagonal in flavor, the quark and antiquark charges are replaced by

$$Q^a_q \rightarrow \sum_{f=1}^{F} Q^a_{q(f)}, \quad Q^a_{\bar{q}} \rightarrow \sum_{f=1}^{F} Q^a_{\bar{q}(f)} , \tag{57}$$

with the charges for different values of the flavor index f commuting. Although the algebras generated by the charges of Eq. (57) are more complicated than those with $F=1$, in acting on a component of a state vector $|f_1 f_2 \cdots f_N\rangle$ with definite quark flavors, all the "wrong flavor" charges can always be commuted through to the right and give a vanishing contribution. Hence all calculations in flavor-conserving processes can be done using just the $F=1$ algebra tables, and exhibit the expected $\mathrm{SU}(F)=\mathrm{SU}(2n_f)$ symmetry.

I wish to conclude with some further comments, in a more speculative vein, about the likely form of the extension of classical algebraic chromodynamics to a full quantum field theory.

(1) Up to this point, there is nothing in what I have done to single out a particular choice of color group $\mathrm{U}(n)$. I suspect that the necessity for taking $n=3$ will appear only when one extends classical algebraic chromodynamics to a full quantum field theory. One strong hint pointing toward the choice $n=3$ is summarized in Table VII, where I work out the complete (as opposed to one-particle-projected) color-charge algebras for the U(2), U(3), and $\mathrm{U}(n)$, $n>3$ cases. This analysis shows that $n=3$ is the largest color group for which the complete color-charge algebra is spanned by charges which are either (a) singlets which commute with all charges, or (b) n^2-plets which obey the simple algebras of Eq. (33). Mathematically, this is simply an elementary statement about triangular numbers: $n=3$ is the largest n for which the binomial coefficients $\binom{n}{j}$, $j=0,\ldots,n$, take on only the values 1 or n.

(2) The fact that the natural charges appearing in classical algebraic chromodynamics are projected charges acting only on states of definite quark number suggests that the structure of the quantized theory may be an unconventional one, taking full advantage of the Fock space structure associated with the color degree of freedom. In particular, I think it probable that the underlying quantum field will be a "prequark," giving rise to both quarks and leptons as different color excitations.[14] From Table VII, we see that when the U(3) color charges are factored into the form $A^\dagger A$, the natural creation operators $A^\dagger$ which appear are quark creation operators $a_1^\dagger(1-n_2)(1-n_3),\ldots,$ diquark creation operators $a_1^\dagger a_2^\dagger(1-n_3),\ldots,$ and a triquark creation operator $a_1^\dagger a_2^\dagger a_3^\dagger$. The triquarks, which are color singlets and which obey the Pauli principle, are natural candidates to be leptons. Note in particular that

$$\begin{aligned} a_3^\dagger a_2^\dagger a_1^\dagger a_1 a_2 a_3 + a_1 a_2 a_3 a_3^\dagger a_2^\dagger a_1^\dagger &= 1-n_1-n_2-n_3+n_1n_2+n_1n_3+n_2n_3 \\ &= 1-(\zeta_1+\zeta_2)\,, \end{aligned} \tag{58}$$

which vanishes on one-quark and two-quark states and is 1 on the vacuum and on triquark states, so lepton fields constructed from the triquark operators will have the right canonical commutation relations. Obviously, for this idea to work the prequark field must be constructed so that the momentum, spin, and mass carried by a lepton are the same as those carried by its three color components, not the sums of those carried by the color components. There is striking, and familiar, empirical evidence for this idea: The masses of the known leptons are $m_\nu \approx 0$, $m_e \approx 0.5$ MeV, $m_\mu \approx 106$ MeV, $m_\tau \approx 1850$ MeV, while the effective quark masses are $m_u \approx m_d \approx$ few MeV, $m_s \approx 100$ MeV, $m_c \approx 1850$ MeV, $\ldots$. Clearly, a crucial empirical test of this idea will be whether heavier quarks, such as the one presumably bound in the Υ, have associated heavy leptons.

