Gauge invariance, minimal coupling, and torsion

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A formalism is given which makes it possible for a modified form of local gauge invariance and minimal coupling to be compatible with torsion. One consequence is a restriction on the possible form of torsion in that it must be determined by the gradient of a scalar function. Furthermore, a dynamical theory is obtained for this scalar and hence allows propagation of torsion in vacuum. The Lagrangian density for interacting electromagnetic, gravitational, torsion, and complex scalar fields is presented, as well as the resulting field equations. The formalism implies the existence of both electric and magnetic currents due to the interaction of the electromagnetic and torsion fields.

I. INTRODUCTION

Gauge invariance is one of the most basic principles of field theory. Another principle that is used extensively to construct theories of interacting fields is that of minimal coupling. This principle is applied in two different contexts: In general relativity one constructs a minimally coupled theory by letting the metric of special relativity $\eta_{\mu\nu}$ go to a general metric $g_{\mu\nu}$ and by replacing ordinary derivatives by covariant derivatives.[1] Minimal coupling of the electromagnetic and charged fields is achieved by adding to partial derivatives in the usual Lagrangian density of the charged fields a term linear in the electromagnetic potential A_μ.[2]

Recently there has been considerable interest in the Einstein-Cartan theory of gravity, which allows nonsymmetric connection coefficients $\Gamma^\mu{}_{\nu\alpha}$. This theory has a nonzero torsion tensor $T^\mu{}_{\alpha\beta}$ (Ref. 3):

$$T^\mu{}_{\nu\alpha} = \Gamma^\mu{}_{\alpha\nu} - \Gamma^\mu{}_{\nu\alpha}. \qquad (1)$$

The usual definition of the electromagnetic field tensor $F_{\mu\nu}$ in general relativity is

$$F_{\mu\nu} = A_{\nu;\mu} - A_{\mu;\nu}, \qquad (2)$$

where a semicolon signifies covariant differentiation involving the connection coefficients $\Gamma^\alpha{}_{\mu\nu}$. As Hehl *et al.*[3] have pointed out, this definition is incompatible with the Einstein-Cartan theory if we want the coupling of electromagnetism to torsion to be invariant under the usual gauge transformation $A_\mu \to A'_\mu = A_\mu + \Lambda_{,\mu}$, where Λ is a scalar function. The solution proposed by Hehl *et al.*[3] is todispense with minimal coupling and to define $F_{\mu\nu}$ as

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} = A_{\nu|\mu} - A_{\mu|\nu}, \qquad (3)$$

where the bar symbol denotes a covariant derivative using the Christoffel symbols of the metric (ignoring torsion).

This type of definition means that photons are decoupled from torsion. If, however, we accept the general principle that spinning particles both generate and react to torsion,[3] it is reasonable to expect that photons should also be coupled to torsion. We should also point out that Hehl *et al.*[3] use Eq. (2) to define the field tensor of the massive vector (Proca) field. It is possible to use this definition because there is no gauge invariance for a massive vector field. We will show in this paper that Eq. (2) may also be used in the pure electromagnetic case in a theory involving torsion.

In this article we keep the idea of minimal coupling between electromagnetism and gravitation when torsion is nonzero in the form of Eq. (2) and also minimal coupling between electromagnetic and charged fields. We also retain the idea of gauge invariance, slightly modified in order to be compatible with minimal coupling when torsion is present. The principles of gauge invariance and minimal coupling give a restriction on the torsion. When torsion vanishes, the usual form of gauge transformations is recovered.

The allowed torsion in this theory is completely determined by the gradient of a scalar field ϕ. This type of torsion may be used to construct a theory of gravitation which includes a dynamical theory of torsion, where torsion is allowed to propagate and to be nonzero in the vacuum. Thus, our theory constitutes a major departure from the Einstein-Cartan theory, where torsion appears only algebraically in the field equations and cannot propagate in the absence of matter.

