Disconnected gauge groups and the global violation of charge conservation

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We show that a gauge group is naturally enlarged to include certain discrete internal symmetries such as charge conjugation. The full gauge group is then disconnected. If space-time is not simply connected, then a mechanism for the global violation of charge conservation results. Examples are given in space dimensions one through three. Gauge theories based on non-Abelian simple compact Lie algebras are also discussed.

I. INTRODUCTION

Our objective is to determine the meaning that can be given to the notion of gauging a discrete internal symmetry. We find that it is natural to include symmetries such as charge conjugation in the gauge group. On space-time manifolds which are not simply connected, this leads to a picture of charge and charge conservation which is strikingly different from the ordinary one. Experts will observe that the discussion can be reexpressed as the problem of reducing the structure group of a principal bundle to the component of the identity. The reader is not assumed to be an expert in differential topology.

The inclusion of a discrete internal symmetry in the gauge group does not lead to an additional associated gauge field. Instead, attention is focused upon the fundamental process of covering the manifold with overlapping patches, each equipped with its own set of conventions for measurement. The problem is to determine whether or not these local conventions are related in such a way that a global convention can be introduced. On simply-connected manifolds, a global convention is always admissible. Non-simply-connected manifolds admit structures which are inconsistent with a global convention. This leaves us with an interesting situation in which charge, for instance, can be defined and is conserved locally, but no global definition of positive charge is possible. This allows for global violations of charge conservation. As bizarre as this seems, we can find in it no inconsistency.

A byproduct of the analysis will be the realization that there is a very natural sense in which ordinary charge conjugation and its non-Abelian generalizations can be considered to be discrete gauge transformations. Or, in another way of putting it, the full gauge symmetry group is disconnected. Of the compact simple Lie algebras, the only one which leads to a gauge group with more than two components is so(8). Here there are six components.

II. FRAMEWORK

In this section, we will review the application of the fundamental concepts of gauge invariance[1] to a Lagrangian with internal symmetry. The process will be strictly conventional. However, we will apply it both to the continuous and to the discrete symmetries of the Lagrangian. At the end, we will understand the consequences of including a discrete internal symmetry in the gauge group.

Rather than confound both the reader and the author by attempting to treat the most general case, we will first consider a simple example. It will then be clear how to treat a large number of more interesting models. Some discussion of these will appear in Sec. V.

Begin with the Lagrangian for two equal-mass free scalar fields:

$$\mathcal{L}=\tfrac{1}{2}\partial_\mu\varphi_1\partial^\mu\varphi_1-\tfrac{1}{2}m^2\varphi_1{}^2+\tfrac{1}{2}\partial_\mu\varphi_2\partial^\mu\varphi_2-\tfrac{1}{2}m^2\varphi_2{}^2\,. \tag{2.1}$$

The internal-symmetry group is O(2) which is disconnected with two components. The component which does not contain the identity contains the orientation-reversing operation

$$d:\begin{cases}\varphi_1\to\varphi_1\,,\\ \varphi_2\to-\varphi_2\,.\end{cases} \tag{2.2}$$

Thus we can also think of the symmetry group as the direct product group

$$O(2)=SO(2)\times\{1,d\}\,. \tag{2.3}$$

Observations of the system are made locally. We therefore think of a large system of observers, each responsible for a small open region U_i of the connected space-time manifold M. We assume that the U_i cover M and that they and all multiple intersections of them are contractible.[2] M, however, may have some nontrivial topology.

Concentrate initially on one particular region U_i. This observer has no way to make an absolute distinction between excitations which are related by a symmetry operation. He must establish an arbitrary convention that defines which excitations are φ_1 and which are φ_2. Since we are dealing with a local field theory, this must be done separately at each point in U_i. We assume that it is done in a smooth way.

Some discussion of this point is necessary. We can think of the establishment of the convention as the process of selecting an orthonormal basis in the internal-symmetry space at each point. It is assumed that there is a process of parallel transport by which the frame at p can be carried to p'. The demand that the convention be smooth is that the relationship between the frame transported to p' and the one at p' should depend smoothly on p'.