To carry this idea just a little further, natural choices for the charge operator are either $Q=-\frac{1}{3}(n_1+n_2+n_3)$ or $Q=1-\frac{1}{3}(n_1+n_2+n_3)$, giving quark, diquark, and lepton charge assignments $-\frac{1}{3}$, $-\frac{2}{3}$, -1 or $\frac{2}{3}$, $\frac{1}{3}$, 0, respectively. Through a suitable Fock space construction, it may be possible to reinterpret the diquark degrees of freedom as quarks. I do not at this point have an identification which gives a satisfactory account of the known quark-lepton spectrum, but I believe it possible that the construction of a consistent quantum algebraic chromodynamics will reveal one. Some other open questions are the following: Are 9 gluons needed, or 18? Is the photon an additional gluon, or is the 0 component of the color field coupled so as to play this role as well? Are weak interactions introduced through an additional flavor-changing gauging at the algebraic level, or does the color field cause flavor-changing as well as flavor-diagonal transitions which lead to an effective $\mathrm{SU}(2)\times\mathrm{U}(1)$ weak-electromagnetic gauge structure at the overlying Yang-Mills level? (Note that flavor-changing color charges have different algebraic properties from flavor-conserving ones, and may not lead to additional strong forces.) Finally, how is the gluon field to be quantized? Here I make a guess, which is that the algebraic gluon field is quantized by quantizing the overlying classical Yang-Mills fields according to the usual rules.

(3) Let me close with some brief remarks about the cosmological implications of unifying quarks and leptons in the manner described above. One of the most striking facts about our universe is that charge neutrality is apparently achieved, not by balancing protons against antiprotons and electrons against positrons, but rather by balancing protons against electrons. While this can simply be accepted as an empirical fact, quark-lepton unification into a prequark field offers the possibility

TABLE VII. Quark color-charge (number-preserving) C^* subalgebras in U(2), U(3), and U(n), $n>3$ color theories.

U(2) color-charge algebra

Quark number of state acted on	1	2
Singlet charge [a]	$\xi^0=\frac{1}{2}\zeta_1,\ \zeta_1=n_1+n_2-2n_1n_2$	$\zeta_2=n_1n_2$
Triplet charge [b]	$\xi^A=\psi_P^\dagger\frac{1}{2}\lambda^A\psi_P$	$\cdots$
U(2) form	$\xi^a=\psi_P^\dagger\frac{1}{2}\lambda^a\psi_P$	ζ_2
Dimensionality	$\binom{2}{1}^2=4$	$\binom{2}{2}^2=1$
Algebra [c]	$\xi^a\xi^b=q^{abc}\xi^c$	$\zeta_2{}^2=\zeta_2$
Decomposition of $\psi^\dagger\frac{1}{2}\lambda^a\psi$:	$\psi^\dagger\frac{1}{2}\lambda^A\psi=\xi^A,\quad \psi^\dagger\frac{1}{2}\lambda^0\psi=\frac{1}{2}(n_1+n_2)=\frac{1}{2}(\zeta_1+2\zeta_2)$	

U(3) color-charge algebra

Quark number of state acted on	1	2	3
Singlet charge [a]	$\xi_+^0=6^{-1/2}\zeta_1,\quad \zeta_1=n_1+n_2+n_3-2(n_1n_2+n_1n_3+n_2n_3)+3n_1n_2n_3$	$\xi_-^0=-6^{-1/2}\zeta_2,\quad \zeta_2=n_1n_2+n_1n_3+n_2n_3-3n_1n_2n_3$	$\zeta_3=n_1n_2n_3$
Octet charge [b]	$\xi_+^A=\psi_P^\dagger\frac{1}{2}\lambda^A\psi_P$	$\xi_-^A=\psi_{P'}^\dagger(-\frac{1}{2}\lambda^{*A})\psi_{P'}$	$\cdots$
U(3) form	$\xi_+^a=\psi_P^\dagger\frac{1}{2}\lambda^a\psi_P$	$\xi_-^a=\psi_{P'}^\dagger(-\frac{1}{2}\lambda^{*a})\psi_{P'}$	ζ_3
Dimensionality	$\binom{3}{1}^2=9$	$\binom{3}{2}^2=9$	$\binom{3}{3}^2=1$
Algebra [c]	$\xi_+^a\xi_+^b=q^{abc}\xi_+^c$	$\xi_-^a\xi_-^b=-q^{bac}\xi_-^c$	$\zeta_3{}^2=\zeta_3$
Decomposition of $\psi^\dagger\frac{1}{2}\lambda^a\psi$:	$\psi^\dagger\frac{1}{2}\lambda^A\psi=\xi_+^A+\xi_-^A,\quad \psi^\dagger\frac{1}{2}\lambda^0\psi=6^{-1/2}(n_1+n_2+n_3)=6^{-1/2}(\zeta_1+2\zeta_2+3\zeta_3)$		