In Sec. II of this paper we discuss the results of requiring gauge invariance and minimal coupling between gravitation, electromagnetism, and a complex scalar field in the presence of torsion. In Sec. III we give a Lagrangian density for the four fields: metric, torsion, electromagnetic, and complex scalar. In Sec. IV we present the field equations and discuss some of their properties. One interesting feature of these equations is a magnetic current due to the interaction of the torsion potential ϕ and the electromagnetic field. Section V is a summary and conclusion.

II. MINIMAL COUPLING AND GAUGE INVARIANCE

In this section we start with the Lagrangian density $\mathfrak{L}_\psi$ for a complex massless scalar field (g is the determinant of the spacetime metric):

$$\mathfrak{L}_\psi = -\frac{1}{4\pi}\psi^*_{,\mu}\psi^{,\mu}\sqrt{-g}\,. \tag{4}$$

We consider, following Utiyama,[4] the group of gauge transformations

$$\psi \to \psi' = e^{iq\Lambda}\psi, \tag{5}$$

where q is the electromagnetic coupling constant. Here Λ is a constant, but we enlarge the group to include all such transformations with Λ a function of spacetime,

$$\Lambda = \Lambda(x). \tag{6}$$

The Lagrangian density $\mathfrak{L}_\psi$ is not invariant under these transformations, but it may be made invariant by replacing the gradient operation by a new operation. The new operation includes the effect of a compensating field.

The compensating field is the electromagnetic four-potential A_μ. (It is natural to have the field ψ coupled to the electromagnetic field since a complex scalar has an associated conserved current.) We interpret the principle of minimal coupling as it applies to the coupling of electromagnetism and the charged scalar field to mean that the new derivative operation should depend linearly on A_μ but not on its derivatives. In the presence of an electromagnetic field, therefore, the Lagrangian density $\mathfrak{L}_\psi$ becomes

$$\mathfrak{L}'_\psi = -\frac{1}{4\pi}\psi^*_{.\mu}\psi^{.\mu}\sqrt{-g}\,, \tag{7}$$

where

$$\psi_{.\mu} = \psi_{,\mu} - iqb_\mu{}^\alpha A_\alpha\psi. \tag{8}$$

The function $b_\mu{}^\alpha$ will in general be a function of spacetime (but not of A_μ), and this is a generalization of the usual procedure. This generalization is required in order to have both gauge invariance and minimal coupling in the presence of torsion. When torsion vanishes, $b_\mu{}^\alpha$ reduces to the usual $\delta_\mu{}^\alpha$.

Under the gauge transformation defined by $\Lambda(x)$, A_μ must also transform in order to leave $\mathfrak{L}'_\psi$ invariant. Let the vector ξ_μ be defined by the A_μ transformation

$$A_\mu \to A'_\mu = A_\mu + \xi_\mu. \tag{9}$$

$\mathfrak{L}'_\psi$ will be invariant provided

$$b_\mu{}^\alpha\xi_\alpha = \Lambda_{,\mu}. \tag{10}$$

Thus the effect of the gauge transformation on A_μ is that A_μ is changed by a linear combination of the derivatives of Λ.

At this point we turn to the coupling of the gravitational field (including torsion) to the electromagnetic field. Minimal coupling in the absence of torsion is taken to mean[1] that the connection is the unique metric-compatible symmetric connection defined by the spacetime metric, which in turn is changed from the flat metric $\eta_{\mu\nu}$ to the general metric $g_{\mu\nu}$. We will denote partial differentiation of a vector field V_μ by

$$V_{\mu,\nu} = \frac{\partial}{\partial x^\nu}V_\mu\,.$$

The symmetric connection determined by $g_{\mu\nu}$ has coefficients $\{^\alpha{}_{\beta\gamma}\}$, and we shall denote this type of covariant derivative with a bar:

$$\begin{aligned} &V_{\mu|\nu} = V_{\mu,\nu} - V_\sigma\{^\sigma{}_{\mu\nu}\},\\ &\{^\sigma{}_{\mu\nu}\} = \tfrac{1}{2}g^{\sigma\tau}(g_{\tau\mu,\nu} + g_{\tau\nu,\mu} - g_{\mu\nu,\tau}). \end{aligned} \tag{11}$$