From this we can deduce two facts: First, all of the frames in U_i must have the same orientation. This follows from continuity and from the fact that U_i is simply connected. Physically, this says that the observer can give an unambiguous definition of positive charge everywhere on U_i. Second, it is possible to introduce the connection (gauge field) in the usual way. This gives the infinitesimal rotation of a frame when it is parallel transported to a neighboring point. We assume that the reader is familiar with the connection and its properties. Only those points which are important to our purpose will be mentioned.

The choice of a basis in the internal-symmetry space over U_i is an arbitrary process upon which the physics should not depend. In order to make the theory invariant under this choice, the connection is introduced into the Lagrangian

$$\begin{aligned} \mathcal{L}_M^{(i)} &= \tfrac{1}{2}[(\partial + A^{(i)})_\mu \varphi^{(i)}]^T[(\partial + A^{(i)})^\mu \varphi^{(i)}] \\ &\quad - \tfrac{1}{2}m^2\varphi^{(i)T}\varphi^{(i)}, \\ \varphi^{(i)} &= \begin{pmatrix} \varphi_1^{(i)} \\ \varphi_2^{(i)} \end{pmatrix}, \\ A_\mu^{(i)} &= a_\mu^{(i)} T, \\ T &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \end{aligned} \tag{2.4}$$

The superscript i identifies quantities associated with the region U_i. The use of (2.1) makes sense only if parallel transport is integrable on U_i. If the internal geometry is to be dynamical, a kinetic energy term for the connection is introduced as usual:

$$\begin{aligned} \mathcal{L}^{(i)} &= \mathcal{L}_G^{(i)} + \mathcal{L}_M^{(i)}, \\ \mathcal{L}_G^{(i)} &= \tfrac{1}{8}\,\mathrm{Tr}[F_{\mu\nu}^{(i)}F^{(i)\mu\nu}], \\ F_{\mu\nu} &= \partial_\mu A_\nu^{(i)} - \partial_\nu A_\mu^{(i)}. \end{aligned} \tag{2.5}$$

Observe that (2.5) is invariant under any change of internal-symmetry basis on U_i. This includes changes in which the orientation of the frames is reversed. Thus we have a gauge symmetry of $\mathcal{L}^{(i)}$ on U_i which is given by

$$\begin{aligned} \varphi &\to g^{-1}\varphi, \\ A &\to g^{-1}(\partial + A)g, \end{aligned} \tag{2.6}$$

with g a smooth map,

$$g:\ U_i \to \mathrm{O}(2), \tag{2.7}$$

which may lie in either component of O(2). A transformation that reverses the orientation at each point can be written

$$\begin{aligned} g &= dg_0, \\ g_0 &: U_i \to \mathrm{SO}(2), \\ d &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \end{aligned} \tag{2.8}$$

It gives

$$\begin{aligned} \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix} &\to g_0^{-1}\begin{pmatrix} \varphi_1 \\ -\varphi_2 \end{pmatrix}, \\ A &\to g_0^{-1}(\partial - A)g_0. \end{aligned} \tag{2.9}$$

We see that it is a combination of charge conjugation and an orientation-preserving gauge rotation. With this enlargement of the gauge group the field strength and the current are not gauge invariant but only gauge covariant, each changing sign under d. (Recall that in non-Abelian theories, field strengths and currents are only gauge covariant even under gauge transformations connected to the identity.)

From studying symmetry on one region U_i, we have been led to two conclusions: First, it is natural to include the discrete transformation of symmetry frames in the gauge group. Second, this does not lead to associating a gauge field with the discrete transformations.

It is time to ask what happens on an overlap region

$$U_{ij} = U_i \cap U_j. \tag{2.10}$$

In such a region, there are two observers studying the same physical system. Each observer has set up his own basis in the internal-symmetry space over U_{ij}. The relationship between these two bases is a gauge transformation

$$g_{ij} : U_{ij} \to \mathrm{O}(2). \tag{2.11}$$

This map may lie in either component of O(2). That is, observers i and j may have opposite charge conventions. This is consistent, since if they have opposite conventions about charge, they will also have opposite conventions about field. They will agree about what an excitation does, but they will disagree about what it is called. The Lagrangians for i and j are each gauge invariant, and the relationship between the i and j descriptions is a gauge transformation. Therefore, observers i and j will agree when observations are expressed gauge invariantly. (We emphasize again that F and the current j are not invariant under the discrete transformations. Physical combinations such as F^2 and $A \cdot j$ are invariant.)