U(n) color-charge algebra $n>3$

Quark number of state acted on	1	$j\neq 1, n-1$	$n-1$	n
U(n) charge [d]	$\xi_+^a=\psi_P^\dagger\frac{1}{2}\lambda^a\psi_P$	$\psi^\dagger_{(j)\alpha}\psi_{(j)\beta}$ $\alpha,\beta=1,\ldots,\binom{n}{j}$	$\xi_-^a=\psi_{P'}^\dagger(-\frac{1}{2}\lambda^{*a})\psi_{P'}$	$\zeta_n=n_1\cdots n_n$
Dimensionality	$\binom{n}{1}^2=n^2$	$\binom{n}{j}^2$	$\binom{n}{n-1}^2=n^2$	1
Algebra	$\xi_+^a\xi_+^b=q^{abc}\xi_+^c$	Contains U(n) representations which are not n^2-plets or singlets [e]	$\xi_-^a\xi_-^b=-q^{bac}\xi_-^c$	$\zeta_n{}^2=\zeta_n$

[a] The ζ's are all idempotent: $\zeta_1{}^2=\zeta_1,\ \ \zeta_2{}^2=\zeta_2,\ \ \zeta_3{}^2=\zeta_3$.

[b] As in the text, $\psi_P=\begin{pmatrix}a_1(1-n_2)\\a_2(1-n_1)\end{pmatrix}$ in the U(2) case, $\psi_P=\begin{pmatrix}a_1(1-n_2)(1-n_3)\\a_2(1-n_1)(1-n_3)\\a_3(1-n_1)(1-n_2)\end{pmatrix}$ in the U(3) case.
The second U(3) spinor is $\psi_{P'}=\begin{pmatrix}a_2a_3(1-n_1)\\a_3a_1(1-n_2)\\a_1a_2(1-n_3)\end{pmatrix}$.

[c] All products of charges acting on states with different quark number vanish.

[d] As above,

$$\psi_P=\begin{pmatrix}a_1\prod_{j\neq 1}(1-n_j)\\ \vdots\end{pmatrix},\quad \psi_{P'}=\begin{pmatrix}\prod_{j\neq 1}a_j(1-n_1)\\ \vdots\end{pmatrix}.$$

The $\psi_{(j)\beta}$ are $\binom{n}{j}$ linearly independent operators of the form $a_{i_1}\cdots a_{i_j}(1-n_{i_{j+1}})\cdots(1-n_{i_n})$, with $i_1,\ldots,i_n$ a permutation of $1,\ldots,n$.

[e] For example, in U(4) there are 36 color charges acting on the two-quark sector. Since $\psi_{(2)\beta}$ spans a $\underline{6}$ representation of SU(4), the 36 charges transform as $\underline{6}\times\underline{6}=\underline{1}+\underline{15}+\underline{20}''$ under SU(4). [See D. Amati *et al.*, Nuovo Cimento $\underline{34}$, 1732 (1964), for a discussion of SU(4) representations.]

of explaining the proton-electron balance as a consequence of a postulate that the prequark quantum number of the universe is zero. If the universe has zero prequark quantum number, then all matter could have been created gravitationally, and if ordinary matter has zero prequark quantum number modulo the addition of neutrinos or antineutrinos, then the problem of particle number nonconservation in black-hole evaporation could be resolved. It will be especially interesting to see, in connection with these speculations, whether 9 gluons suffice to construct a satisfactory theory of strong interactions, or whether 18 are needed. As I have already noted, the 9-gluon theory gives an asymmetry between quark and antiquark forces in color-nonsinglet states.[15] If such states were predominant in the early very hot stages of the universe, then a 9-gluon theory might well predict an asymmetric condensation of matter with zero net prequark number, yielding color-singlet states with nonzero, and balancing lepton and baryon numbers.

Added note. After this manuscript was submitted for publication, I learned from R. Giles and L. McLerran that they are studying a related (but not identical) algebraic approach to the problem of finding a semiclassical approximation to QCD. See R. Giles and L. McLerran (unpublished).

NOTES ADDED IN PROOF

(1) Although the definitions of Eqs. (47)–(49) are satisfied by the outer and inner products $P(u,v)$ and $S(u,v)$, the n^2-plet elements u,v are not elements of a field (they are operators with a multiplication rule defined only implicitly by their construction), and hence the standard definition of a Lie algebra (with a trace) is not satisfied. I wish to thank V. Rittenberg for pointing this out. This means that all of Conjecture (ii) remains to be proved, including the assertion that the diagonalized P table is composed entirely of Lie algebras.