In the presence of torsion, the coefficients of covariant differentiation will be denoted by $\Gamma^\alpha{}_{\beta\gamma}$, and the derivative will be expressed with a semicolon:

$$\begin{aligned} &V_{\mu;\alpha} = V_{\mu,\alpha} - V_\sigma\Gamma^\sigma{}_{\mu\alpha},\\ &\Gamma^\sigma{}_{\mu\alpha} = \{^\sigma{}_{\mu\alpha}\} - \tfrac{1}{2}(T^\sigma{}_{\mu\alpha} - T_\mu{}^\sigma{}_\alpha - T_\alpha{}^\sigma{}_\mu). \end{aligned} \tag{12}$$

The torsion tensor $T^\sigma{}_{\mu\alpha}$ is defined in Eq. (1). This connection is metric compatible, as is $\{^\alpha{}_{\beta\gamma}\}$:

$$g_{\mu\nu;\gamma} = 0 = g_{\mu\nu|\gamma}\,.$$

When torsion is present we have at least two possibilities for the expression of the minimal-coupling principle. As a matter of fact some authors[3] use both of the above prescriptions for covariant differentiation concomitantly. They do so in order to retain gauge invariance of the electromagnetic field in its usual form. We take as the principle of minimal coupling the replacement of partial differentiation by covariant differentiation using the $\Gamma^{\alpha}{}_{\beta\gamma}$ connection for all fields. We are able also to keep gauge invariance in the general form as given above.

Thus the electromagnetic field tensor $F_{\mu\nu}$ is given by

$$\begin{aligned} F_{\mu\nu} &= A_{\nu;\mu} - A_{\mu;\nu} \\ &= A_{\nu|\mu} - A_{\mu|\nu} - A_\sigma T^\sigma{}_{\mu\nu} \\ &= A_{\nu,\mu} - A_{\mu,\nu} - A_\sigma T^\sigma{}_{\mu\nu}. \end{aligned} \tag{13}$$

We have seen that under a gauge transformation, A_μ transforms as

$$A_\mu \to A'_\mu = A_\mu + c_\mu{}^\alpha \Lambda_{,\alpha}, \tag{14}$$

where $c_\mu{}^\alpha$ is a set of functions of spacetime. Parenthetically, we should add that a natural extension of the usual gauge transformation law would demand that A'_μ depend linearly on the derivatives of the function Λ. Gauge invariance for a complex scalar field restricts the dependence to the first derivatives of Λ.

The electromagnetic field is gauge invariant provided that

$$c_\nu{}^\alpha \Lambda_{,\alpha\mu} - c_\mu{}^\alpha \Lambda_{,\alpha\nu} + (c_\nu{}^\alpha{}_{,\mu} - c_\mu{}^\alpha{}_{,\nu} - c_\sigma{}^\alpha T^\sigma{}_{\mu\nu})\Lambda_{,\alpha} = 0 \tag{15}$$

for all Λ. Consequently, it follows that

$$c_\nu{}^{(\alpha}\delta_\mu{}^{\beta)} - c_\mu{}^{(\alpha}\delta_\nu{}^{\beta)} = 0\,, \tag{16}$$

$$c_\nu{}^\alpha{}_{,\mu} - c_\mu{}^\alpha{}_{,\nu} - c_\sigma{}^\alpha T^\sigma{}_{\mu\nu} = 0\,. \tag{17}$$

The first of the above equations requires that $c_\mu{}^\sigma$ be proportional to $\delta_\mu{}^\sigma$, and we write

$$c_\mu{}^\sigma = f\delta_\mu{}^\sigma, \quad f = \tfrac{1}{4}c_\mu{}^\mu\,. \tag{18}$$

Equation (17) gives

$$fT^\alpha{}_{\mu\nu} = \delta_\nu{}^\alpha f_{,\mu} - \delta_\mu{}^\alpha f_{,\nu}\,. \tag{19}$$

If $f = 0$ at a point, its derivatives are zero at that point, and $c_\mu{}^\alpha = 0$. We discard this singular case and treat the case where f is of one sign everywhere. In order to agree with Eq. (10) and the limiting case $b_\mu{}^\alpha \to \delta_\mu{}^\alpha$ when torsion vanishes, we choose a positive sign and write $f = e^\phi$. The scalar field ϕ may also be defined from a spinorial point of view. Its study, particularly in the presence of other general forms of matter, will be the subject of other papers.[5] The tentative name "tlaplon" (derived from a Náhuatl root) has been given to the field.