We have now established a framework in which to operate. The theory lives on a manifold M which is covered by the $\{U_i\}$. Over each point of each U_i, a basis in the internal-symmetry space is chosen. On the overlap regions $\{U_{ij}\}$, there are gauge transformations $\{g_{ij}\}$ which relate the i and j bases. All of this is entirely standard except that we included the discrete symmetry in the gauge group. This shows up in the transition functions $\{g_{ij}\}$. They tell us whether or not the i and j observers have the same charge convention.

III. SIMPLY-CONNECTED MANIFOLDS

In the last section, we studied the consequences of including a discrete internal symmetry in the gauge group. We find our space-time manifold M covered with regions U_i, each with its own charge convention. This does not seem to agree with ordinary experience. Charges can be moved around over a great range of distances. No difficulty is experienced in keeping track of their identities. How can this be understood?

Begin with a positive charge in region U_i. Transport it to a neighboring region U_j. If observer j thinks it is positive, fine. If not, reverse the orientation of his symmetry frames. It will then appear positive. Continue to another region. In this way, we can attempt to extend the convention to all of M. If we succeed, then one charge convention applies to all of M, and we have effectively reduced the gauge group from O(2) to SO(2). What conditions guarantee that this is possible?

Suppose we attempt to orient all the frames by beginning in region U_1. To orient the frames on U_j, a charge which is positive in U_1 is transported to U_j. The orientation of the U_j frames is reversed or not so that the charge appears positive there. Now if all other paths to U_j from U_1 give the same orientation for U_j, everything is fine. If this is true for all regions U_j, we have succeeded in giving a global orientation to the internal-symmetry space. The only possible hitch is that there may be a region U_j for which a path P_1 from U_1 leads to one choice of orientation on U_j and another path P_2 leads to the opposite choice. However, if P_1 and P_2 are homotopic, this is impossible. Thus, if M is simply connected, it is always possible by an appropriate choice of orientation over each U_i to give a global definition of charge.

If space-time is simply connected, our conventional use of the term "positive charge" makes sense. Over a very broad range of distances space-time does appear to be simply connected.

IV. NON-SIMPLY-CONNECTED MANIFOLDS

This section will study the notion of charge on some manifolds which are not simply connected. In order to keep the discussion as intuitive as possible, we will deal with examples in which space-time is a product manifold

$$M_{ST} = M_S \times T\,. \tag{4.1}$$

From now on, we will refer to the space manifold M_S simply as M. Furthermore, we will look at static electric fields on M.

It is important to keep in mind that over a given manifold M there may be many inequivalent internal-symmetry spaces, and for each of these, many ways to choose the bases. On a simply-connected manifold, all choices will be equivalent to a global charge convention. Over a manifold which is not simply connected, an internal-symmetry space may or may not admit a global convention.

Before we discuss the examples, another general point can be made. A compact manifold (without boundary) on which there is a global definition of charge must be neutral. Intuitively, there can be no net charge on such a manifold because the electric field lines have nowhere to go. Formally, we integrate Gauss's law[3]

$$*d*E = \rho \tag{4.2}$$

to obtain

$$Q = \int_M *\rho = \int_M d*E = \int_{\partial M} *E = 0 \tag{4.3}$$

since

$$\partial M = 0\,. \tag{4.4}$$

Thus simply-connected compact manifolds such as S^n, $n \geq 2$ can never be charged. However, on compact manifolds which are not simply connected S^1, T^n, etc., we can circumvent this result. This will be seen in the examples.