(2) The assertion following Eq. (46), that the general algebraic gauge transformation of Eq. (45) leaves the relative orientations of the effective charges unchanged, needs amplification. This assertion is true if, as I tacitly assumed, the effective charges are always defined as the decomposition on the diagonalizing bases of the original quark and antiquark charges, which satisfy the color commutation relations $[Q^A_{(n)}, Q^B_{(m)}] = \delta_{nm} i f^{ABC} Q^C_{(n)}$. These are the charges which specify the state vectors for the quantum mechanical color problem. If, alternatively, one defines effective charges as the decomposition of the gauge transformed charges w_1', w_2' on the diagonalizing bases, these effective charges remain invariant in magnitude, but the angle between them changes for general local algebraic gauge transformations. However, the new charges w_1', w_2' do not in general satisfy the color commutation relations, and thus do not provide a unitarily equivalent quantum mechanical description of the color state of the system. If the class of allowed gauge transformations is restricted to that which gives $w_{1,2}'$ which satisfy the color commutation relations, then the effective charges defined by $w_{1,2}'$ do have the same relative orientation as the original effective charges. I wish to thank R. Gonsalves, R. Giles and L. McLerran for a discussion of this point.

(3) A simple calculation shows that $S(Q_q, Q_q) = n/2$, $S(Q_{\bar q}, Q_{\bar q}) = (n/2)^3$ which implies that the symmetry between quark and antiquark persists in the overlying classical structures, irrespective of color state. Thus, Conjecture (*iv*?) is in fact false; an 18 gluon version of the theory is needed to give a charge conjugation symmetric theory of strong forces.

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[1] I follow the metric convention $\eta_{00} = 1$, $\eta_{ij} = -\delta_{ij}$, with contravariant spatial components identified with physical vectors.

[2] J. E. Mandula, Phys. Rev. D 14, 3497 (1976).

[3] Related peculiarities of gauge theories, having to do with their vacuum structure, have been studied by C. Callan, R. Dashen, and D. Gross, Phys. Lett. 63B, 334 (1976); R. Jackiw and C. Rebbi, Phys. Rev. Lett. 37, 172 (1976); G. 't Hooft, *ibid.* 37, 8 (1976). After this work was completed, I learned from C. Sommerfield that he and M. K. Prasad had studied matrix solutions of classical Yang-Mills equations in M. K. Prasad and C. M. Sommerfield, Nucl. Phys. B110, 153 (1976).

[4] C. N. Yang and R. L. Mills, Phys. Rev. 96, 191 (1954); R. Shaw (unpublished).

[5] I have proved that $\pm q^{abc}$ and $\pm q^{*abc}$ are the only linear combinations of f^{abc} and d^{abc} which satisfy Eq. (7).

[6] S. K. Wong, Nuovo Cimento 55A, 689 (1970).

[7] Taking

$$\mathcal{L}_{(\text{interaction+quark kinetic})} = -gS(b_\mu, J^\mu) - \int d\tau \sum_n m_{(n)} [-u_n^\alpha(\tau) u_{n\alpha}(\tau)]^{1/2}$$

satisfies Eq. (25a), but does not satisfy Eq. (25b). A Dirac-field construction is needed for this part of the Lagrangian. See in this connection Ref. 6.

[8]Consequently, if I had started from a P product constructed with $-q^{*abc}$ replacing q^{abc}, the nonvanishing shift would have appeared in the charge Q_q.

[9]A. J. Macfarlane *et al.*, Commun. Math. Phys. 11, 77 (1968), Eq. (2.7) and Eqs. (2.12)–(2.18); P. Cvitanović, Phys. Rev. D 14, 1536 (1976).

[10]I wish to thank A. Borel for pointing this out to me.

[11]A. A. Albert, Trans. Am. Math. Soc. 64, 552 (1948).

[12]R. M. Santilli, Nuovo Cimento 51A, 570 (1967); Suppl. Nuovo Cimento 6, 1225 (1968).

[13]See, e. g., H. Fritzsch *et al.*, Phys. Lett. 47B, 365 (1973).

[14]For a review of more conventional ideas about quark-lepton unification, see H. Fritzsch, lectures given at the International Summer Institute for Theoretical Phys Physics, Bielefeld, Germany, 1976 (unpublished); M. Gell-Mann, P. Ramond, and R. Slansky, Rev. Mod. Phys. (to be published); and J. G. Pati, Univ. of Maryland Report No. 78-073, 1978 (unpublished).

[15]In choosing which excitations to label quarks and which to label antiquarks, I have assumed that the larger $\langle D\rangle$ values appearing in what I have called the qq case (as compared with the $\bar{q}\bar{q}$ case) imply stronger color forces.