We thus have

$$c_\mu{}^\sigma = e^\phi \delta_\mu{}^\sigma, \tag{20}$$

$$T^\alpha{}_{\mu\nu} = \delta_\nu{}^\alpha \phi_{,\mu} - \delta_\mu{}^\alpha \phi_{,\nu}\,. \tag{21}$$

The special case ϕ = constant corresponds to the case of zero torsion. (The usual forms of gauge invariance and minimal coupling are possible only when torsion is zero.)

Therefore we see that minimal coupling and gauge invariance are consistent with nonvanishing torsion of a particular type. Using the above results, we have for the gauge transformations of the scalar and electromagnetic fields

$$\begin{aligned} &\psi \to \psi' = e^{iq\Lambda}\psi, \\ &A_\mu \to A'_\mu = A_\mu + e^\phi \Lambda_{,\mu}\,, \end{aligned} \tag{22}$$

and the minimal electromagnetic coupling is defined by the prescription

$$\psi_{,\mu} \to \psi_{\cdot\mu} = \psi_{,\mu} - iqe^{-\phi}A_\mu\psi\,. \tag{23}$$

The tlaplon field ϕ serves as a potential for torsion, and we will thus be able to define a theory with propagating torsion in contrast to the Einstein-Cartan theory.

III. LAGRANGIAN DENSITIES

Making use of Eqs. (22) and (23), we can now write the gauge-invariant Lagrangian density $\mathcal{L}'_\psi$ for the complex scalar field ψ as

$$\begin{aligned} \mathcal{L}'_\psi &= -\frac{1}{4\pi}\psi^*_{\cdot\mu}\psi^{\cdot\mu}\sqrt{-g} \\ &= -\frac{1}{4\pi}(\psi^*_{,\mu} + iqe^{-\phi}A_\mu\psi^*)(\psi^{,\mu} - iqA^\mu\psi)\sqrt{-g}\,. \end{aligned} \tag{24}$$

The Lagrangian density $\mathcal{L}_{g\phi}$ for the gravitational field with torsion is[3]

$$\mathcal{L}_{g\phi} = \frac{1}{16\pi}R'\sqrt{-g}\,, \tag{25}$$

where R' is the scalar curvature derived from the connection $\Gamma^\alpha{}_{\beta\gamma}$, which includes torsion. If we denote by R the scalar curvature derived from the symmetric connection $\{^{\alpha}_{\beta\gamma}\}$, we have

$$\mathcal{L}_{g\phi} = \mathcal{L}_g + \mathcal{L}_\phi + \text{ total divergence}, \tag{26}$$

where

$$\mathcal{L}_g = \frac{1}{16\pi}R\sqrt{-g}\,, \tag{27}$$

$$\mathcal{L}_\phi = -\frac{3}{8\pi}\phi_{,\alpha}\phi^{,\alpha}\sqrt{-g}\,. \tag{28}$$

We eliminate the total divergence term in going from $\mathcal{L}_{g\phi}$ to $\mathcal{L}_g + \mathcal{L}_\phi$.

We now turn to the Lagrangian density for the electromagnetic field. No further interaction Lagrangian densities are allowed among the four fields we are considering $(\psi, \phi, g_{\mu\nu}, A_\mu)$ according to the principle of minimal coupling. The usual electromagnetic Lagrangian density is[1]

$$\mathcal{L}_A = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} \sqrt{-g}\,. \qquad (29)$$

We retain this form since it is the embodiment of the minimal coupling between gravitation plus torsion and electromagnetism.