Example 1

Let M be a circle,

$$M = S^1\,. \tag{4.5}$$

Although we should cover S^1 with three regions, it is sufficient and easier to use only two. Thus let U_1 be the bottom of S^1 and U_2 the top so that

$$S^1 = U_1 \cup U_2 , \tag{4.6}$$

and the intersection is two disconnected regions,

$$\begin{aligned} &U_{12} = U_1 \cap U_2 = V_1 \cup V_2 , \\ &V_1 \cap V_2 = \phi . \end{aligned} \tag{4.7}$$

From the point of view of the discrete gauge symmetry, there are essentially two different structures possible. If the internal-symmetry space has no "twists" in it, then on V_1 and V_2 the relative orientation of the frames of U_1 and U_2 will be either both the same or both opposite. If the latter, reverse the orientation on U_2. This gives a global charge convention. On U_{12} the charge and field direction conventions of U_1 and U_2 agree. Thus the electric field is global and by (4.3) the net charge must vanish.

We will now take a brief look at the behavior of charges in this case. For simplicity, assume that initially there is no background field; the field strength vanishes and the charge density is zero. A positive and negative charge can be introduced close to each other so that the field is constant between them the short way and zero between them the long way. They will attract. Carry the positive charge the short way around to the other side of the negative charge. There will still be a constant field between them the short way and a zero field the long way. They will attract in the short direction. Now return to the original arrangement and carry the positive charge the long way around to the opposite side of the negative charge. There will now be a zero field between them the short way and a constant field the long way. They will attract in the *long* direction. Thus knowledge of the initial background field and the positions of charges does not predict the forces. Homotopy information about the field must also be specified. String aficionados may find this appealing.

Now consider another possible structure with a twist. In this case, the orientation of the frames of U_1 and U_2 will agree on one of the overlaps and will disagree on the other. Arrange these to be V_2 and V_1, respectively. On V_2 there will be agreement on the sign of the charge and the direction of the field. On V_1, U_1 and U_2 will have opposite conventions; however, they will agree on physical things such as the direction of the force on a charge.

In this case, a background electric field is *not* allowed. Nonzero total charge *is* allowed. An electric field can be consistently associated with each charge, and the total field is the sum of the fields from each charge. While this seems satisfactory and in accord with ordinary experience, no global charge convention is possible. Some rather bizarre global properties of charge result.

In each of regions U_1 and U_2, Maxwell's equations are separately valid. So local charge conservation is good, and the net charge seen by U_i can change only when currents flow across the boundaries of U_i. The global situation is not so familiar. A positive charge in U_1 carried into U_2 through V_1 will enter U_2 as a negative charge. If the charge is now carried across U_2 and into U_1 through V_2, it will reenter U_1 as a negative charge. Thus, while neither observer has seen any local violation of charge conservation, the charge in U_1 plus the charge in U_2 has changed. This is possible because there is no proper global definition of charge. On V_1, U_1 and U_2 do not agree. Thus a charge carried around a circle does not maintain its identity. This is no more unusual than the fact that a Möbius strip is not an orientable surface.

Example 2

Let M be a cylinder,

$$M = S^1 \times R^1 . \tag{4.8}$$

This example is basically the same as the previous one. It is included to show that the results are not special to one dimension and to help introduce the subsequent models.

Let the regions U_1 and U_2, and V_1 and V_2 be the product of R^1 with the corresponding regions of the previous example. Assume first that the conventions on V_1 and V_2 agree. Since field lines can escape to infinity, it is possible to have a net charge on the cylinder. As a consequence, the total field on M can be regarded as the sum of the fields from each of the charges and a possible background field which we will assume is zero.

The field of a point charge is easily exhibited. Begin with the plane R^2 and an infinite line of equal equally spaced point charges. Solve for the field of this set of charges. Identify all points separated by a translation parallel to the line and equal to the charge spacing. This gives a cylinder and the correct field.