The scalar field "feels" torsion only in the presence of the electromagnetic field: If $A_\mu = 0$, there is no torsion term in $\mathcal{L}'_\psi$. If $A_\mu \neq 0$, it is tempting to define B_μ by

$$B_\mu = e^{-\phi} A_\mu\,, \qquad (30)$$

so that

$$\mathcal{L}'_\psi = -\frac{1}{4\pi}(\psi^*_{,\mu} + iqB_\mu \psi^*)(\psi^{,\mu} - iqB^\mu \psi)\sqrt{-g}\,. \qquad (31)$$

This field B_μ obeys the gauge-transformation law

$$B_\mu \to B'_\mu = B_\mu + \Lambda_{,\mu}\,. \qquad (32)$$

The gauge-invariant field which is derived from B_μ is

$$H_{\mu\nu} = B_{\nu,\mu} - B_{\mu,\nu}\,. \qquad (33)$$

It is readily verified that

$$F_{\mu\nu} = e^{\phi} H_{\mu\nu}\,. \qquad (34)$$

For a Lagrangian density we have two choices, proportional to $F_{\mu\nu}F^{\mu\nu}\sqrt{-g}$ or to $H_{\mu\nu}H^{\mu\nu}\sqrt{-g}$. These two choices are physically different. The second choice, with $H_{\mu\nu}$, involves the usual form of gauge invariance but does not obey the gravitational minimal-coupling principle that derivatives should involve the connection $\Gamma^\alpha{}_{\beta\nu}$. This choice has been taken by Hehl *et al.*[3] We have chosen the Lagrangian density involving $F_{\mu\nu}$ because we can then retain both gauge invariance and minimal coupling (in a modified sense), and in this case the electromagnetic field and torsion interact.

IV. FIELD EQUATIONS

The total Lagrangian density is

$$\mathcal{L} = \frac{\sqrt{-g}}{4\pi}(R - 6\phi_{,\alpha}\phi^{,\alpha} - F_{\mu\nu}F^{\mu\nu} - 4\psi^*_{\cdot\mu}\psi^{\cdot\mu})\,. \qquad (35)$$

Using the previously derived form for the torsion, we list the definitions

$$\psi_{\cdot\mu} = \psi_{,\mu} - iqe^{-\phi}A_\mu\psi\,, \qquad (36)$$

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} + A_\mu\phi_{,\nu} - A_\nu\phi_{,\mu}\,. \qquad (37)$$

The connection is

$$\Gamma^\alpha{}_{\beta\gamma} = \{^\alpha{}_{\beta\gamma}\} - \delta^\alpha{}_\gamma\phi_{,\beta} + g_{\beta\gamma}\phi^{,\alpha}\,, \qquad (38)$$

where $\{^\alpha{}_{\beta\gamma}\}$ is the symmetric connection generated by the metric $g_{\mu\nu}$. With the above form of the connection, the definitions (11) and (12) become

$$V_{\alpha|\beta} = V_{\alpha,\beta} - V_\sigma\{^\sigma{}_{\alpha\beta}\}\,,$$
$$V_{\alpha;\beta} = V_{\alpha,\beta} - V_\sigma\Gamma^\sigma{}_{\alpha\beta} = V_{\alpha|\beta} + V_\beta\phi_{,\alpha} - g_{\alpha\beta}V_\sigma\phi^{,\sigma}\,. \qquad (39)$$

The field equations are obtained from independent variation of $g_{\mu\nu}$, A_μ, ϕ, ψ, and ψ^* in $\mathcal{L}$. They are ($G^{\mu\nu}$ is the Einstein tensor)

$$\delta g_{\mu\nu}:\ G^{\mu\nu} - 6(\phi^{,\mu}\phi^{,\nu} - \tfrac{1}{2}g^{\mu\nu}\phi_{,\alpha}\phi^{,\alpha}) - 2(F^{\mu\alpha}F^\nu{}_\alpha - \tfrac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta}) - 4(\psi^{*\cdot(\mu}\psi^{\cdot\nu)} - \tfrac{1}{2}g^{\mu\nu}\psi^{*\cdot\alpha}\psi_{\cdot\alpha}) = 0\,, \qquad (40)$$

$$\delta A_\mu:\ F^{\mu\nu}{}_{|\nu} + F^{\mu\nu}\phi_{,\nu} - iqe^{-\phi}(\psi^{*\cdot\mu}\psi - \psi^{\cdot\mu}\psi^*) = 0, \qquad (41)$$

$$\delta\phi:\ 3\phi^{|\alpha}{}_\alpha + (F^{\mu\nu}A_\mu)_{|\nu} + iqe^{-\phi}A_\mu(\psi^{\cdot\mu}\psi^* - \psi^{*\cdot\mu}\psi) = 0\,, \qquad (42)$$

$$\delta\psi:\ \psi^{*\cdot\mu}{}_{|\mu} + iqe^{-\phi}A_\mu\psi^{*\cdot\mu} = 0\,, \qquad (43)$$

$$\delta\psi^*,\ \ \psi^{\cdot\mu}{}_{|\mu} - iqe^{-\phi}A_\mu\psi^{\cdot\mu} = 0\,. \qquad (44)$$

Equation (42) simplifies when the field Eq. (41) is used:

$$\phi^{|\alpha}{}_\alpha - \tfrac{1}{6}F^{\mu\nu}F_{\mu\nu} = 0\,. \qquad (45)$$

The above equations are gauge invariant. This invariance is obvious for Eqs. (40) and (41). In Eq. (42), a gauge transformation adds a term which vanishes because of Eq. (41). A gauge transformation of Eq. (43) or (44) produces an additional term which vanishes because of (43) or (44) itself. Also note that Eq. (44) is of course the complex conjugate of Eq. (43).

The field equations (40) through (45) are not independent, but are related by the five identities which arise from gauge invariance of the electromagnetic and gravitational fields.[6] In the electromagnetic case, the field equation (41) is obtained by variation of the action $S = \int \mathcal{L} d^4x$:

$$\int \frac{\delta S}{\delta A_\mu}\,\delta A_\mu d^4x = 0\,. \qquad (46)$$

When δA_μ is of the form $e^\phi\Lambda_{,\mu}$, the resulting equation is an identity. When $\delta A_\mu = e^\phi\Lambda_{,\mu}$ is inserted and an integration by parts is made, we have

$$\int \left(\frac{\delta S}{\delta A_\mu}e^\phi\right)_{,\mu}\Lambda d^4x = 0 \qquad (47)$$

for all Λ. The equation which results is

$$\left(\frac{\delta S}{\delta A_\mu}e^\phi\right)_{,\mu} = \left(\frac{\delta S}{\delta A_\mu}\right)_{,\mu} e^\phi + \frac{\delta S}{\delta A_\mu}e^\phi\phi_{,\mu} = 0\,. \qquad (48)$$

The second term vanishes by Eq. (41), and the remaining equation,

$$\left(\frac{\delta S}{\delta A_\mu}\right)_{,\mu} = 0\,, \qquad (49)$$

is the identity corresponding to the ordinary electric-current-conservation law in a vacuum. We will return to this law below.

In the gravitational case, an infinitesimal gauge transformation is the same as an infinitesimal coordinate transformation. Such a transformation is generated by a vector field ζ_μ and results in a variation of $g_{\mu\nu}$ of the form

$$\delta g_{\mu\nu} = \zeta_{(\mu|\nu)} \,. \tag{50}$$

When this form is inserted into the variational principle, we find

$$\int \frac{\delta S}{\delta g_{\mu\nu}} \zeta_{\mu|\nu}\, d^4x = 0\,, \tag{51}$$

since the $\delta S/\delta g_{\mu\nu}$ term picks out the symmetric part of $\zeta_{\mu|\nu}$. Since $\delta S/\delta g_{\mu\nu}$ is a tensor density, we may integrate by parts and obtain the usual identities

$$\left(\frac{\delta S}{\delta g_{\mu\nu}}\right)_{|\nu} = 0\,. \tag{52}$$

The above conservation laws can be derived directly from the field equations. In doing so, the following equations are used:

$$\psi_{.\alpha|\beta} - \psi_{.\beta|\alpha} = -iqe^{-\phi}(F_{\beta\alpha}\psi + A_\alpha\psi_{.\beta} - A_\beta\psi_{.\alpha})\,, \tag{53}$$

$$F_{[\mu\nu|\gamma]} = F_{[\mu\nu}\phi_{,\gamma]}\,. \tag{54}$$

These equations follow from the definitions of $\psi_{.\mu}$ and $F_{\mu\nu}$. The conservation laws are useful in the development of a full theory of gravity including general sources, and they will be discussed further in a forthcoming paper.[5]

Although we will not exhibit exact solutions here, it is important to note some of the effects due to torsion. Equation (45) shows that the electromagnetic field serves as a source for torsion and that torsion obeys a wave equation. Thus torsion need not vanish in a vacuum, and torsion and electromagnetism interact. These are some of the differences between our theory and the usual Einstein-Cartan theory. The electromagnetic source term for torsion includes terms quadratic in ϕ. This nonlinearity leads one to expect a rich structure of solutions.

Other effects of torsion can be observed in the equations for the scalar and electromagnetic fields. The effect of torsion on ψ hinges on a nonzero value of A_μ. To show the effects of torsion on the electromagnetic field, Eqs. (41) and (54) have been written in the form most useful for observational purposes. In fact, since uncharged spinless particles follow paths which are extremals based on the metric,[7] a freely falling observer would use a reference frame in which $\{^{\alpha}{}_{\beta\gamma}\}$ vanishes along his timelike line. Such an observer would use $F^{\mu\nu}{}_{,\nu}$ and $F_{[\mu\nu,\gamma]}$ to define the electric and magnetic currents in this coordinate patch. In other coordinates, therefore, $F^{\mu\nu}{}_{|\nu}$ and $F_{[\mu\nu|\gamma]}$ define these currents.

Explicitly, we see that the torsion in combination with $F_{\mu\nu}$ generates both electric and magnetic currents and therefore acts as part of the source for electromagnetism. This electric current is

$$j_\phi{}^\mu = -F^{\mu\nu}\phi_{,\nu}\,. \tag{55}$$

This current is conserved provided ψ vanishes:

$$j_\phi{}^\mu{}_{|\mu} = 0 \quad \text{if } \psi = 0\,. \tag{56}$$

If ψ does not vanish, it is the total electric current which is conserved, as usual, because of the identity (49).

The magnetic current associated with ϕ is

$$m_\phi{}^\mu = {}^*F^{\mu\nu}\phi_{,\nu}\,, \tag{57}$$

where ${}^*F^{\mu\nu}$ is the dual of $F_{\mu\nu}$. This current is conserved:

$$m_\phi{}^\mu{}_{|\mu} = 0\,. \tag{58}$$

The total magnetic charge associated with $m_\phi{}^\mu$ vanishes, however. To prove this fact we define the magnetic charge M in terms of a spacelike hypersurface Σ, with the associated volume element d^3x_μ:

$$M = \int_\Sigma \sqrt{-g}\, m_\phi{}^\mu d^3x_\mu\,. \tag{59}$$

If Σ is taken as normal to the vector $(1, 0, 0, 0)$, then $d^3x_\mu = \delta_\mu{}^0 d^3x$, and M becomes

$$\begin{aligned} M &= \int \sqrt{-g}\, {}^*F^{0\nu}\phi_{,\nu} d^3x \\ &= -\tfrac{1}{2} \int \epsilon^{0\nu\alpha\beta} F_{\alpha\beta}\phi_{,\nu} d^3x\,, \end{aligned} \tag{60}$$

where $\epsilon^{0\nu\alpha\beta}$ is the numerical completely antisymmetric symbol defined by $\epsilon^{0123} = 1$. With the definition of $F_{\alpha\beta}$, M may be written as

$$\begin{aligned} M &= -\int \epsilon^{0\nu\alpha\beta} A_{\beta,\alpha}\phi_{,\nu} d^3x \\ &= -\int (\epsilon^{orst} A_{t,s}\phi)_{,r} d^3x \quad \text{with } r, s, t, = 1, 2, 3\,. \end{aligned} \tag{61}$$

Since M is the integral of a pure divergence, it may be expressed as an integral over an infinitely large two-surface. With an assumption of asymptotic zero values for the fields, and in a simply connected space, the total value for M becomes zero.