Now assume that the orientations of U_1 and U_2 are opposite on V_1. The discussion given for this case in Example 1 applies here. We will content ourselves with constructing the field of a single point charge. Use a construction similar to the previous one, except that the sign of the charges on the line in the plane should be alternated. Solve for the field in the plane with the charges on the x axis at positions

$$Q=+1 \quad \text{at} \quad x=0, \pm 2, \pm 4, \pm 6, \ldots,$$
$$Q=-1 \quad \text{at} \quad x=\pm 1, \pm 3, \pm 5, \ldots . \tag{4.9}$$

Cut out the strip bounded by the lines

$$x=\tfrac{2}{6} \text{ and } x=-\tfrac{2}{6}, \tag{4.10}$$

and call this U_1. For region U_2 use the strip between

$$x=\tfrac{1}{6} \text{ and } x=\tfrac{5}{6}. \tag{4.11}$$

Form these strips into a cylinder by overlapping the $x=\frac{1}{6}$ edge of U_2 to $x=\frac{1}{6}$ on U_1 and the $x=\frac{5}{6}$ edge of U_2 to $x=-\frac{1}{6}$ on U_1. The first overlap will be V_2 with the fields agreeing, and the second will be V_1 with the fields opposite.

Example 3

Take

$$M=T^2=S^1\times S^1. \tag{4.12}$$

This manifold differs from S^1 and $S^1\times R^1$ in that the fundamental group has two generators rather than one. This gives us three possibilities. There may be a global charge convention. The charge may reverse when carried around the torus on a path corresponding to one of these generators but not the other. The charge may reverse when carried around in either way.

Consider the second possibility. Let the regions U_1 and U_2 be two finite-length cylinders on which the charge convention is global. Join them into a torus by overlapping one end from each in a region V_2 so that the conventions agree there. When the other pair of ends are overlapped to form V_1, a twist is put in the internal-symmetry space so that opposite conventions result in this region. Then if a charge is carried around a loop which links the hole in the torus (thought of as embedded in R^3), it will return unchanged. However, a path that goes around through V_1 and V_2 will cause the charge to return reversed.

We can change this example into one which is more interesting. Deform the torus until it looks like a very large S^2 with a small handle attached. Do it so that V_1 is in the handle. If a point on this S^2 far from the handle is identified as infinity, the next example is suggested.

Example 4

Begin with R^2. Cut out two D^2 disks. The boundaries of the holes which are left are two S^1's. Join these two S^1's with a handle $S^1\times D^1$. Thus the manifold M is a plane with a handle attached. Arrange to have a reversal of charge take place in the handle. Moving a charge around in any way on the plane will leave it unchanged. However, if it is carried through the handle, it will come out with the opposite charge.

Example 5

Our last example is the most relevant, but is not as easy to visualize as the others. Generalize Example 4 to one higher dimension. From R^3 remove two three-disks D^3. The surfaces of the resulting holes are spheres S^2. Connect these boundaries with a "handle" $S^2\times D^1$. Proceed as above.

The conclusion of this section is that manifolds which are not simply connected may have a "twist" in their internal-symmetry spaces which is inconsistent with a global convention for charge. As a result, a charge carried over some nontrivial loop may reverse its sign. At the level of classical gauge fields, matter fields, and particles on manifolds of given geometry, this is consistent.

We have not discussed the dynamics of the manifold geometry or the quantum mechanics of these configurations. We do not know how this would affect the classical discussion that we have given. However, some vague comments will be inserted anyway.

(1) The topology of space-time on a cosmological scale is not known.

(2) The topology of space-time on a scale smaller than the 10^{-15} cm associated with present high-energy experiments is not known. Indeed, it has been emphasized that quantum fluctuations of the topology of space-time should occur at distances comparable to the Planck length.[4]

(3) Fluctuations in the topology of the internal-symmetry space may also be possible. A change of relative orientation may occur with no cost in kinetic or potential energy whenever all fields vanish on the overlap region.

(4) Suppose that a small handle exists and that a positive charge passes through it and emerges as a negative charge. From a large distance, it will not appear that charge conservation has been violated. Rather, it will appear that two positive charges have been left behind in the handle. However, a close examination of the inside of the handle will not reveal the existence of any such charges.

(5) Electrodynamics on manifolds which are not simply connected has been discussed previously.[5] It was assumed that the field strength two-form is global. The possibility that we are considering was thereby excluded.

V. NON-ABELIAN GROUPS

The discussion in the previous sections can be generalized to any discrete internal symmetry.

The discrete symmetries associated with gauge theories based on simple compact Lie groups will be considered here.