We should mention that when $\psi = 0$, the transformation $\phi \to \phi' = \phi + c$ with $c = \text{const}$, preserves

$\mathcal{L}$. The current $\partial\mathcal{L}/\partial\phi_{,\mu}$ is then a conserved current. In general, if $\psi \neq 0$, the transformation $\phi \to \phi' = \phi + c$, with $c = \text{const}$, along with $A_\mu \to A'_\mu = e^c A_\mu$, $g_{\mu\nu} \to g'_{\mu\nu} = e^{2c} g_{\mu\nu}$, $\psi \to \psi' = \psi$, changes $\mathcal{L}$ to $\mathcal{L}' = e^{4c}\mathcal{L}$. This transformation also preserves the field equations.

V. SUMMARY AND CONCLUSION

The basic results of this paper are the development of a formalism making torsion compatible with the principles of gauge invariance and minimal coupling and a dynamical theory of torsion which allows its propagation in a vacuum.

In order to have both minimal coupling and gauge invariance along with torsion, the form of gauge transformations must be modified according to Eq. (22). This modification, however, still preserves the essential character of the electromagnetic gauge-transformation group, and our modified form reduces to the usual gauge transformations when torsion vanishes. We have found that torsion must have a restricted form, given by Eq. (21). Torsion is generated by a scalar function ϕ.

The existence of the function ϕ, the potential function for torsion, has allowed us to formulate a dynamical theory of torsion. Our starting point is the usual Lagrangian densities of gravitation, electromagnetism, and a charged scalar field. The total Lagrangian is constructed in accord with the principles of minimal coupling of the electromagnetic field and of the gravitation field to other fields. The specific forms of minimal coupling are expressed in Eqs. (8) and (13).

The field equation for ϕ is a wave equation whose source is a nonlinear combination of the electromagnetic potential A_μ and ϕ, which vanishes when $A_\mu = 0$. The source term is gauge invariant; indeed, the Lagrangian density itself was chosen to be explicitly gauge invariant. The field equation is for the potential of torsion and not torsion itself. It involves derivatives of the spin angular momentum tensor and thus clears up a troublesome point in the usual Einstein-Cartan theory: In that theory, electromagnetism must be treated as exempt from minimal coupling.[3]

Some of the novel features of our theory include the way torsion affects other fields and itself. The charged scalar field ψ, for example, is affected by torsion only in the presence of an electromagnetic field, as shown in Eqs. (43) and (44). In the case of the electromagnetic field, there are electric and magnetic current terms, $j_\phi{}^\mu$ and $m_\phi{}^\mu$, respectively, involving the product of the electromagnetic field and torsion. The total electric current, including contributions from charged fields, is conserved. The torsion-induced magnetic current is conserved by itself, but it gives rise only to a total magnetic charge of zero (in a simply connected space where the fields fall off to zero sufficiently fast at spatial infinity).

At this stage we can only speculate as to the physical meaning of the current terms $j_\phi{}^\mu$ and $m_\phi{}^\mu$, which are defined in Eqs. (55) and (57). The magnetic current $m_\phi{}^\mu$ may prove to be unobservable because of the zero value of the total magnetic charge. In a localized area of torsion, perhaps associated with a particle-like ("tlaplon") state of the ϕ field, the magnetic charge vanishes. It may be, however, that magnetic charge effects can be observed in a large region of nonzero torsion.

In future papers we will be developing this theory into a true gravitational theory involving general sources. We will also be investigating the properties of specific solutions. The nonlinear character of our field equations means that these solutions may be expected to have a rich structure.

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