An SU(N) gauge theory with matter fields in the defining representation of the group has a discrete internal symmetry which is the natural generalization of charge conjugation in a U(1) theory. In the SU(N) case, the name particle conjugation is more appropriate in that the symmetry interchanges particles and antiparticles while some of the non-Abelian charges change sign and some do not. If the T_i are the $N \times N$ anti-Hermitian matrix generators of SU(N), then the gauge field is conveniently expressed as the matrix one-form

$$A = A^i_\mu T_i dx^\mu . \tag{5.1}$$

The action of the particle conjugation symmetry on the gauge field is then given by

$$A \to -A^T. \tag{5.2}$$

Since (5.2) is an automorphism of the Lie algebra, it is a symmetry of the SU(N) Yang-Mills action. For $N=2$, this transformation of the gauge field can be obtained by a global SU(2) gauge rotation. But for all larger N, the transformation (5.2) is disconnected from the identity in the group of automorphisms of the Lie algebra.[6] Thus, the SU(N) case is very similar to the U(1) case. It may not be possible to give a global particle-antiparticle convention on non-simply-connected manifolds.

Some general statements can be made about SO(N) theories. Individual attention must then be given to several special cases. Gauge theories based on the algebra of

$$SO(2N+1)\,, \quad N = 1, 2, \ldots \tag{5.3}$$

with a single multiplet of matter fields carrying the defining representation of the group will have the full O($2N+1$) symmetry. However, the discrete transformation

$$d = -1\,, \tag{5.4}$$

which lies in the component of O($2N+1$) disconnected from the identity, is a multiple of the identity. As a consequence, it acts as the identity on the gauge fields and on the

$$\tfrac{1}{2}(2N+1)(2N) = N(2N+1) \tag{5.5}$$

conserved charges. Thus it can be simultaneously diagonalized with the commuting charges. States will transform into themselves under the discrete symmetry. Thus, even in the cases in which a global convention for this discrete symmetry cannot be established, the effects are not as striking as in the previous examples.

Among the so($2N$) algebras there are three special cases. SO(2) is not semisimple. We have already treated this case. SO(4) is semisimple but not simple. In the algebra,

$$so(4) = su(2) \oplus su(2)\,. \tag{5.6}$$

SO(8) is simple, but it is special in that it is the only simple algebra which has six components to its automorphism group.[6] All others have one or two. The general results will apply to these special cases. Consequences of the special properties of SO(4) and SO(8) will be discussed at the end.

Gauge theories based on the algebra of SO($2N$) with matter fields in the defining representation of the group will have a full O($2N$) symmetry. We can select, as a standard element in the component disconnected from the identity, the matrix

$$d = \text{diag}\ (1, 1, \ldots, 1, -1)\,. \tag{5.7}$$

This acts nontrivially on the charges and the gauge fields. A basis can be chosen in which $2N-1$ of each change sign and the rest do not. This symmetry cannot be obtained by a gauge rotation connected to the identity.[6] SO($2N$) has N commuting charges and N independent invariant polynomials in the charges. In both cases, these can be chosen so that one changes sign under (5.7) and the remainder do not. Refer to the charge which changes sign as Q_- and the (Nth-order) invariant which changes sign as C_-. Those which do not change sign will be referred to collectively as Q_+ and C_+, respectively. The stable particles and the resonances of the theory will fall into multiplets. (We treat the case of normal rather than broken symmetry.) The multiplets will be labeled by the values of C_+ and C_-, and the members of a multiplet will be distinguished by their values of Q_+ and Q_-. Let D be the symmetry of the theory associated with the matrix of (5.7). Consider the action of D on the multiplets. If C_- vanishes for the multiplet and Q_- is zero for all members of the multiplet, then we are dealing with a singlet, and D will carry the state into itself (with a multiplication by -1 possibly). This is similar to the SO($2N+1$) case. If C_- vanishes for the multiplet, but Q_- takes on nonzero values, then D will permute the members of the multiplet. States with opposite values of Q_- will be interchanged. Thus, the action of D on any particular member of the multiplet could be obtained by an appropriate combination of global symmetry rotations. Finally, if C_- is not zero for the multiplet, then there will be another related multiplet with the opposite value of C_- and D will interchange these multiplets of particles. This situation is similar to the charge-conjugation example. C_- plays the role of charge and D that of charge conjugation. On non-simply-connected manifolds, it may not be possible to give a global convention for the C_- labeling of states.

For $N>2$, the fundamental gauge fields have C_- zero. For $N>1$, the fundamental matter fields have C_- zero. Thus, if particles with C_- nonzero exist, they will be bound states unless N is 1 or 2.

The SO(4) theory has six gauge fields and four real scalar fields. Although the algebra is a direct sum (5.6), this representation of SO(4) is irreducible. Thus, the gauge part of the action is a sum of contributions from the two subalgebras, but the matter part of the action is not. The symmetry (5.7) corresponds to interchanging the two subalgebras. Thus, when the manifold is not simply connected, it may not be possible to give a global convention for dividing the six gauge bosons into pairs of three corresponding to (5.6).

The SO(8) theory may have more discrete symmetries. The gauge-field action has the full discrete symmetry group of order 6. However, the amount of symmetry in the matter-field action seems to depend on the representation of the group that acts on the matter fields. For a single multiplet of matter fields in the defining representation of SO(8), we could find no way to implement a symmetry larger than O(8). However, for a multiplet of matter fields in the adjoint representation, all of the discrete symmetries can be represented. The full symmetry group has six components.

Let $\mathfrak{D}$ represent the order-6 finite group of discrete symmetries. The particles of the theory come in multiplets. An element of $\mathfrak{D}$ either carries the multiplet into itself or produces another degenerate multiplet. Now it is possible to have as many as six degenerate multiplets related to each other by the action of $\mathfrak{D}$. When the manifold is not simply connected, there will be many possibilities. There will be as many possibilities as there are inequivalent homomorphisms of the fundamental group of the manifold into $\mathfrak{D}$.[7]

The remaining simple Lie algebras can also be used to construct gauge theories. These will not be treated in any detail. However, two simple observations can be made. The Yang-Mills action for an E_6 gauge field has a disconnected symmetry group with two components. We can expect a similarity with the O(2N) case if the matter fields are in a real representation, and with the SU(N) case if the representation is complex. The Yang-Mills actions for theories based on the algebras C_n, G_2, F_4, E_7, and E_8 have connected symmetry groups. We can expect these cases to be similar to the SO($2N+1$) case.

Thus much of the discussion of the previous sections carries over to the non-Abelian theories. However, the details of the physical interpretation depend upon the group, the representation of the group carried by the matter fields, and the bound-state spectrum of the model.

VI. CONCLUSION

We have found that it is natural to include certain discrete internal symmetries such as charge conjugation in a gauge group. The full gauge group is then disconnected. Particularly interesting examples arise in gauge theories based on the algebras su(N) and so(2N). SO(8) is a special case. The Yang-Mills action has a symmetry group with six components.

This enlargement of the gauge group does not lead to new results if space-time is simply connected. However, on non-simply-connected manifolds, it may be impossible to give a global internal-symmetry convention. The simplest and most graphic example is electric charge. A particle could leave as a positive charge for a trip around a noncontractible loop and return as a negative charge. Although a handle in space with this property would be useful, we cannot imagine how one might be constructed.

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[1]Discussions from a similar point of view can be found in Tai Tsun Wu and Chen Ning Yang, Phys. Rev. D 12, 3845 (1975); Nucl. Phys. B107, 365 (1976).

[2]A manifold is contractible if it is homotopic to a point. Thus, all of its homotopy groups must be trivial.

[3]With attention restricted to the space manifold, the electric field can be viewed as a one-form. The dual (*) and exterior derivative (d) operators are those associated with the space manifold rather than space-time.

[4]Some discussion and references appear in C. W. Misner, K. S. Thorne, and J. A. Wheeler, *Gravitation* (Freeman, San Francisco, 1973), Chap. 44.

[5]C. W. Misner and J. A. Wheeler, in *Geometrodynamics*, edited by J. A. Wheeler (Academic, New York, 1962).

[6]N. Jacobson, *Lie Algebras* (Wiley, New York, 1962).

[7]N. Steenrod, *The Topology of Fiber Bundles* (Princeton Univ. Press, Princeton, New Jersey, 1951).